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On extremal domains and codomains for convolution of distributions and fractional calculus

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Abstract
It is proved that the class of c-closed distribution spaces contains extremal domains and codomains to make convolution of distributions a well-defined bilinear mapping. The distribution spaces are systematically endowed with topologies and bornologies that make convolution hypocontinuous whenever defined. Largest modules and smallest algebras for convolution semigroups are constructed along the same lines. The fact that extremal domains and codomains for convolution exist within this class of spaces is fundamentally related to quantale theory. The quantale theoretic residual formed from two c-closed spaces is characterized as the largest c-closed subspace of the corresponding space of convolutors. The theory is applied to obtain maximal distributional domains for fractional integrals and derivatives, for fractional Laplacians, Riesz potentials and for the Hilbert transform. Further, maximal joint domains for families of these operators are obtained such that their composition laws are preserved.

Keywords Convolution · Spaces of distributions · Quantales · Fractional calculus

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1 Introduction

A recurring problem in applications of distribution theory is to find an optimal domain for a given convolution operator or semigroups of such [1,20,34,56]. Here, "optimal" means as large as admissible for convolution of distributions [45,50,52,55]. In other

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words, optimal spaces are defined as the convolution duals (originally called "c-duals" [24,58]) of a given set of distributions.

The main goal of this work is to develop a systematic method to construct optimal domains and codomains of distributions. Our method applies to general sets or semigroups of convolution operators.

Different from most investigations on this topic we study sets of function spaces endowed with a composition of spaces that is naturally induced by convolution. The emerging algebraic and order theoretic structures are investigated in detail, and they reveal interesting links to quantale theory [47], [17, p. 114-116]. Subsequently, the distribution spaces are endowed with topologies and bornologies in a uniform way, which ensures continuity and boundedness properties of convolutions between these spaces. The application of our general theory is exemplified for convolution operators used in fractional calculus and for the Hilbert transform.

To explain the crosslinks with quantale theory recall first the general order theoretic concept of Galois connections on power sets (also called "polarities") [3, p. 122], [17, p. 116-120]. According to [3, Thm. 19 & Cor., p. 123] any symmetric binary relation $R \subseteq X \times X$ on a set X induces a "symmetric Galois connection"

$$G_R \colon \mathfrak{P}(X) \to \mathfrak{P}(X), \quad Y \mapsto G_R(Y) := \{z \in X\,;\, \forall y \in Y : yRz\} \tag{1.1}$$

and an associated closure operator $H_R := G_R \circ G_R$. Taking the set $X := \mathscr{D}'$ of distributions and the relation $yRz :=$ "y and z are convolvable" furnishes convolution duals $G_R = (-)^*_{\mathscr{D}'}$ and perfections $H_R = (-)^{**}_{\mathscr{D}'}$ [58, p. 20]. The elements of $(\mathfrak{P}(\mathscr{D}'))^{**}_{\mathscr{D}'} = \{U \subseteq \mathscr{D}'\,;\, (U)^{**}_{\mathscr{D}'} = U\}$, the closure system associated to $(-)^{**}_{\mathscr{D}'}$, are called convolution perfect spaces. These spaces are linear by the definition of convolvability and $(\mathfrak{P}(\mathscr{D}'))^{**}_{\mathscr{D}'}$ constitutes a complete lattice [3].

The link between convolution perfect spaces and quantales emerges when studying extremality of inclusions $U * V \subseteq W$, where $U, V, W \subseteq \mathscr{D}'$ are convolution perfect. This becomes clear from our extremality Theorem 1.

Theorem 1 *Let U, V and W be convolution perfect spaces such that convolution of distributions is well-defined as a bilinear mapping $* \colon U \times V \to W$. Then, there exists a largest convolution perfect space U' containing U and a smallest convolution perfect space W' contained in W such that $* \colon U' \times V \to W$ and $* \colon U \times V \to W'$ are well-defined bilinear mappings in the same sense.*

The second part of our extremality theorem is immediate from linearity properties of convolution and the fact that convolution perfect spaces constitute a closure system. Let $\widetilde{*}$ denote the partially defined composition

$$U \,\widetilde{*}\, V := (U * V)^{**}_{\mathscr{D}'} \tag{1.2}$$

of convolution perfect spaces U and V that is defined if and only if U und V are convolvable elementwise. Clearly, the space $U \,\widetilde{*}\, V$ is the unique solution for W' in Theorem 1. This applies to convolution operators in the following way: Any set of distributions $U \subseteq \mathscr{D}'$ induces a bilinear mapping

$$*\colon (U)^{*}_{\mathscr{D}'} \times (U)^{**}_{\mathscr{D}'} \to (U)^{*}_{\mathscr{D}'} \mathbin{\widetilde{*}} (U)^{**}_{\mathscr{D}'}. \tag{1.3}$$

The space $(U)^{*}_{\mathscr{D}'}$ serves as the largest joint domain for convolution operators with kernels from U and $(U)^{**}_{\mathscr{D}'}$ is the smallest convolution perfect space containing the kernels. Note, that when all convolution operators K with kernel $u \in U$ are represented by $K(v) = u * v = B(u, v)$ for all $v \in (U)^{*}_{\mathscr{D}'}$ with some fixed bilinear convolution mapping B between convolution perfect spaces, then $U \mathbin{\widetilde{*}} V$ is the smallest possible codomain for this mapping B.

The partially defined operation (1.2) furnishes an algebraic and order theoretic structure $\big((\mathfrak{P}(\mathscr{D}'))^{**}_{\mathscr{D}'}, \subseteq, \widetilde{*}\big)$. This triple is identified as a "commutative quantale with partially defined operation", as explained in the following.

Up to formalities a *commutative quantale* [47,48] is a triple $(Q, \leq, \bullet)$ with $(Q, \leq)$ a complete lattice and $(Q, \bullet)$ a commutative semigroup such that

$$\sup(A \bullet b) = (\sup A) \bullet b \qquad \text{for all } A \subseteq Q,\ b \in Q. \tag{1.4}$$

Now, consider for $(Q, \leq)$ the set of convolution perfect distribution spaces, ordered by inclusion, enriched with the null vector space $\{0\}$ and an artificially adjoined largest element "∞". Define $\bullet$ as the extension of $\widetilde{*}$ to Q by $\{0\} \bullet U = \{0\}$ for $U \in Q$, by $U \bullet V = \infty$ for non-convolvable $U, V \in Q \setminus \{\infty\}$ and by $U \bullet \infty = \infty$ for $U \in Q \setminus \{\{0\}\}$. It will be proved that this defines a quantale $(Q, \leq, \bullet)$ and we will call it the *quantale associated to* $\big((\mathfrak{P}(\mathscr{D}'))^{**}_{\mathscr{D}'}, \subseteq, \widetilde{*}\big)$.

Commutative quantales $(Q, \leq, \bullet)$ possess *residuals* (using the nomenclature from [48, p. 922]) [47, p. 15]. The residual called "c by b" is defined as

$$c \not\bullet b := \sup\{a \in Q\,;\ a \bullet b \leq c\} \quad \text{for all } b, c \in Q. \tag{1.5}$$

The property (1.4) and the definition (1.5) result in the equivalence

$$a \bullet b \leq c \qquad \Leftrightarrow \qquad a \leq c \not\bullet b \qquad \text{for all } a, b, c \in Q, \tag{1.6}$$

see [47, p. 15], [17, Def. 3]. Because the quantale associated to $\big((\mathfrak{P}(\mathscr{D}'))^{**}_{\mathscr{D}'}, \subseteq, \widetilde{*}\big)$ possesses residuals it follows that the residual "W by V", given by the set

$$\bigcup \left\{U \in (\mathfrak{P}(\mathscr{D}'))^{**}_{\mathscr{D}'}\,;\ U * V \text{ exists and } U * V \subseteq W\right\}, \tag{1.7}$$

is a well-defined convolution perfect space if $V \subseteq W$ and is empty otherwise. Here $V \subseteq W$ is equivalent to $\mathscr{E}' \mathbin{\widetilde{*}} V \subseteq W$, because $\mathscr{E}'$ is identified as the neutral element for $\widetilde{*}$. By construction, the space defined by (1.7) is the unique solution for U' in Theorem 1. Wrapping up, Theorem 1 reflects the closure system property of convolution perfect spaces and the quantale property (1.4) of the composition $\widetilde{*}$ (more precisely, of its extension $\bullet$ from above).

Let us now outline the paper section by section, thereby introducing notations and highlighting some important results. Beginning in Sect. 2 we introduce and study the so called *Φ-absolute value*

$$|-|_\Phi : \mathscr{D}' \to \mathscr{C}^+,\ f \mapsto |f|_\Phi := \sup\{|\phi * f|\,;\ \phi \in \Phi\} \qquad \text{for } \Phi \in \mathfrak{B}(\mathscr{D}), \tag{1.8}$$

where $\mathfrak{B}(\mathscr{D}) := \{B \subseteq \mathscr{D}\,;\ B \text{ bounded}\}$. Note, that $|-|_\Phi$ is absolutely homogeneous and subadditive, just like a seminorm. In Theorems 2 and 3, these functionals are used to characterize mapping properties of convolution of distributions via convolution on $\mathscr{I}_+ := \{f : \mathbb{R}^d \to \overline{\mathbb{R}}_+ \text{ lower semicontinuous}\}$.

The first part of Sect. 3 studies the quantale structure induced by convolution on the set system $\mathfrak{I}_\mathrm{T}$, which consist of the *cone ideals*, i.e. non-empty additively closed downsets $I \subseteq \mathscr{I}_+$, that are *moderated*, i.e.

$$\overline{\mathrm{T}}_K I \subseteq I \qquad \text{for all compact } K \subseteq \mathbb{R}^d. \tag{1.9}$$

Here $\overline{\mathrm{T}}_K f := \sup\{\mathrm{T}_y f\,;\ y \in K\}$ is the *translation shell* of $f : \mathbb{R}^d \to \overline{\mathbb{R}}_+$ with T_y denoting translation by $y \in \mathbb{R}^d$. A composition $\check{*}$ is introduced on $\mathfrak{I}_\mathrm{T}$ via

$$I \,\check{*}\, J := (I * J)^\curlyvee \qquad \text{for } I, J \in \mathfrak{I}_\mathrm{T}, \tag{1.10}$$

where $I * J = \{f * g\,;\ f \in I, g \in J\}$ with $f * g$ declared pointwise by upper integrals and where $(H)^\curlyvee$ denotes the smallest cone ideal containing $H \subseteq \mathscr{I}_+$. Proposition 4 establishes that $(\mathfrak{I}_\mathrm{T}, \subseteq, \check{*})$ is a commutative quantale.

The second part of Sect. 3 transports the quantale structure to a class of distribution spaces: To $U \subseteq \mathscr{D}'$ one associates the moderated cone ideal

$$I = |U|_\mathfrak{B}^\curlyvee := (\{|u|_\Phi\,;\ u \in U,\ \Phi \in \mathfrak{B}(\mathscr{D})\})^\curlyvee\,. \tag{1.11}$$

The *regularization-solid closure* of U is defined as the set $V \subseteq \mathscr{D}'$ adjoint to the cone ideal $I = |U|_\mathfrak{B}^\curlyvee$ with respect to the mapping $|-|_\mathfrak{B}^\curlyvee$, in other words

$$V := \{v \in \mathscr{D}'\,;\ \forall \Phi \in \mathfrak{B}(\mathscr{D}) : |v|_\Phi \in I\}. \tag{1.12}$$

Proposition 6 establishes a bijection between regularization-solid spaces and moderated cone ideals $I \subseteq \mathscr{I}_\mathrm{lb}^+ := \{f \in \mathscr{I}_+\,;\ f \text{ locally bounded}\}$, that is interpreted via order theoretic adjoints. Further, Theorem 5 states that any convolution inclusion of regularization-solid spaces is equivalent to the convolution inclusion of the corresponding moderated cone ideals.

Section 4 is devoted to algebraic and order theoretic properties of convolution duals by studying the quantale $(\mathfrak{I}_\mathrm{T}, \subseteq, \check{*})$. Proposition 9 yields a correspondence between the convolution dual $(-)^*_{\mathscr{D}'}$ and the duality operation

$$I \mapsto (I)^*_\mathrm{T} := \mathscr{I}_\mathrm{lb}^+ \not{*}_\mathrm{T}\, I, \tag{1.13}$$

where $\not*_{\mathrm{T}}$ denotes residuals (1.5) formed in $(\mathfrak{I}_{\mathrm{T}}, \subseteq, \check{*})$. The correspondence is induced by the bijection from Proposition 6, mentioned above. Defining

$$\mathfrak{I}_{\mathrm{T}}^{**} := \{I \in \mathfrak{I}_{\mathrm{T}}\,;\ (I)_{\mathrm{T}}^{**} = ((I)_{\mathrm{T}}^{*})_{\mathrm{T}}^{*} = I\} \tag{1.14}$$

the convolution perfect distribution spaces are found to map bijectively to the *non-degenerate* $J \in \mathfrak{I}_{\mathrm{T}}^{**}$ (this means $J \in \mathfrak{I}_{\mathrm{T}}^{**}$ with $\{0\} \neq J \subseteq \mathscr{I}_{\mathrm{lb}}^{+}$). With the composition $I \mathbin{\widetilde{*}} J := (I * J)_{\mathrm{T}}^{**}$ for $I, J \in \mathfrak{I}_{\mathrm{T}}^{**}$ this gives rise to a Girard-quantale $(\mathfrak{I}_{\mathrm{T}}^{**}, \subseteq, \widetilde{*})$ with dualizing element $\mathscr{I}_{\mathrm{lb}}^{+}$ [47, Sec. 6]. Identifying the space $\mathscr{I}_{+}$ with ∞, it follows that $(\mathfrak{I}_{\mathrm{T}}^{**}, \subseteq, \widetilde{*})$ is isomorphic to the quantale associated to $\big((\mathfrak{P}(\mathscr{D}'))_{\mathscr{D}'}^{**}, \subseteq, \widetilde{*}\big)$, defined below Equation (1.4). These results are applied to maximal domains for composites of convolution operators in Theorem 6.

The construction (1.2) by itself does not yield optimal modules and algebras associated to semigroups of convolution operators, as has been observed for the Hilbert transform in [20, p. 301]. Therefore, Sect. 5 studies universal constructions of maximal modules and associated algebras. Let $A \subseteq \mathscr{D}'$ be *totally convolvable*, that is, every p-tuple from A is convolvable. The space

$$(A)_{\mathscr{D}'}^{*\mathrm{M}} := \{m \in \mathscr{D}'\,;\ \forall a_1, \ldots, a_p \in A,\ p \in \mathbb{N} : a_1 * \cdots * a_p * m \text{ exists}\} \tag{1.15}$$

and related constructions are studied in Sect. 5. This construction was introduced and applied to the mathematical modeling of fractional relaxation as linear translation invariant systems involving fractional derivatives on distributional domains in [31]. Theorem 7 establishes, that $(A)_{\mathscr{D}'}^{*\mathrm{M}}$ defines a convolution module over $(A)_{\mathscr{D}'}^{*\mathrm{A}} := ((A)_{\mathscr{D}'}^{*\mathrm{M}})_{\mathscr{D}'}^{*}$, improving an earlier result by the authors [31, Thm. 8]. The latter constitutes a *perfect convolution algebra*: A convolution perfect space A that is totally convolvable and closed with respect to convolution. With Proposition 13 it is obtained that perfect convolution algebras correspond to idempotent elements [47, Def. 2.1.3] of $(\mathfrak{I}_{\mathrm{T}}^{**}, \subseteq, \widetilde{*})$. The construction (1.15) is applied to the Hilbert transform in Example 9.

In Sect. 6 topological structures are introduced and investigated. Inspired by Köthes' "normale Topologie" on perfect sequence spaces [33, §30] convolution perfect spaces U will be endowed systematically with a weighted L^1-type topology $\mathfrak{T}^*(U)$ and a bornology $\mathfrak{B}^*(U)$. For these definitions we obtain two functional analytic results. Theorem 8: Convolution between convolution perfect spaces is hypocontinuous whenever well-defined in the algebraic sense. Theorem 9: For given convolution perfect spaces $V \subseteq W$, the quantale theoretic residual "W by V" from Equation (1.7) is equal to the largest regularization-solid space contained in the space of convolutors $\mathscr{O}'_C(V, W)$ [2, Def. 12]. The space $\mathscr{O}'_C(V, W)$ corresponds to the continuous convolution operators $V \to W$.

Section 7 treats applications to causal fractional integrals and derivatives considered as translation invariant convolution operators. Their domains are significantly extended and the index laws are generalized to larger classes of distributions. Special cases of these results were recently discussed in [31] and in [22] with reference to desiderata for fractional integrals and derivatives. The negative fractional Laplacian $(-\Delta)^{\alpha/2}, \alpha > 0$ is discussed in Sect. 8. We prove that the operator $(-\Delta)^{\alpha/2}, \alpha > 0$ defines a continuous

Table 1 Locations of the definition for uncommon mathematical symbols

Notations	Location of definition
$\lvert u\rvert_\Phi$, $\overline{\mathrm{T}}_K f$	Definition 1 on page 7
$\mathfrak{I}$, $\mathfrak{I}_\mathrm{T}$, $(-)^\curlyvee$, $(-)^\curlyvee_\mathrm{T}$, $\check{*}$, $\not{*}_\mathfrak{I}$, $\not{*}_\mathrm{T}$	Definition 2 on page 11
$\mathscr{I}^+_\mathrm{c}$, $\mathscr{I}^+_\mathrm{b}$, $\mathscr{I}^+_{\mathrm{i},p}$, $\mathscr{I}^+_\mathrm{p}$, $\mathscr{I}^+_{*\mathrm{p}}$, $\mathscr{I}^+_+$	Example 1 on page 15
$\lvert-\rvert^\curlyvee_\mathfrak{B}$, $(-)^\preceq_{\mathscr{D}'}$, $(-)^{\blacksquare}_{\mathscr{D}'}$	Definition 3 on page 13
$(-)^*_\mathrm{T}$, $(-)^{**}_\mathrm{T}$, $\mathfrak{I}^{**}_\mathrm{T}$	Definition 4 on page 16
$(-)^*_{\mathscr{D}'}$, $(-)^{**}_{\mathscr{D}'}$	Definition 5 on page 17
$(-)^{*\mathrm{a}}_{\mathscr{I}_+}$, $(-)^{*\mathrm{a}}_{\mathscr{D}'}$, $(-)^{*\mathrm{s}}_{\mathscr{D}'}$, $(-)^{*\blacksquare\mathrm{a}}_{\mathscr{D}'}$	Definition 6 on page 19
$(-)^{*\mathrm{M}}_\mathrm{T}$, $(-)^{*\mathrm{A}}_\mathrm{T}$, $(-)^{*\mathrm{M}}_{\mathscr{D}'}$, $(-)^{*\mathrm{A}}_{\mathscr{D}'}$	Definition 7 on page 20
$\mathfrak{T}^*(U)$, $\mathfrak{B}^*(U)$	Definition 8 on page 22

linear endomorphism of $\mathscr{D}'_\alpha$ by applying the maximal domain operator. This result was also stated in the recent "Handbook of Fractional Calculus with Applications" [34], but the characterization of the distributional domain, which is provided therein, is not accurate due to a subtle error in its functional analytic construction.

Some remarks on notation: We always treat distributions on $\mathbb{R}^d$ and suppress the attribute $(\mathbb{R}^d)$ with the exception of Sect. 7 and the space of causal distributions $\mathscr{D}'_+ = \mathscr{D}'_+(\mathbb{R})$. The reader's familiarity with notations for function spaces from [51], such as $\mathscr{E}'$, $\mathscr{D}$, $\mathscr{B}$ etc. is assumed. The compact and bounded subsets of a locally convex space E are denoted as $\mathfrak{K}(E)$ and $\mathfrak{B}(E)$. The abbreviation $\mathfrak{K} := \mathfrak{K}(\mathbb{R}^d)$ is used and we write $\mathscr{D}_K := \{\phi \in \mathscr{D}\,;\ \operatorname{supp}\phi \subseteq K\}$ for $K \subseteq \mathbb{R}^d$. Translation, reflection and the support of a distribution u are denoted as $\mathrm{T}_x u$, $\check{u}$ and $\operatorname{supp} u$ respectively. The absolute convex closure of a subset B of a linear space is denoted by $\operatorname{acx} B$. Dual pairings are denoted by $\langle -, - \rangle$. The shorthand $\{f = g\}$ stands for $\{x \in \mathbb{R}^d\,;\ f(x) = g(x)\}$. Because many uncommon or new notations are used in this work, we included Table 1 for convenient reference.

2 Generalized absolute values and convolution of distributions

This Section summarizes properties of generalized absolute values and translation shells and then describes their application to convolution.

2.1 Basic properties of generalized absolute values and translation shells

Definition 1 Let $u \in \mathscr{D}'$ and $\Phi \in \mathfrak{B}(\mathscr{D})$. Define the *$\Phi$-absolute value* of u as

$$\lvert u\rvert_\Phi(x) := \sup\{\lvert\phi * u\rvert\,;\ \phi \in \Phi\} \qquad \text{for all } x \in \mathbb{R}^d. \tag{2.1}$$

Let $f \in \mathscr{I}_+$ and $K \subseteq \mathbb{R}^d$. The *K-translation shell* of f is defined as

$$\overline{\mathrm{T}}_K f(x) := \sup\{\mathrm{T}_y f(x)\,;\ y \in K\} \qquad \text{for all } x \in \mathbb{R}^d. \tag{2.2}$$

The Φ-absolute value $|u|_\Phi$ of a distribution u is a locally Lipschitz continuous function $\mathbb{R}^d \to \mathbb{R}_+$. In particular $|u|_\Phi \in \mathscr{I}_{\mathrm{lb}}^+$ and $|u|_\Phi$ is a regular distribution. Note, that the mapping $\mathscr{D}' \ni u \mapsto |u|_\Phi \in \mathscr{I}_+$ is absolutely homogeneous and subadditive, just like a seminorm. Similar to the definition of the spaces $\mathscr{D}'_{L^p}$ the definition of generalized absolute values depends on the Lie group structure of $\mathbb{R}^d$. The fact that

$$\mathrm{T}_K \Phi = \{\mathrm{T}_x \phi\,;\ \phi \in \Phi, x \in K\} \in \mathfrak{B}(\mathscr{D}) \qquad \text{for all } K \in \mathfrak{K},\ \Phi \in \mathfrak{B}(\mathscr{D}), \tag{2.3}$$

makes generalized absolute values more convenient to use in connection with translation shell operators as compared to absolute values of regularizations. The latter could be called "weak generalized absolute values".

Formation of K-translation shells equals supremal convolution with the indicator function 1_K. Supremal convolution arises in the context of convolution operators of measures on weighted spaces of continuous functions [30]. By virtue of [30, Prop. 3] $\overline{\mathrm{T}}_K \mathscr{I}_+ \subseteq \mathscr{I}_+$ and $\overline{\mathrm{T}}_K \mathscr{I}_{\mathrm{lb}}^+ \subseteq \mathscr{I}_{\mathrm{lb}}^+$ for all $K \in \mathfrak{K}$.

Let us summarize some readily verified relations. Generalized absolute values and compact translation shells are connected by the relations

$$|u|_{(\mathrm{T}_K \Phi)} = \overline{\mathrm{T}}_K(|u|_\Phi) \qquad \text{for all } u \in \mathscr{D}',\ K \in \mathfrak{K},\ \Phi \in \mathfrak{B}(\mathscr{D}) \tag{2.4a}$$

and

$$|f|_\Phi \le \sup\{\|\phi\|_1\,;\ \phi \in \Phi\} \cdot \overline{\mathrm{T}}_{(-\overline{\bigcup \operatorname{supp} \Phi})} f \qquad \text{for all } f \in \mathscr{I}_{\mathrm{lb}}^+,\ \Phi \in \mathfrak{B}(\mathscr{D}). \tag{2.4b}$$

Moreover, for all $K \in \mathfrak{K}$ with non-empty interior one finds $\phi \in \mathscr{D}_K$ such that

$$f \le |\overline{\mathrm{T}}_K f|_{\{\phi\}} \qquad \text{for all } f \in \mathscr{I}_{\mathrm{lb}}^+. \tag{2.4c}$$

Generalized absolute values preserve supports up to compact sets, that is

$$\operatorname{supp}|u|_\Phi \subseteq \operatorname{supp} u - \overline{\bigcup \operatorname{supp} \Phi} \quad \text{for all } u \in \mathscr{D}',\ \Phi \in \mathfrak{B}(\mathscr{D}). \tag{2.5a}$$

Further, if $f \in \mathscr{E}$ is such that $\{f = 1\} \supseteq \operatorname{supp} u + (K \cup \{0\})$, then

$$|u|_\Phi = |f| \cdot |u|_\Phi = |f \cdot u|_\Phi. \tag{2.5b}$$

For $1 \le p \le +\infty$ and $u \in \mathscr{D}'$ one readily derives from [51, Thm. XXV] that

$$u \in \mathscr{D}'_{L^p} \qquad \Leftrightarrow \qquad |u|_\Phi \in L^p \text{ for all } \Phi \in \mathfrak{B}(\mathscr{D}). \tag{2.6}$$

Proposition 1 *Let (θ_n) be an approximate unit and $\Phi \in \mathfrak{B}(\mathscr{D})$. Then*

$$|(1-\theta_n)u|_\Phi \overset{n\to\infty}{\longrightarrow} 0, \quad |\theta_n u|_\Phi \overset{n\to\infty}{\longrightarrow} |u|_\Phi \quad \text{within } L^\infty_{loc} \text{ for all } u \in \mathscr{D}', \tag{2.7a}$$

and there exists $\Psi \in \mathfrak{B}(\mathscr{D})$ such that

$$|(1-\theta_n)u|_\Phi \le |u|_\Psi, \quad |\theta_n u|_\Phi \le |u|_\Psi \quad \text{for all } n \in \mathbb{N} \text{ and } u \in \mathscr{D}'. \tag{2.7b}$$

Proof Let $x \in \mathbb{R}^d$ and $u \in \mathscr{D}'$. Then the functions from the set $(1-\theta_n)\mathrm{T}_x\check{\Phi}$ converge to zero uniformly within $\mathscr{D}$ for $n \to \infty$ and $|(1-\theta_n)u|_\Phi(x)$ is equal to $\sup\{|\langle u, (1-\theta_n)\mathrm{T}_x\check{\phi}\rangle| \,;\, \phi \in \Phi\}$ by transposition. It follows $|(1-\theta_n)u|_\Phi(x) \to 0$ for $n \to \infty$. Further, $|\theta_n u|_\Phi \to |u|_\Phi$ point-wise for $n \to \infty$, because $|{-}|_\Phi$ satisfies the inverse triangle inequality. Replacing Φ by $\mathrm{T}_K\Phi$ and using (2.4a) yields (2.7a). Because the set of functions $B := \{1-\mathrm{T}_x\check{\theta}_n, \mathrm{T}_x\check{\theta}_n \,;\, x \in \mathbb{R}^d, n \in \mathbb{N}\}$ belongs to $\mathfrak{B}(\mathscr{B})$, the set of test functions $\Psi := B \cdot \Phi$ belongs to $\mathfrak{B}(\mathscr{D})$. Now, Equation (2.7b) follows by construction of Ψ. □

Proposition 2 *The following three sets of functions $\mathscr{D}' \to \mathscr{I}_+$*

$$\{u \mapsto |u|_\Phi \,;\, \Phi \in \mathfrak{B}(\mathscr{D})\} \tag{2.8a}$$
$$\left\{u \mapsto 1_K * |u|_\phi \,;\, \phi \in \mathscr{D},\ K \in \mathfrak{K}\right\} \tag{2.8b}$$
$$\left\{u \mapsto \overline{T}_K(|u|_\phi) \,;\, \phi \in \mathscr{D},\ K \in \mathfrak{K}\right\} \tag{2.8c}$$

generate the same cone ideal of functions $\mathscr{D}' \to \mathscr{I}_+$.

Proof It is known that every finite $\Phi \subseteq \mathscr{D}$ is contained in $\operatorname{acx}(\Psi * \Psi)$ for some other finite $\Psi \subseteq \mathscr{D}$ [14]. An examination of the proof in [14] reveals, that every $\Phi \in \mathfrak{B}(\mathscr{D})$ is contained in $\operatorname{acx}(\Psi * \Theta)$ for some $\Psi \in \mathfrak{B}(\mathscr{D})$ and some finite $\Theta \subseteq \mathscr{D}$ (see also [12], where this is further generalized). Using this, one estimates

$$\sup_{\phi\in\Phi}|\phi * u| \le \sup_{\psi\in\Psi}\sum_{\theta\in\Theta}|\psi * \theta * u| \le \sum_{\theta\in\Theta}\left(\sup_{\psi\in\Psi}|\psi|\right) * |u|_\theta \le C\sum_{\theta\in\Theta}\overline{\mathrm{T}}_K|u|_\theta \tag{2.9}$$

for some $C < \infty$ and $K \in \mathfrak{K}$ with a finite sum on the right-hand side. For the converse, note that $C \cdot \mathrm{T}_K\theta \in \mathfrak{B}(\mathscr{D})$ and apply the relation (2.4a). □

Proposition 3 *Let $\Phi, \Psi, \Theta \in \mathfrak{B}(\mathscr{D})$. There exist $\tilde{\Phi}, \tilde{\Psi}, \tilde{\Theta} \in \mathfrak{B}(\mathscr{D})$ such that*

$$||u|_\Phi|_\Psi \le |u|_{\tilde{\Theta}} \quad \text{and} \quad |u|_\Theta \le ||u|_{\tilde{\Phi}}|_{\tilde{\Psi}} \quad \text{for all } u \in \mathscr{D}'. \tag{2.10}$$

Proof Let $u \in \mathscr{D}'$. Let $L \in \mathfrak{K}$ and $\psi \in \mathscr{D}_L$ with $\int \psi(x)\,\mathrm{d}x = 1$. Using (2.4c) and (2.4a) one obtains

$$|u|_\Theta \le |\overline{\mathrm{T}}_L(|u|_\Theta)|_{\{\psi\}} = ||u|_{(\overline{\mathrm{T}}_L\Theta)}|_{\{\psi\}}.$$

Define K and λ as in (2.4b). By virtue of (2.4b) and (2.4a) one obtains

$$||u|_{\Phi}|_{\Psi} \leq \lambda \cdot \overline{\mathrm{T}}_K(|u|_{\Phi}) = |u|_{\lambda \cdot \overline{\mathrm{T}}_K \Phi}.$$

Thus $\tilde{\Phi} := \mathrm{T}_L \Theta$, $\tilde{\Psi} := \{\psi\}$ and $\tilde{\Theta} := \lambda \cdot \mathrm{T}_K \Phi$ satisfy (2.10). □

2.2 Characterization of convolution in terms of generalized absolute values

In the following we recall some properties of convolution $*$ on the set $\mathscr{I}_+$ of lower semicontinuous functions $\mathbb{R}^d \to \overline{\mathbb{R}}_+$. For definitions and properties of convolution of distributions we refer to [13,27,39,42,43,46,52,59,60]. In particular, for convolution of p-tuples see [27,43,46,52,60]. The most common definition for *convolvability* of p-tuples seems to be condition (a) in Theorem 2 below.

The convolution $f * g$ of $f, g \in \mathscr{I}_+$ is defined by the formula

$$(f * g)(x) := \int f(x-y)g(y)\,\mathrm{d}y \qquad \text{for all } x \in \mathbb{R}^d, \tag{2.11}$$

via an upper integral [5]. By virtue of Fubini's Theorem for lower semicontinuous functions [19, p. 55 (a)], one obtains that $(\mathscr{I}_+, *)$ is a semigroup. Further, $*$ is homogeneous with respect to $\overline{\mathbb{R}}_+$-scalar multiplication, additive, isotone, reflection invariant and commutes argumentwise with translations. Conveniently, convolution on $\mathscr{I}_+$ does not require a convolvability condition for well behavedness. Isotony implies the inequality

$$\overline{\mathrm{T}}_{K+L}(f * g) \leq \overline{\mathrm{T}}_K f * \overline{\mathrm{T}}_L g \qquad \text{for all } f, g \in \mathscr{I}_+,\ K, L \subseteq \mathbb{R}^d, \tag{2.12}$$

where $\overline{\mathrm{T}}_K f := \sup\{\mathrm{T}_x f\,;\ x \in K\}$ as in Definition 1. The inequality (2.12) with compact K, L is fundamental for the quantale structures studied in Sect. 3.

Theorem 2 *Let $u_1, \ldots, u_p \in \mathscr{D}'$, $p \in \mathbb{N}$. The following are equivalent:*

(a) The inclusion $\theta^{\Delta p}(u_1 \otimes \cdots \otimes u_p) \in \mathscr{D}'_{L^1}(\mathbb{R}^{dp})$ holds for all $\theta \in \mathscr{D}$.
(b) The inclusion $|u_1|_{\Phi} * \cdots * |u_p|_{\Phi} \in \mathscr{I}^+_{lb}$ holds for all $\Phi \in \mathfrak{B}(\mathscr{D})$.
(c) The convolution $|u_1|_{\{\phi_1\}} * \cdots * |u_p|_{\{\phi_p\}}$ is finite-valued for all $\phi_1, \ldots, \phi_p \in \mathscr{D}$.

Proof It suffices to give a proof for $p = 2$, the generalization to $p \in \mathbb{N}$ is straightforward. Denote $u = u_1$ and $v = u_2$. Due to (2.6) condition (a) implies

$$\forall \theta \in \mathscr{D}, \Phi \in \mathfrak{B}(\mathscr{D}(\mathbb{R}^{2d})) \quad : \quad \left|\theta^{\Delta} \cdot (u \otimes v)\right|_{\Phi} \in L^1(\mathbb{R}^{2d}). \tag{2.13}$$

Using Equations (2.5a) and (2.5b), and the fact that θ^{Δ} has uniformly bounded derivatives for $\theta \in \mathscr{D}$, this is found to be equivalent to

$$\forall \theta \in \mathscr{D}, \Phi \in \mathfrak{B}(\mathscr{D}(\mathbb{R}^{2d})) \quad : \quad \left|\theta^{\Delta}\right| \cdot |u \otimes v|_{\Phi} \in L^1(\mathbb{R}^{2d}). \tag{2.14}$$

According to [54, Theorem 51.7], $\Phi \in \mathfrak{B}(\mathscr{D}(\mathbb{R}^{2d}))$ can be replaced by $\Psi \otimes \Psi$ with $\Psi \in \mathfrak{B}(\mathscr{D})$ in (2.14). Using $|u \otimes v|_{\Psi\otimes\Psi} = |u|_\Psi \otimes |v|_\Psi$ it follows that

$$\forall \theta \in \mathscr{D}, \Psi \in \mathfrak{B}(\mathscr{D}) \ : \ \int |\theta(x+y)|\, |u|_\Psi(x)|v|_\Psi(y)\,\mathrm{d}(x,y) < \infty \tag{2.15}$$

is equivalent to (2.14). The integral in (2.15) is rewritten as

$$\int |\theta(x+y)|\, |u|_\Psi(x)|v|_\Psi(y)\,\mathrm{d}(x,y) = \int |\theta(x)|(|u|_\Psi * |v|_\Psi)(x)\,\mathrm{d}x \tag{2.16}$$

and therefore (2.15) is equivalent to $|u|_\Psi * |v|_\Psi \in L^1_{\mathrm{loc}}$ for all $\Psi \in \mathfrak{B}(\mathscr{D})$. Due to Equations (2.4a) and (2.12), this entails $|u|_\Psi * |v|_\Psi \in L^\infty_{\mathrm{loc}}$ for all $\Psi \in \mathfrak{B}(\mathscr{D})$, which is Condition (b). For trivial reasons "(b) $\Rightarrow$ (c)".

Note, that (c) is equivalent to $(\psi * u)\cdot(\phi * \check{v}) \in L^1$ for $\psi, \phi \in \mathscr{D}$ when $p = 2$. Therefore "(c) $\Rightarrow$ (a)" is a corollary of [52, Thm. 2]. □

Theorem 3 *Let $\Phi \in \mathfrak{B}(\mathscr{D})$ and $p \in \mathbb{N}$. There exists $\Psi \in \mathfrak{B}(\mathscr{D})$ such that*

$$|u_1 * \cdots * u_p|_\Phi \le |u_1|_\Psi * \cdots * |u_p|_\Psi \tag{2.17}$$

for all convolvable tuples $(u_1, \ldots, u_p)$ with $u_1, \ldots, u_p \in \mathscr{D}'$.

Proof Assume $p = 2$. Let $u, v \in \mathscr{D}'$ convolvable and $\Psi \in \mathfrak{B}(\mathscr{D})$ such that $\Phi \subseteq \operatorname{acx}(\Psi * \Psi)$ (see proof of Proposition 2). Using [46, Prop. 1] one estimates

$$|u * v|_\Phi \le |u * v|_{\Psi*\Psi} \le \sup_{\psi_1,\psi_2\in\Psi} |(\psi_1 * u) * (\psi_2 * v)| \le |u|_\Psi * |v|_\Psi . \tag{2.18}$$

□

In the remaining part of this section we demonstrate a possible utilization of generalized absolute values by giving another proof for the associativity properties of convolution that were obtained in [27,52,60].

Lemma 1 *Let $u_1, \ldots, u_p \in \mathscr{D}'$, $p \in \mathbb{N}$ with $(u_1, \ldots, u_p)$ convolvable and (ϕ_n) an approximate unit. Let $\Phi \in \mathfrak{B}(\mathscr{D})$. There exists $\Theta \in \mathfrak{B}(\mathscr{D})$ such that*

$$|u_1 * \cdots * u_p - \phi_n u_1 * \cdots * \phi_n u_p|_\Phi \le |u_1|_\Theta * \cdots * |u_p|_\Theta \tag{2.19a}$$

for all $n \in \mathbb{N}$. Moreover,

$$|u_1 * \cdots * u_p - \phi_n u_1 * \cdots * \phi_n u_p|_\Phi \overset{n\to\infty}{\longrightarrow} 0 \tag{2.19b}$$

uniformly on compact sets.

Proof Assume $p = 2$. Let $\Psi \in \mathfrak{B}(\mathscr{D})$ such that (2.17) holds and set

$$\Theta := \sqrt{2}\cdot\Big\{(\mathrm{T}_x\phi_n)\mathrm{T}_y\psi,\ (1-\mathrm{T}_x\phi_n)\mathrm{T}_y\psi\ ;\ n \in \mathbb{N},\ \psi \in \Psi,\ x \in \mathbb{R}^d,\ y \in K\Big\}.$$

Clearly, $\Theta \in \mathfrak{B}(\mathscr{D})$. Using Equation (2.17) one estimates

$$\begin{aligned}|u_1 * u_2 - (\phi_n u_1) * (\phi_n u_2)|_\Phi &\le |(1-\phi_n)u_1|_\Psi * |u_2|_\Psi + |u_1|_\Psi * |(1-\phi_n)u_2|_\Psi \\ &\le |u_1|_\Theta * |u_2|_\Theta .\end{aligned}$$

Lebesgue's Theorem of dominated convergence and Proposition 1 yield (2.19b). The generalization to general $p \in \mathbb{N}$ is straightforward. □

Corollary 1 *Under the assumptions of Lemma 1 the sequence $\phi_n u_1 * \cdots * \phi_n u_p$ converges to $u_1 * \cdots * u_p$ for $n \to \infty$ with respect to the strong topology of $\mathscr{D}'$.*

Theorem 4 *Let $u_0, u_1, \ldots, u_p \in \mathscr{D}'$, $p \in \mathbb{N}$. Convolvability of all the three tuples $(u_0, u_1, \ldots, u_p)$, $(u_1, \ldots, u_p)$ and $(u_0, u_1 * \cdots * u_p)$ implies*

$$u_0 * u_1 * \cdots * u_p = u_0 * \left(u_1 * \cdots * u_p\right). \tag{2.20}$$

In addition, if the tuple $(u_0, u_1, \ldots, u_p)$ is convolvable and $u_0 \neq 0$, then the tuples $(u_1, \ldots, u_p)$ and $(u_0, u_1 * \cdots * u_p)$ are convolvable as well.

Proof It holds $\circledast_{l=0}^{p}(\phi_n u_l) = \phi_n u_0 * (\circledast_{k=1}^{p}\phi_n u_k)$ by associativity of $(\mathscr{E}', *)$ [54, Thm. 27.7]. Using Lemma 1 multiple times, and then Corollary 1, yields that this equation holds in the limit. Now, assume $u_0 \neq 0$ and let $\Phi \in \mathfrak{B}(\mathscr{D})$ with $|u_0|_\Phi \neq 0$. If at least one of the tuples $(u_1, \ldots, u_p)$ and $(u_0, u_1 * \cdots * u_p)$ is non-convolvable then Theorem 3, Theorem 2 and the associative law for $(\mathscr{I}_+, *)$ imply that $(u_0, u_1, \ldots, u_p)$ is non-convolvable as well. □

Corollary 2 *Convolvable p-tuples of non-zero distributions can be arbitrarily rewritten by introducing parentheses without changing the result.*

Corollary 3 *Let $u_1, \ldots, u_p \in \mathscr{D}'$, $v_1, \ldots, v_p \in \mathscr{E}'$, $p \in \mathbb{N}$. If $(u_1, \ldots, u_p)$ is convolvable then $(v_1 * u_1, \ldots, v_p * u_p)$ is convolvable and*

$$\left(u_1 * \cdots * u_p\right) * \left(v_1 * \cdots * v_p\right) = (u_1 * v_1) * \cdots * \left(u_p * v_p\right). \tag{2.21}$$

Proof Use "(a) ⇔ (c)" from Theorem 2, $f * \mathscr{I}_{\mathrm{lb}}^+ \subseteq \mathscr{I}_{\mathrm{lb}}^+$ for all $f \in \mathscr{I}_{\mathrm{lb}}^+$ with compact support and Corollary 2. □

3 Regularization-solid spaces and moderated cone ideals

This section first describes the convolution quantale structure on the set system of moderated cone ideals $\mathfrak{I}_{\mathrm{T}}$. Then regularization-solid distribution spaces are introduced using generalized absolute values. Convolution inclusions between such spaces are then characterized by the convolution inclusions of the corresponding moderated cone ideals via Proposition 6 and Theorem 5.

3.1 The convolution quantale of moderated cone ideals

Definition 2 A non-empty subset $I \subseteq \mathscr{I}_+$ will be called *cone ideal* if

$$h \leq \lambda \sup\{f, g\} \;\Rightarrow\; h \in I \qquad \text{for all } h \in \mathscr{I}_+,\ \lambda \in \mathbb{R}_+ \text{ and } f, g \in I. \tag{3.1}$$

A cone ideal $I \subseteq \mathscr{I}_+$ is called *moderated* if

$$\overline{\mathrm{T}}_K f \in I \qquad \text{for all } f \in I,\ K \in \mathfrak{K}. \tag{3.2}$$

The set system of (moderated) cone ideals and the corresponding closure operator are denoted by $\mathfrak{I}$ ($\mathfrak{I}_\mathrm{T}$) and $(-)^\curlyvee$ ($(-)^\curlyvee_\mathrm{T}$). One defines the binary operation

$$I \,\breve{*}\, J := (I * J)^\curlyvee \qquad \text{for all } I, J \in \mathfrak{I}. \tag{3.3}$$

The residual operator of $(\mathfrak{I}, \subseteq, \breve{*})$ and $(\mathfrak{I}_\mathrm{T}, \subseteq, \breve{*})$ is denoted by $\not{*}_\mathfrak{I}$ and $\not{*}_\mathrm{T}$, respectively. Equations (3.6) below extend $\not{*}_\mathfrak{I}$ and $\not{*}_\mathrm{T}$ to arbitrary subsets of $\mathscr{I}_+$. Ideals $I \in \mathfrak{I}_\mathrm{T}$ are called *non-degenerate* if $\mathscr{I}_\mathrm{c}^+ \subseteq I \subseteq \mathscr{I}_\mathrm{lb}^+$, where

$$\mathscr{I}_\mathrm{c}^+ := \{f \in \mathscr{I}_\mathrm{lb}^+ \,;\ \operatorname{supp} f \text{ compact}\}. \tag{3.4}$$

Remark 1 1. Cone ideals are precisely the non-empty additively closed down sets of $\mathscr{I}_+$. The set systems $\mathfrak{I}$ and $\mathfrak{I}_\mathrm{T}$ are closure systems over $\mathscr{I}_+$. The set system $\mathfrak{I}_\mathrm{T}$ is a complete sublattice of $\mathfrak{I}$.
2. The closure operators $(-)^\curlyvee$ and $(-)^\curlyvee_\mathrm{T}$ can be described explicitly as

$$(I)^\curlyvee = \{f \in \mathscr{I}_+ \,;\ \exists G \subseteq I \text{ finite},\ \lambda \in \mathbb{R}_+ : f \leq \lambda \cdot \sup G\}, \tag{3.5a}$$
$$= \left\{f \in \mathscr{I}_+ \,;\ \exists g_1, \ldots, g_n \in I,\ n \in \mathbb{N}_0 : f \leq \textstyle\sum_{k=1}^n g_k\right\}, \tag{3.5b}$$
$$(I)^\curlyvee_\mathrm{T} = \left(\left\{\overline{\mathrm{T}}_K f \,;\ f \in I, K \in \mathfrak{K}\right\}\right)^\curlyvee, \tag{3.5c}$$

for $I \subseteq \mathscr{I}_\mathrm{lb}^+$. Here "sup" denotes the pointwise supremum of sets of functions in $\mathscr{I}_+$. Note, that $\overline{\mathrm{T}}_K(\sup F) = \sup\{\overline{\mathrm{T}}_K f : f \in F\}$ for $F \subseteq \mathscr{I}_+$ and $K \in \mathfrak{K}$, because suprema over two independent variables commute.
3. Equation (3.2) is a typical mild assumption for weight function systems to obtain well behaved weighted (ultra-)distibution spaces and relates to the condition [wM] for weight function systems from [11].

Proposition 4 *The triple* $(\mathfrak{I}, \subseteq, \breve{*})$ *is a commutative quantale. The triple* $(\mathfrak{I}_T, \subseteq, \breve{*})$ *is a subquantale of* $(\mathfrak{I}, \subseteq, \breve{*})$ *in the sense of [47, Def. 3.1.3]. The residuals formed in* $(\mathfrak{I}, \subseteq, \breve{*})$ *can be described by the formula*

$$I \not{*}_\mathfrak{I} J = \{h \in \mathscr{I}_+ \,;\ \forall g \in J : g * h \in I\}. \tag{3.6a}$$

Residuals formed in $(\mathfrak{I}_T, \subseteq, \breve{*})$ *can be described by the formula*

$$I \not{*}_T J = \{h \in \mathscr{I}_+ \,;\ \forall\, g \in J, K \in \mathfrak{K} : \overline{T}_K g * \overline{T}_K h \in I\}. \tag{3.6b}$$

The quantale $(\mathfrak{I}_T, \subseteq, \check{*})$ *is unitary with unit* $\mathscr{I}_c^+$*, that is*

$$\mathscr{I}_c^+ \check{*} I = I \qquad \textit{for all } I \in \mathfrak{I}_T. \tag{3.7}$$

Further, it holds

$$I \check{*} J \supseteq I + J \qquad \textit{for all } I, J \in \mathfrak{I}_T \textit{ with } I, J \supseteq \mathscr{I}_c^+. \tag{3.8}$$

Proof Using isotony and additivity of convolution one obtains the inclusion

$$(I)^{\curlyvee} * (J)^{\curlyvee} \subseteq (I * J)^{\curlyvee} \qquad \text{for all } I, J \subseteq \mathscr{I}_+. \tag{3.9}$$

Equation (3.9) and the fact that $*$ is associative and respects unions as an operation on $\mathfrak{P}(\mathscr{I}_+)$ [47, Exa. (10), p. 18] imply the quantale property (1.4), by virtue of [47, Thm. 3.3.1]. Equation (2.12) and Remark 1 entail that $\mathfrak{I}_\mathrm{T}$ is a subquantale. The Equations (3.6) are immediate from the definitions.

For unitarity of $(\mathfrak{I}_\mathrm{T}^{**}, \subseteq, \check{*})$, let $0 \neq f \in \mathscr{I}_\mathrm{c}^+$, $I \in \mathfrak{I}_\mathrm{T}$, $K := \operatorname{supp} f$ and $\lambda := \int f(x)\,\mathrm{d}x$. Using the inequalities (2.4) one estimates

$$(f/\lambda) * g \leq \overline{\mathrm{T}}_K g \leq (f/\lambda) * \overline{\mathrm{T}}_{K-K} g \qquad \text{for all } g \in I, \tag{3.10}$$

which implies $\left(\mathscr{I}_\mathrm{c}^+ * I\right)^{\curlyvee} = I$. Now, for $I, J \in \mathfrak{I}_\mathrm{T}$ with $I, J \supseteq \mathscr{I}_\mathrm{c}^+$ the inclusion (3.8) follows from (3.7), because $\check{*}$ is monotone and because $I + J$ is equal to the supremum of I and J formed in $\mathfrak{I}_\mathrm{T}$. □

3.2 Regularization-solid distribution spaces and convolution inclusions

Definition 3 The *moderated cone ideal associated to* $U \subseteq \mathscr{D}'$ is defined as

$$|U|_{\mathfrak{B}}^{\curlyvee} := \left\{ f \in \mathscr{I}_+ ;\ \exists V \subseteq U \text{ finite},\ \Psi \in \mathfrak{B}(\mathscr{D}) : f \leq \sup_{v \in V} |v|_\Psi \right\}, \tag{3.11a}$$

and *the regularization-solid set of distributions associated to* $I \subseteq \mathscr{I}_+$ is

$$(I)_{\mathscr{D}'}^{\preceq} := \{u \in \mathscr{D}' ;\ \forall \Phi \in \mathfrak{B}(\mathscr{D})\, \exists f \in I : |u|_\Phi \leq f\}. \tag{3.11b}$$

The *regularization-solid closure* is the composite operator

$$(-)_{\mathscr{D}'}^{\blacksquare} := (|-|_{\mathfrak{B}}^{\curlyvee})_{\mathscr{D}'}^{\preceq} \tag{3.12}$$

and its fixed points are called *regularization-solid distribution spaces*.

Remark 2 Due to Proposition 2 one has

$$(I)_{\mathscr{D}'}^{\preceq} = \{u \in \mathscr{D}' ;\ \forall \phi \in \mathscr{D} : |\phi * u| \in I\} \tag{3.13}$$

for every cone ideal $I \subseteq \mathscr{I}_+$ that satisfies $1_K * I \subseteq I$ for all $K \in \mathfrak{K}$, which holds for all $I \in \mathfrak{I}_\mathrm{T}$ in particular. This yields another proof for Equation (2.6).

Proposition 5 *It holds* $||U|_{\mathfrak{B}}^{\curlyvee}|_{\mathfrak{B}}^{\curlyvee} = |U|_{\mathfrak{B}}^{\curlyvee}$ *and* $|U|_{\mathfrak{B}}^{\curlyvee} \subseteq (U)_{\mathscr{D}'}^{\blacksquare}$ *for all* $U \subseteq \mathscr{D}'$.

Proof These are direct consequences of Proposition 3. □

By an *adjoint pair* we will refer to a tuple (f, g) of isotone mappings $f\colon X \to Y$ and $g\colon Y \to X$ between to ordered sets X and Y such that $f(x) \le y$ if and only if $x \le g(y)$ for all $x \in X$ and $y \in Y$ (this is called "Galois connection" in [10, 7.23], [17]). The range of f is the associated kernel system (also called "interior system") in Y and the range of g is the associated closure system in X. Kernel systems are the closure systems with respect to the reversed order.

Proposition 6 *One has an adjoint pair*

$$|-|_{\mathfrak{B}}^{\curlyvee}\colon \mathfrak{P}(\mathscr{D}') \to \mathfrak{I},\ U \mapsto |U|_{\mathfrak{B}}^{\curlyvee}, \quad (-)_{\mathscr{D}'}^{\precsim}\ \colon \mathfrak{I} \to \mathfrak{P}(\mathscr{D}'),\ I \mapsto (I)_{\mathscr{D}'}^{\precsim}. \tag{3.14}$$

The associated kernel system in $\mathfrak{I}$ *is equal to* $\mathfrak{I}_T \cap \mathfrak{P}(\mathscr{I}_{lb}^{+})$.

Proof Clearly, the criteria (Gal1) and (Gal2) from [10, 7.26] are satisfied, implying adjointness. Equation (2.4a) implies $|U|_{\mathfrak{B}}^{\curlyvee} \in \mathfrak{I}_\mathrm{T}$ for all $U \subseteq \mathscr{D}'$ and it remains to prove $I = |(I)_{\mathscr{D}'}^{\precsim}|_{\mathfrak{B}}^{\curlyvee}$ for $I \in \mathfrak{I}_\mathrm{T}$ with $I \subseteq \mathscr{I}_\mathrm{lb}^{+}$. Here, adjointness implies "$\subseteq$" and the reverse inclusion follows from (2.4b). □

Corollary 4 *The assignment* $U \mapsto (U)_{\mathscr{D}'}^{\blacksquare}$ *defines a closure operator on* $\mathfrak{P}(\mathscr{D}')$ *and* $U \mapsto |U|_{\mathfrak{B}}^{\curlyvee}$ *defines an order isomorphism from the set system of regularization-solid distribution spaces to the set system* $\mathfrak{I}_T \cap \mathfrak{P}(\mathscr{I}_{lb}^{+})$, *which constitutes a kernel system in* $\mathfrak{I}$ *with kernel operator* $I \mapsto |(I)_{\mathscr{D}'}^{\precsim}|_{\mathfrak{B}}^{\curlyvee}$.

Proof These are standard order theoretic conclusions found in [10, 7.27]. □

Corollary 5 *It holds* $|I|_{\mathfrak{B}}^{\curlyvee} = I$ *for all* $I \in \mathfrak{I}_T$ *with* $I \subseteq \mathscr{I}_{lb}^{+}$.

Lemma 2 *Let* $p \in \mathbb{N}$. *Convolution, regularization-solid closures and associated moderated cone ideals satisfy the compatibility relations*

$$|I_1 * \cdots * I_p|_{\mathfrak{B}}^{\curlyvee} = I_1 \check{*} \cdots \check{*} I_p \quad \text{for all } I_k \in \mathfrak{I}_T \cap \mathfrak{P}(\mathscr{I}_{lb}^{+}), \tag{3.15a}$$

$$|(U_1)_{\mathscr{D}'}^{\blacksquare} * \cdots * (U_p)_{\mathscr{D}'}^{\blacksquare}|_{\mathfrak{B}}^{\curlyvee} = |U_1|_{\mathfrak{B}}^{\curlyvee} \check{*} \cdots \check{*} |U_p|_{\mathfrak{B}}^{\curlyvee} \qquad \text{for all } U_k \subseteq \mathscr{D}'. \tag{3.15b}$$

Proof Assume $p = 2$, let $I, J \in \mathfrak{I}_\mathrm{T}$ and $U, V \subseteq \mathscr{D}'$. The inequalities (2.4c) and (2.12) yield $|I * J|_{\mathfrak{B}}^{\curlyvee} \supseteq I * J$. Conversely, Theorem 3 and Corollary 5 yield $|I * J|_{\mathfrak{B}}^{\curlyvee} \subseteq |I|_{\mathfrak{B}}^{\curlyvee} \check{*} |J|_{\mathfrak{B}}^{\curlyvee} = I \check{*} J$ and that $|(U)_{\mathscr{D}'}^{\blacksquare} * (V)_{\mathscr{D}'}^{\blacksquare}|_{\mathfrak{B}}^{\curlyvee} \subseteq |U|_{\mathfrak{B}}^{\curlyvee} \check{*} |V|_{\mathfrak{B}}^{\curlyvee}$. The proof is completed by using the inclusions $|I * J|_{\mathfrak{B}}^{\curlyvee} \in \mathfrak{I}_\mathrm{T}$, $(U)_{\mathscr{D}'}^{\blacksquare} \supseteq |U|_{\mathfrak{B}}^{\curlyvee}$, $(V)_{\mathscr{D}'}^{\blacksquare} \supseteq |V|_{\mathfrak{B}}^{\curlyvee}$ and applying the definition of $\check{*}$. □

Theorem 5 *Let* U, V, W *be regularization-solid distribution spaces with corresponding moderated cone ideals* I, J, K *(in the sense of Proposition 6). Then*

$$(U, V) \text{ is convolvable and } U * V \subseteq W \qquad \Leftrightarrow \qquad I * J \subseteq K. \tag{3.16}$$

Proof Theorem 2 and Theorem 3 imply "⇐". In order to prove "⇒", assume $U * V \subseteq W$ with the left-hand side well defined. Proposition 5 yields $I \subseteq U$ and $J \subseteq V$ and thus $I * J \subseteq W$. Applying $|-|^{\curlyvee}_{\mathfrak{B}}$ on both sides of the inclusion $I * J \subseteq W$ and using Lemma 2 yields $I * J \subseteq I \check{*} J = |I * J|^{\curlyvee}_{\mathfrak{B}} \subseteq K$. □

Example 1 Consider the moderated cone ideals

$$\mathscr{I}^+_{\mathrm{c}} := \{f \in \mathscr{I}^+_{\mathrm{lb}}\,;\ \operatorname{supp} f \text{ compact}\}, \tag{3.17a}$$

$$\mathscr{I}^+_{+} := \{f \in \mathscr{I}^+_{\mathrm{lb}}(\mathbb{R})\,;\ \inf \operatorname{supp} f > -\infty\}, \tag{3.17b}$$

$$\mathscr{I}^+_{\mathrm{i},p} := \{f \in \mathscr{I}^+_{\mathrm{lb}}\,;\ \forall K \in \mathfrak{K} : \overline{\mathrm{T}}_K f \in L^p\}. \tag{3.17c}$$

Clearly, $\mathscr{D}' = (\mathscr{I}^+_{\mathrm{lb}})^{\preceq}_{\mathscr{D}'}$. Using (2.5a) it is readily seen that $\mathscr{E}' = (\mathscr{I}^+_{\mathrm{c}})^{\preceq}_{\mathscr{D}'}$ and $\mathscr{D}'_+ = (\mathscr{I}^+_+)^{\preceq}_{\mathscr{D}'}$. Conversely, Proposition 6 implies $|\mathscr{E}'|^{\curlyvee}_{\mathfrak{B}} = \mathscr{I}^+_{\mathrm{c}}$, $|\mathscr{D}'|^{\curlyvee}_{\mathfrak{B}} = \mathscr{I}^+_{\mathrm{lb}}$ and $|\mathscr{D}'_+|^{\curlyvee}_{\mathfrak{B}} = \mathscr{I}^+_+$. Equation (2.6) implies $\mathscr{D}'_{L^p} = (L^p \cap \mathscr{I}^+_{\mathrm{lb}})^{\preceq}_{\mathscr{D}'}$ for $p \in [1, +\infty]$. Conversely, however, $|\mathscr{D}'_{L^p}|^{\curlyvee}_{\mathfrak{B}} = \mathscr{I}^+_{\mathrm{i},p} \subsetneq L^p \cap \mathscr{I}^+_{\mathrm{lb}}$. Using Theorem 5 the well known inclusions $\mathscr{E}' * \mathscr{E}' \subseteq \mathscr{E}'$, $\mathscr{E}' * \mathscr{D}' \subseteq \mathscr{D}'$, $\mathscr{D}'_+ * \mathscr{D}'_+ \subseteq \mathscr{D}'_+$ and $\mathscr{D}'_{L^p} * \mathscr{D}'_{L^q} \subseteq \mathscr{D}'_{L^r}$ for $1/p + 1/q = 1 + 1/r$ can be derived from those for the corresponding ideals. For example, $\mathscr{I}^+_{\mathrm{i},p} * \mathscr{I}^+_{\mathrm{i},q} \subseteq \mathscr{I}^+_{\mathrm{i},r}$ follows from Youngs' inequality and Equation (2.12).

Example 2 With the notations from Example 1 it holds

$$\mathscr{I}^+_{\mathrm{i},p} \not*_{\mathrm{T}} \mathscr{I}^+_{\mathrm{i},p} = \mathscr{I}^+_{\mathrm{i},1} \qquad \text{for all } 1 \leq p \leq \infty. \tag{3.18}$$

This is clear for $p = \infty$ because constant functions belong to $\mathscr{I}^+_{\mathrm{i},\infty}$. For $p < \infty$ one uses [36, Thm. 3.6.1] with $G = \mathbb{Z}^d$, the inequality $\overline{\mathrm{T}}_Q f * \overline{\mathrm{T}}_Q g \geq \overline{\mathrm{T}}_{2Q}(f * g)$ and the equivalence $f \in \ell^{p,+}(\mathbb{Z}^d) \Leftrightarrow \overline{\mathrm{T}}_Q f \in \mathscr{I}^+_{\mathrm{i},p}(\mathbb{R}^d)$. Here $Q = [-1, 1]^d$ and f is extended to a function $\mathbb{R}^d \to \overline{\mathbb{R}}_+$ by zero.

Example 3 The slowly increasing and rapidly decreasing lower semicontinuous functions P and P^* are moderated cone ideals. The corresponding regularization-solid spaces are the tempered distributions $\mathscr{S}' = (P)^{\preceq}_{\mathscr{D}'}$ and the convolutors $\mathscr{O}'_C = (P^*)^{\preceq}_{\mathscr{D}'}$. The space of very rapidly decreasing distributions $\mathscr{O}'_M$ is not regularization-solid, because $P^* \subseteq \mathscr{O}'_M$, but $\mathscr{O}'_M \subsetneq \mathscr{O}'_C$.

Example 4 Let $A \subseteq \mathbb{R}^{d \times d}$ be positive definite and $p \in \mathbb{R}_+$. Consider the function $g_{A,p}(x) := \exp((x^t A x)^p)$. It holds $\big(\{g_{A,p}\}\big)^{\curlyvee} \in \mathfrak{I}_{\mathrm{T}}$ if only if $p \leq 1/2$. For $p = 1$ one calculates $\big(\{g_{A,1}\}\big)^{\curlyvee}_{\mathrm{T}} = \big(\{g_{A,1} \cdot g_{\lambda \mathrm{I},1/2}\,;\ \lambda \in \mathbb{R}_+\}\big)^{\curlyvee}$ where $\mathrm{I} \in \mathbb{R}^{d \times d}$ denotes the unit matrix. Note, that adjoining infinitesimal translates of $g_{A,1}$ does not suffice to turn $\big(\{g_{A,p}\}\big)^{\curlyvee}$ into a moderated cone ideal because $\big(g_{A,1} \cdot P\big)^{\curlyvee} \subsetneq \big(\{g_{A,1}\}\big)^{\curlyvee}_{\mathrm{T}}$.

4 Duals and perfections with respect to convolution

Convolution duals and perfections are investigated and interpreted with reference to quantale theory. Theorem 6 provides a systematic method to define the domain of composed convolution operators $A \circ B$ such that the composition law $A(B(x)) = (A \circ B)(x)$ holds for all x from the domain of $A \circ B$. Here, the convolution kernel of $A \circ B$ is the convolution of the kernels of A and B.

4.1 Convolution perfect moderated cone ideals as a Girard quantale

Definition 4 Let $I \subseteq \mathscr{I}_+$. Define the operators

$$(I)^*_{\mathrm{T}} := \{g \in \mathscr{I}_+ \,;\, \forall f \in I, K \in \mathfrak{K} : \overline{\mathrm{T}}_K f * \overline{\mathrm{T}}_K g \in \mathscr{I}^+_{\mathrm{lb}}\}, \tag{4.1a}$$

$$(I)^{**}_{\mathrm{T}} := ((I)^*_{\mathrm{T}})^*_{\mathrm{T}}, \tag{4.1b}$$

and denote the set $\mathfrak{I}^{**}_{\mathrm{T}} := \{I \in \mathfrak{I}_{\mathrm{T}} \,;\, (I)^{**}_{\mathrm{T}} = I\}$.

Remark 3 Convolution duals can be interpreted as residuals formed in the quantale $(\mathfrak{I}_{\mathrm{T}}, \subseteq, \check{*})$. Comparing Equations (4.1a) and (3.6) one notices

$$(I)^*_{\mathrm{T}} = \mathscr{I}^+_{\mathrm{lb}} \not{*}_{\mathrm{T}}\, I \qquad \text{for all } I \subseteq \mathscr{I}_+. \tag{4.2}$$

Proposition 7 *Let $I, J \in \mathfrak{I}_T$ and $K \subseteq \mathscr{I}_+$. Then*

$$I * J \subseteq K \quad \Rightarrow \quad I * (K)^*_T \subseteq (J)^*_T. \tag{4.3}$$

Proof Let $f \in I, h' \in (K)^*_{\mathrm{T}}$ and $g \in J$. It suffices to prove $(f * h') * g \in \mathscr{I}^+_{\mathrm{lb}}$ because $(K)^*_{\mathrm{T}} \in \mathfrak{I}_{\mathrm{T}}$. Associativity and commutativity imply $(f * h') * g = (f * g) * h'$. By assumption $f * g \in K$ and thus $(f * g) * h' \in \mathscr{I}^+_{\mathrm{lb}}$. □

Corollary 6 *For moderated cone ideals I, J, K such that $I * J \subseteq K$ it follows*

$$(I)^{**}_T * (K)^*_T \subseteq (J)^*_T, \qquad\qquad (I)^{**}_T * (J)^{**}_T \subseteq (K)^{**}_T. \tag{4.4a}$$

Inserting $K = (I * J)^{\curlyvee}$ one obtains

$$(I)^{**}_T * (I * J)^*_T \subseteq (J)^*_T, \qquad\qquad (I)^{**}_T * (J)^{**}_T \subseteq (I * J)^{**}_T. \tag{4.4b}$$

Remark 4 Corollary 6 implies that $\left((\mathfrak{I}_{\mathrm{T}})^{**}_{\mathrm{T}}, \subseteq, \tilde{*}\right)$ with

$$I \,\tilde{*}\, J := (I \,\check{*}\, J)^{**}_{\mathrm{T}} = (I * J)^{**}_{\mathrm{T}} \tag{4.5}$$

is a quotient quantale of $(\mathfrak{I}_{\mathrm{T}}, \subseteq, \check{*})$ [47, Def. 3.1.1 & p. 32] and a Girard quantale with dualizing element $\mathscr{I}^+_{\mathrm{lb}}$ [47, Sec. 6]. See also [15, Thm. 2.6.13].

Proposition 8 *It holds $K \not{*}_T\, J = (J * (K)^*_T)^*_T$ for all $K \in \mathfrak{I}^{**}_T$ and $J \in \mathfrak{I}_T$.*

Proof First note, that $K \not{*}_{\mathrm{T}}\, J = (K \not{*}_{\mathrm{T}}\, J)^{**}_{\mathrm{T}}$ holds due to Equation (4.4b). The proposition then follows by calculating, for all $I \in \mathfrak{I}_{\mathrm{T}}$, that

$$
\begin{array}{llllll}
 & I \subseteq K \not\ast_{\mathrm{T}} J & \Leftrightarrow & (I)^{**}_{\mathrm{T}} \subseteq K \not\ast_{\mathrm{T}} J & \Leftrightarrow & (I)^{**}_{\mathrm{T}} * J \subseteq K \\
\Leftrightarrow & J * (K)^{*}_{\mathrm{T}} \subseteq (I)^{*}_{\mathrm{T}} & \Leftrightarrow & (I)^{**}_{\mathrm{T}} \subseteq (J * (K)^{*}_{\mathrm{T}})^{*}_{\mathrm{T}} & \Leftrightarrow & I \subseteq (J * (K)^{*}_{\mathrm{T}})^{*}_{\mathrm{T}}.
\end{array}
$$

Here it was used that $(-)^{**}_{\mathrm{T}}$ is a closure operator, Equations (1.6) and (4.4a) and that $(-)^{*}_{\mathrm{T}}$ is a Galois connection. ◻

4.2 Implications for convolution duals and perfections of distributions

Definition 5 The *convolution dual* and *perfection* of $U \subseteq \mathscr{D}'$ are defined as

$$
(U)^{*}_{\mathscr{D}'} := \{v \in \mathscr{D}' \,;\, \forall\, u \in U : (u, v) \text{ convolvable}\}, \quad (U)^{**}_{\mathscr{D}'} := ((U)^{*}_{\mathscr{D}'})^{*}_{\mathscr{D}'}. \tag{4.6}
$$

Spaces U such that $(U)^{**}_{\mathscr{D}'} = U$ are called *convolution perfect*.

The mapping $\mathfrak{P}(\mathscr{D}') \ni U \mapsto (U)^{*}_{\mathscr{D}'}$ reverses inclusions and $U \subseteq ((U)^{*}_{\mathscr{D}'})^{*}_{\mathscr{D}'}$ holds. Thus $(-)^{*}_{\mathscr{D}'}$ is a Galois connection on $\mathfrak{P}(\mathscr{D}')$ with $(-)^{**}_{\mathscr{D}'}$ the associated closure [58, p. 20]. The same holds for $(-)^{*}_{\mathrm{T}}$ and $(-)^{**}_{\mathrm{T}}$. Convolution duals of classical distribution spaces were calculated in [58, Thm. 5].

Lemma 3 *The convolution dual satisfies*

$$
(U)^{*}_{\mathscr{D}'} = ((|U|^{\curlyvee}_{\mathfrak{B}})^{*}_{T})^{\preceq}_{\mathscr{D}'} \quad \textit{for all } U \subseteq \mathscr{D}'. \tag{4.7}
$$

Proof Let $v \in \mathscr{D}'$. One derives the chain of equivalences

$$
\begin{array}{llcl}
v \in (U)^{*}_{\mathscr{D}'} \overset{(i)}{\Leftrightarrow} & \forall \Phi \in \mathfrak{B}(\mathscr{D}), u \in U & : & |v|_{\Phi} * |u|_{\Phi} \in \mathscr{I}^{+}_{\mathrm{lb}} \overset{(ii)}{\Leftrightarrow} \\
 & \forall \Phi \in \mathfrak{B}(\mathscr{D}), f \in |U|^{\curlyvee}_{\mathfrak{B}} & : & |v|_{\Phi} * f \in \mathscr{I}^{+}_{\mathrm{lb}} \overset{(iii)}{\Leftrightarrow} \\
 & \forall \Phi \in \mathfrak{B}(\mathscr{D}) & : & |v|_{\Phi} \in (|U|^{\curlyvee}_{\mathfrak{B}})^{*}_{\mathrm{T}} \Leftrightarrow v \in ((|U|^{\curlyvee}_{\mathfrak{B}})^{*}_{\mathrm{T}})^{\preceq}_{\mathscr{D}'}
\end{array}
$$

by using Definitions 3 and 5 , and (i) Theorem 2; (ii) additivity and monotonicity of convolution; (iii) Equation (2.4a) and $|U|^{\curlyvee}_{\mathfrak{B}} \in \mathfrak{I}_{\mathrm{T}}$. ◻

Proposition 9 *The convolution dual operators $(-)^{*}_{T}$ and $(-)^{*}_{\mathscr{D}'}$ correspond to each other in the sense that*

$$
((I)^{\preceq}_{\mathscr{D}'})^{*}_{\mathscr{D}'} = ((I)^{*}_{T})^{\preceq}_{\mathscr{D}'} \quad \textit{for all non-degenerate } I \in \mathfrak{I}_{T}, \tag{4.8a}
$$

$$
(|U|^{\curlyvee}_{\mathfrak{B}})^{*}_{T} = |(U)^{*}_{\mathscr{D}'}|^{\curlyvee}_{\mathfrak{B}} \quad \textit{for all } U \subseteq \mathscr{D}' \textit{ with } U \nsubseteq \{0\}. \tag{4.8b}
$$

The space $(U)^{*}_{\mathscr{D}'}$ is regularization-solid for all $U \subseteq \mathscr{D}'$ and

$$
(U)^{*}_{\mathscr{D}'} = ((U)^{*}_{\mathscr{D}'})^{\blacksquare}_{\mathscr{D}'} = ((U)^{\blacksquare}_{\mathscr{D}'})^{*}_{\mathscr{D}'}. \tag{4.9}
$$

Proof Equation (4.8a) follows from Equation (4.7) by inserting $U = (I)^{\preceq}_{\mathscr{D}'}$ and using $I = |(I)^{\preceq}_{\mathscr{D}'}|^{\curlyvee}_{\mathfrak{B}}$ for $I \in \mathfrak{I}_{\mathrm{T}}$, which holds by Corollary 4. Equation (4.8b) follows by applying $|-|^{\curlyvee}_{\mathfrak{B}}$ on both sides of Equation (4.7) and using $(|U|^{\curlyvee}_{\mathfrak{B}})^{*}_{\mathrm{T}} = |((|U|^{\curlyvee}_{\mathfrak{B}})^{*}_{\mathrm{T}})^{\preceq}_{\mathscr{D}'}|^{\curlyvee}_{\mathfrak{B}}$ for $U \subseteq \mathscr{D}'$, $U \nsubseteq \{0\}$, which holds by Corollary 4.

Equation (4.7) implies that $(U)^*_{\mathscr{D}'}$ is regularization-solid for any $U \subseteq \mathscr{D}'$. Using this fact and Equations (4.8) yields

$$(U)^*_{\mathscr{D}'} = ((U)^*_{\mathscr{D}'})^{\blacksquare}_{\mathscr{D}'} = (|(U)^*_{\mathscr{D}'}|^{\curlyvee}_{\mathfrak{B}})^{\preceq}_{\mathscr{D}'} = ((|U|^{\curlyvee}_{\mathfrak{B}})^{\preceq}_{\mathscr{D}'})^*_{\mathscr{D}'} = ((U)^{\blacksquare}_{\mathscr{D}'})^*_{\mathscr{D}'}. \tag{4.10}$$

□

Proposition 10 *Let $U, V, W \subseteq \mathscr{D}'$ be regularization-solid and $V \neq \{0\}$. Then:*

$$\begin{matrix} (U, V) \text{ is convolvable} \\ \text{and } U * V \subseteq W \end{matrix} \quad \Rightarrow \quad \begin{matrix} \big(U, (W)^*_{\mathscr{D}'}\big) \text{ is convolvable} \\ \text{and } U * (W)^*_{\mathscr{D}'} \subseteq (V)^*_{\mathscr{D}'}. \end{matrix} \tag{4.11}$$

Proof Let I, J, K be the moderated cone ideals corresponding to U, V, W. By Theorem 5, the left-hand side of (4.11) is equivalent to the left-hand side of (4.3) and the right-hand side of (4.3) follows from Proposition 7. It holds $J \neq \{0\}$ and thus $(J)^*_{\mathrm{T}} \subseteq \mathscr{I}^+_{\mathrm{lb}}$. Theorem 5, Theorem 2 and Proposition 9 imply the right-hand side of (4.11). □

Theorem 6 *Let $U, V \subseteq \mathscr{D}'$ be regularization-solid and non-zero with (U, V) convolvable. Then $((U * V)^*_{\mathscr{D}'}, U)$ and $((U * V)^*_{\mathscr{D}'}, V)$ are convolvable and*

$$(U * V)^*_{\mathscr{D}'} \subseteq (U)^*_{\mathscr{D}'} \cap (V)^*_{\mathscr{D}'}, \qquad \begin{matrix} (U * V)^*_{\mathscr{D}'} * U \subseteq (V)^*_{\mathscr{D}'}, \\ (U * V)^*_{\mathscr{D}'} * V \subseteq (U)^*_{\mathscr{D}'}. \end{matrix} \tag{4.12a}$$

Moreover, for $u \in U$, $v \in V$ and $w \in (U * V)^*_{\mathscr{D}'}$, one has the associative law

$$(u * v) * w = u * (v * w) = v * (u * w). \tag{4.12b}$$

Proof Equation (4.12a) follows from Corollary 6 and Proposition 4 applied in the light of Theorem 5 and Proposition 9. The inclusions (4.12a) imply that triples (u, v, w) in (4.12b) are convolvable. Thus, Theorem 4 implies (4.12b). □

Example 5 The most illustrative example of a non-perfect regularization-solid distribution space is the space $\dot{\mathscr{B}}' = (\mathscr{I}^+_{\mathrm{v}})^{\preceq}_{\mathscr{D}'}$ of distributions vanishing at infinity. The convolution perfection of $\dot{\mathscr{B}}'$ is the space $\mathscr{B}' = (\mathscr{I}^+_{\mathrm{b}})^{\preceq}_{\mathscr{D}'}$ of uniformly bounded distributions [58, Theorem 5].

Example 6 An important example of weighted distribution spaces are spaces of power-logarithmic growth in dimension $d \in \mathbb{N}$. Define the sets

$$P^{\mu;k} := \Big\{ f \in \mathscr{I}_+(\mathbb{R}^d) \, ; \, \exists C > 0 : f \leq C \cdot w_{\mu,k} \Big\}, \tag{4.13a}$$

$$w_{\mu,k}(x) := (1 + |x|)^{\mu} (\log(e + |x|))^k \qquad \text{for } x \in \mathbb{R}^d, \tag{4.13b}$$

and the functions of μ-power growth $P^{\mu} := P^{\mu;0}$. As sets, the associated distribution spaces and their convolution duals coincide with the spaces

$$\mathscr{D}'_{L^\infty,-\mu,-k} = (P^{\mu;k})^{\blacksquare}_{\mathscr{D}'}, \quad \mathscr{D}'_{L^1,\mu,k} = ((P^{\mu;k})^*_{\mathrm{T}})^{\blacksquare}_{\mathscr{D}'}, \tag{4.14}$$

that were studied in [1,20,44,57]. For $\mu \in (-\infty, \infty]$ we define further

$$P^{<\mu} := \bigcup_{\nu<\mu} P^{\mu}, \qquad P^{\mu;\infty} := \bigcup_{k\in\mathbb{N}_0} P^{\mu;k}, \tag{4.15a}$$

$$P := P^{<\infty}, \qquad Q := P^{<-d}, \qquad R := P^{-d;\infty}. \tag{4.15b}$$

When $d = 1$ the notation

$$I_+ := I \cap \mathscr{I}_+^+ \tag{4.16}$$

will be used. All the defined sets belong to $\mathfrak{I}_\mathrm{T}$. It can be calculated that

$$P^{\mu;k} \check{*} P^{\nu;l} = P^{\mu+\nu+d;k+l} \quad \text{for all } \mu, \nu > -d,\ k, l \in \mathbb{N}_0, \tag{4.17a}$$

$$P^{\mu;k} \check{*} P^{-d;l} = P^{\mu;k+l+1} \quad \text{for all } \mu \geq -d,\ k, l \in \mathbb{N}_0, \tag{4.17b}$$

$$P^{\mu;k} \check{*} P^{\nu;l} = P^{\mu;k} \quad \text{for all } \nu < -d,\ k, l \in \mathbb{N}_0, \begin{cases} \mu > \nu \text{ or} \\ \mu = \nu,\ k \geq l. \end{cases} \tag{4.17c}$$

whenever $\mu + \nu < -d$ and $P^{\mu;k} * P^{\nu;l} = (\{+\infty\})^{\curlyvee}$ otherwise. The relations (4.17) hold for all $\mu, \nu \in \mathbb{R}$ with $d = 1$ when $P^{\mu;k}$ is replaced by $P_+^{\mu;k}$. As many of the relations (4.17) were proved previously in [4,8,41], [9, Lemma 2.2] [1, Lemma 2.8] and [18, VIII.8] we omit the calculations.

5 Largest distributional modules for convolution semigroups

Operators that generate algebras and modules of distributions, $(-)^{*\blacksquare\mathrm{a}}_{\mathscr{D}'}$, $(-)^{*\mathrm{A}}_{\mathscr{D}'}$ and $(-)^{*\mathrm{M}}_{\mathscr{D}'}$, are introduced and investigated parallel to their counterparts $(-)^{*\mathrm{a}}_{\mathscr{I}_+}$, $(-)^{*\mathrm{A}}_{\mathrm{T}}$ and $(-)^{*\mathrm{M}}_{\mathrm{T}}$. Theorem 7 characterizes $(A)^{*\mathrm{M}}_{\mathscr{D}'}$ as convolution module over the associative algebra $(A)^{*\mathrm{A}}_{\mathscr{D}'}$ for given totally convolvable $A \subseteq \mathscr{D}'$.

5.1 Regularization-solid algebras and modules

Definition 6 1. Let $I \subseteq \mathscr{I}_+$. The smallest cone ideal J that satisfies the inclusions $J * J \subseteq J$ and $I \subseteq J$ is denoted by $(I)^{*\mathrm{a}}_{\mathscr{I}_+}$.

2. Let $A \subseteq \mathscr{D}'$. The set A is called *totally convolvable* if for all $p \in \mathbb{N}$ all tuples $(a_1, \ldots, a_p)$ with $a_1, \ldots, a_p \in A$ are convolvable. If, in addition, $A * A \subseteq A$ (and $\delta \in A$) then A is called a *convolution semigroup (monoid)*. Linear convolution monoids are called *convolution algebras*. The smallest convolution semigroup and regularization-solid convolution algebra containing A are denoted $(A)^{*\mathrm{s}}_{\mathscr{D}'}$ and $(A)^{*\blacksquare\mathrm{a}}_{\mathscr{D}'}$, respectively.

Proposition 11 *1. For $I \in \mathfrak{I}_T$ it holds $(I)^{*a}_{\mathscr{I}_+} \in \mathfrak{I}_T$.*

2. A set $A \subseteq \mathscr{D}'$ is totally convolvable iff $(|A|^{\curlyvee}_{\mathfrak{B}})^{*a}_{\mathscr{I}_+} \subseteq \mathscr{I}^+_{lb}$.

3. *If $A \subseteq \mathscr{D}'$ is totally convolvable then the same holds for $(A)^{\blacksquare}_{\mathscr{D}'}$. Moreover, $(A)^{*s}_{\mathscr{D}'}$, $((A)^{\blacksquare}_{\mathscr{D}'})^{*s}_{\mathscr{D}'}$ and $(A)^{*\blacksquare a}_{\mathscr{D}'}$ exist and*

$$(A)^{*\blacksquare a}_{\mathscr{D}'} = (((A)^{\blacksquare}_{\mathscr{D}'})^{*s}_{\mathscr{D}'})^{\blacksquare}_{\mathscr{D}'}. \tag{5.1}$$

4. *The operators $(-)^{*\blacksquare a}_{\mathscr{D}'}$ and $(-)^{*a}_{\mathscr{I}_+}$ correspond to each other via*

$$|(A)^{*\blacksquare a}_{\mathscr{D}'}|^{\curlyvee}_{\mathfrak{B}} = (|A|^{\curlyvee}_{\mathfrak{B}})^{*a}_{\mathscr{I}_+} \quad \text{for all totally convolvable } A \subseteq \mathscr{D}', \tag{5.2a}$$

$$((I)^{\preceq}_{\mathscr{D}'})^{*\blacksquare a}_{\mathscr{D}'} = ((I)^{*a}_{\mathscr{I}_+})^{\preceq}_{\mathscr{D}'} \quad \text{for all } I \in \mathfrak{I}_T \text{ with } (I)^{*a}_{\mathscr{I}_+} \subseteq \mathscr{I}^{+}_{lb}. \tag{5.2b}$$

Proof Part 1 follows from Equation (2.12) and Part 2 from Theorem 2.

Part 3: Part 2 and the equation $|(A)^{\blacksquare}_{\mathscr{D}'}|^{\curlyvee}_{\mathfrak{B}} = |A|^{\curlyvee}_{\mathfrak{B}}$, furnished by Proposition 6, imply that $(A)^{\blacksquare}_{\mathscr{D}'}$ is totally convolvable. Now, the existence of $(A)^{*s}_{\mathscr{D}'}$ and $((A)^{\blacksquare}_{\mathscr{D}'})^{*s}_{\mathscr{D}'}$ is clear. Once it is proved that $(((A)^{\blacksquare}_{\mathscr{D}'})^{*s}_{\mathscr{D}'})^{\blacksquare}_{\mathscr{D}'}$ is a regularization-solid convolution algebra containing A it is clear that this is the smallest such set of distributions, proving Equation (5.1). Now, Lemma 2 implies

$$|(A)^{\blacksquare}_{\mathscr{D}'} * \overset{p\text{-times}}{\cdots} * (A)^{\blacksquare}_{\mathscr{D}'}|^{\curlyvee}_{\mathfrak{B}} = |A|^{\curlyvee}_{\mathfrak{B}} \check{*} \overset{p\text{-times}}{\cdots} \check{*}\, |A|^{\curlyvee}_{\mathfrak{B}} \qquad \text{for all } p \in \mathbb{N}. \tag{5.3}$$

Forming the supremum over $p \in \mathbb{N}$ within the complete lattice $\mathfrak{I}_{\mathrm{T}}$ results in $(|A|^{\curlyvee}_{\mathfrak{B}})^{*\mathrm{a}}_{\mathscr{I}_+}$ on the right-hand side. The supremum on the left-hand side can be moved inside of $|-|^{\curlyvee}_{\mathfrak{B}}$ turning into a union of sets, due to Proposition 6 and [10, Prop. 7.31]. Thus, the left-hand side becomes $|((A)^{\blacksquare}_{\mathscr{D}'})^{*\mathrm{s}}_{\mathscr{D}'}|^{\curlyvee}_{\mathfrak{B}}$. Further,

$$(((A)^{\blacksquare}_{\mathscr{D}'})^{*\mathrm{s}}_{\mathscr{D}'})^{\blacksquare}_{\mathscr{D}'} = ((|A|^{\curlyvee}_{\mathfrak{B}})^{*\mathrm{a}}_{\mathscr{I}_+})^{\preceq}_{\mathscr{D}'} \tag{5.4}$$

follows from Proposition 6. Finally, Theorem 5 applied to (5.4) yields that the left-hand side is a convolution algebra.

Part 4: Inserting (5.1) in (5.4), applying $|-|^{\curlyvee}_{\mathfrak{B}}$, and using Corollary 4 yields Equation (5.2a). Let $I \in \mathfrak{I}_{\mathrm{T}}$ with $(I)^{*\mathrm{a}}_{\mathscr{I}_+} \in \mathscr{I}^{+}_{\mathrm{lb}}$ and $A := (I)^{\preceq}_{\mathscr{D}'}$. By Part 2 A is totally convolvable. Now insert $A = (I)^{\preceq}_{\mathscr{D}'}$ into (5.2a) and apply the operator $(-)^{\preceq}_{\mathscr{D}'}$ on both sides. Because $(A)^{*\blacksquare\mathrm{a}}_{\mathscr{D}'} = ((I)^{\preceq}_{\mathscr{D}'})^{*\blacksquare\mathrm{a}}_{\mathscr{D}'}$ is regularization-solid and $(I)^{*\mathrm{a}}_{\mathscr{I}_+} \in \mathfrak{I}_{\mathrm{T}}$, another application of Corollary 4 yields (5.2b). □

5.2 Convolution perfect algebras and modules

Definition 7 Let $I \subseteq \mathscr{I}_+$. Define the operators

$$(I)^{*\mathrm{M}}_{\mathrm{T}} := ((I)^{*\mathrm{a}}_{\mathscr{I}_+})^{*}_{\mathrm{T}}, \qquad (I)^{*\mathrm{A}}_{\mathrm{T}} := ((I)^{*\mathrm{M}}_{\mathrm{T}})^{*}_{\mathrm{T}}. \tag{5.5}$$

The *maximal convolution module* for totally convolvable $A \subseteq \mathscr{D}'$ is defined as

$$(A)^{*\mathrm{M}}_{\mathscr{D}'} := \left\{ m \in \mathscr{D}' \,;\, \forall a_1, \ldots, a_p \in A,\, p \in \mathbb{N} : a_1 * \ldots * a_p * m \text{ exists} \right\}. \tag{5.6a}$$

The *convolution perfect algebra* generated by A is defined as

$$(A)^{*\mathrm{A}}_{\mathscr{D}'} := ((A)^{*\mathrm{M}}_{\mathscr{D}'})^{*}_{\mathscr{D}'}. \tag{5.6b}$$

Proposition 12 *Let $A \subseteq \mathscr{D}'$ be totally convolvable. Then:*

$$(A)^{*M}_{\mathscr{D}'} = ((A)^{*\blacksquare a}_{\mathscr{D}'})^{*}_{\mathscr{D}'}, \quad A \subseteq (A)^{*A}_{\mathscr{D}'} = ((A)^{*\blacksquare a}_{\mathscr{D}'})^{**}_{\mathscr{D}'}. \tag{5.7}$$

Proof Let $B := (A)^{*\blacksquare \mathrm{a}}_{\mathscr{D}'}$ and $0 \neq m \in \mathscr{D}'$. Using the inclusion $|B|^{\curlyvee}_{\mathfrak{B}} * \cdots * |B|^{\curlyvee}_{\mathfrak{B}} \subseteq B$ and Theorem 2 the statement "$(b_1, \ldots, b_p, m)$ is convolvable for all $b_1, \ldots, b_p \in B$, $p \in \mathbb{N}$" is seen to be equivalent to "(b, m) is convolvable for all $b \in B$". This equivalence means just that $(B)^{*}_{\mathscr{D}'} = (B)^{*\mathrm{M}}_{\mathscr{D}'}$. Using the latter equation, one obtains

$$(B)^{*\mathrm{A}}_{\mathscr{D}'} = ((B)^{*\mathrm{M}}_{\mathscr{D}'})^{*}_{\mathscr{D}'} = ((B)^{*}_{\mathscr{D}'})^{*}_{\mathscr{D}'} = (B)^{**}_{\mathscr{D}'}$$

which completes the proof. □

Proposition 13 *The operators $(-)^{*M}_{T}$ and $(-)^{*A}_{T}$ are related to $(-)^{*M}_{\mathscr{D}'}$ and $(-)^{*A}_{\mathscr{D}'}$, respectively, analogously to Equations (4.8) and (5.2).*

Proof This follows from Propositions 9 and 11 . □

Theorem 7 *Let $A \subseteq \mathscr{D}'$ be totally convolvable. Then $(A)^{*A}_{\mathscr{D}'}$ is totally convolvable as well and convolution defines bilinear operations*

$$*\colon (A)^{*A}_{\mathscr{D}'} \times (A)^{*A}_{\mathscr{D}'} \to (A)^{*A}_{\mathscr{D}'}, \qquad *\colon (A)^{*A}_{\mathscr{D}'} \times (A)^{*M}_{\mathscr{D}'} \to (A)^{*M}_{\mathscr{D}'} \tag{5.8}$$

that are associative. That is, for all $a, b, c \in (A)^{*A}_{\mathscr{D}'}$ and $m \in (A)^{*M}_{\mathscr{D}'}$ we have

$$a * (b * c) = (a * b) * c, \quad a * (b * m) = (a * b) * m. \tag{5.9}$$

Proof Let $B := (A)^{*\blacksquare \mathrm{a}}_{\mathscr{D}'}$. By Theorem 2, the pair $(B, (A)^{*\mathrm{M}}_{\mathscr{D}'})$ is convolvable and $(A)^{*\mathrm{M}}_{\mathscr{D}'} = (B)^{*\mathrm{M}}_{\mathscr{D}'}$. Therefore B instead of A can be considered. Let $b_1, \ldots, b_p, b \in B$, $p \in \mathbb{N}, b_1 \neq 0$ and let $m \in (B)^{*\mathrm{M}}_{\mathscr{D}'}$. By definition of $(-)^{*\mathrm{M}}_{\mathscr{D}'}$ the tuple $(b_1, \ldots, b_p, b, m)$ is convolvable. Using Theorem 4 one concludes that the tuple $(b_1, \ldots, b_p, b * m)$ is convolvable as well. This proves $B * (B)^{*\mathrm{M}}_{\mathscr{D}'} \subseteq (B)^{*\mathrm{M}}_{\mathscr{D}'}$. Using Proposition 10 with $U, W = (B)^{*\mathrm{M}}_{\mathscr{D}'}$, $V = B$ and Proposition 11 one obtains

$$(B)^{*\mathrm{M}}_{\mathscr{D}'} * (B)^{*\mathrm{A}}_{\mathscr{D}'} = (B)^{*\mathrm{M}}_{\mathscr{D}'} * ((B)^{*\mathrm{M}}_{\mathscr{D}'})^{*}_{\mathscr{D}'} \subseteq (B)^{*}_{\mathscr{D}'} = (B)^{*\mathrm{M}}_{\mathscr{D}'}. \tag{5.10}$$

Applying Proposition 10 to (5.10) one obtains $(B)^{*\mathrm{A}}_{\mathscr{D}'} * (B)^{*\mathrm{A}}_{\mathscr{D}'} \subseteq (B)^{*\mathrm{A}}_{\mathscr{D}'}$.

Having proved well definedness of (5.8), the associative laws (5.9) are immediate from Theorem 4 and the definitions of $(A)^{*\mathrm{A}}_{\mathscr{D}'}$ and $(A)^{*\mathrm{M}}_{\mathscr{D}'}$. □

Example 7 The pairs $(\mathscr{E}', \mathscr{D}')$, $(\mathscr{D}'_{L^1}, \mathscr{B}')$, $(\mathscr{O}'_C, \mathscr{S}')$, $(\mathscr{D}'_+, \mathscr{D}'_+)$ and $(\mathscr{D}'_-, \mathscr{D}'_-)$ are classical examples of the form $((U)^{*\mathrm{A}}_{\mathscr{D}'}, (U)^{*\mathrm{M}}_{\mathscr{D}'})$. The spaces $\mathscr{S}'_+$ and $(\mathscr{S}'_+)^*_{\mathscr{D}'}$ arise naturally in causal fractional calculus on the real line, see Sect. 7. It is well known that $\mathscr{S}'_+$ is a convolution algebra of distributions [55]. The space $(\mathscr{S}'_+)^*_{\mathscr{D}'}$ consists of distributions vanishing rapidly for $t \to -\infty$.

Example 8 Continuing Example 6 consider an arbitrary sum (supremum of ideals) $\sum \mathcal{P}$ formed from a subset $\mathcal{P}$ of $\{P_+^{\nu;l}\ ;\ \nu \in \mathbb{R}, l \in \mathbb{N}_0\}$. Equations (4.17) imply that of such sums precisely the sets $P_+^{-\mu}$, $P_+^{<-\mu}$, $P_+^{-\mu;k}$, $P_+^{-\mu;\infty}$, P_+, Q_+ and R_+, where $\mu \in (1, +\infty)$ and $k \in \mathbb{N}_0$, are closed with respect to convolution. Similar statements are true for the sets $P^{\mu,k}$. Due to Proposition 11, the corresponding regularization-solid distribution spaces are regularization-solid convolution algebras.

6 Hypocontinuity with respect to a weighted L^1-type topology

Topologies and bornologies are now introduced on every convolution perfect distribution space using generalized absolute values and convolution. Hypocontinuity [26, Ch. 4, §7] and boundedness [25, 1:2] of convolution are established in Theorem 8 for these topologies and bornologies. A residual formed in $(\mathfrak{I}^{**}_{\mathrm{T}}, \subseteq, \widetilde{*})$ is characterized as the largest regularization-solid space contained in the corresponding space of convolutors in Theorem 9. The Hilbert transform on $\mathscr{D}'_{L^p}$ is a convolutor not contained in this subspace, see Example 9.

Recall, that the space of convolutors $\mathscr{O}'_C(V, W)$ from V to W consists of the distributions u with the property that the mapping $\mathscr{D} \ni \phi \mapsto u * \phi$ extends to a continuous linear mapping $V \to W$, see [2, Def. 12]. Here V and W are normal distribution spaces in the sense of [26, p. 319].

Definition 8 Let U be a convolution perfect distribution space. The locally convex topology $\mathfrak{T}^*(U)$ is generated by the seminorms

$$u \mapsto (|u|_\Phi * |v|_\Phi)(0) = \int |u|_\Phi(x)|v|_\Phi(-x)\,\mathrm{d}x, \qquad \Phi \in \mathfrak{B}(\mathscr{D}),\ v \in (U)^*_{\mathscr{D}'}. \tag{6.1a}$$

The bornology $\mathfrak{B}^*(U)$ is defined as the set of subsets $B \subseteq U$ such that

$$\sup\{|b|_\Phi\ ;\ b \in B\} \in U \qquad \text{for all } \Phi \in \mathfrak{B}(\mathscr{D}). \tag{6.1b}$$

Remark 5 The inclusion $(U, \mathfrak{T}^*(U), \mathfrak{B}^*(U)) \to (V, \mathfrak{T}^*(V), \mathfrak{B}^*(V))$ is continuous and bounded for convolution perfect $U \subseteq V$ due to $(U)^*_{\mathscr{D}'} \supseteq (V)^*_{\mathscr{D}'}$.

Remark 6 The topology $\mathfrak{T}^*(\mathscr{D}')$ coincides with the strong topology on $\mathscr{D}'$ due to $\mathscr{I}_{\mathrm{c}}^+ = |(\mathscr{D}')^*_{\mathscr{D}'}|^{\curlyvee}_{\mathfrak{B}}$ and Proposition 2.

Remark 7 Any convolution perfect space $U \subseteq \mathscr{D}'$ defines a normal space of distributions $U = (U, \mathfrak{T}^*(U))$: Clearly, the space $\mathscr{E}'_K$ is continuously included in U for any compact $K \subseteq \mathbb{R}^d$, and thus, $\mathscr{E}'$ is continuously included in U by [26, Thm. 1, p. 321]. Proposition 1 and Lebesgue's theorem of dominated convergence yield that $\mathscr{E}'$, and thus $\mathscr{D}$, is dense in U. Remarks 5 and 6 yield that U is continuously included in $\mathscr{D}'$.

Remark 8 The seminorm in (6.1a) is equal to $\sup_{K\in\mathfrak{K}} \int_K |u|_\Phi(x)|v|_\Phi(-x)\,\mathrm{d}x$ and therefore Remark 6 implies that any $\mathfrak{T}^*(U)$-neighborhood is $\mathscr{D}'$-closed. By virtue of [28, Thm. 3.2.4] it follows, that any $\mathfrak{T}^*(U)$-Cauchy-filter that has a $\mathscr{D}'$-limit, has the same $\mathfrak{T}^*(U)$-limit.

Theorem 8 *Let U, V and W be convolution perfect distribution spaces and assume that (U, V) is convolvable and $U * V \subseteq W$. Then, convolution defines a hypocontinuous and bounded bilinear mapping*

$$(U, \mathfrak{T}^*(U), \mathfrak{B}^*(U)) \times (V, \mathfrak{T}^*(V), \mathfrak{B}^*(V)) \to (W, \mathfrak{T}^*(W), \mathfrak{B}^*(W)). \tag{6.2}$$

Proof Let $I := |U|^{\curlyvee}_{\mathfrak{B}}$, $J := |V|^{\curlyvee}_{\mathfrak{B}}$ and $K := |W|^{\curlyvee}_{\mathfrak{B}}$. Theorem 5 and Corollary 6 yield the inclusion $(I)^{**}_{\mathrm{T}} * (J)^{**}_{\mathrm{T}} \subseteq (K)^{**}_{\mathrm{T}}$. Then, an application of Theorem 3 yields the inclusion $\mathfrak{B}^*(U) * \mathfrak{B}^*(V) \subseteq \mathfrak{B}^*(W)$, that is, (6.2) is bounded.

Let $B \in \mathfrak{B}^*(U)$, $v \in V$, $w' \in (W)^*_{\mathscr{D}'}$ and $\Phi \in \mathfrak{B}(\mathscr{D})$. Theorem 3 yields

$$\sup\{(|b * v|_\Phi * |w'|_\Phi)(0)\,;\ b \in B\} \leq \big(|v|_\Psi * \sup\{|b|_\Psi\,;\ b \in B\} * |w'|_\Phi\big)(0) \tag{6.3}$$

with $\Psi \in \mathfrak{B}(\mathscr{D})$ independent of B, v and w'. Corollary 6 implies

$$\sup\{|b|_\Psi\,;\ b \in B\} * |w'|_\Phi \in (I)^{**}_{\mathrm{T}} * (K)^*_{\mathrm{T}} \subseteq (J)^*_{\mathrm{T}}. \tag{6.4}$$

Thus (6.3) implies hypocontinuity. □

In the following, when a particular bornology [25, 1:1] is specified on the spaces V, W then $\mathscr{O}'_C(V, W)$ is endowed with the topology of uniform convergence with respect to the bornology on V [28, Sec. 8.4]. The bornology on $\mathscr{O}'_C(V, W)$ is defined as the sets of mappings $L \subseteq \mathscr{O}'_C(V, W)$ such that $L(B)$ is bounded in W for any bounded $B \subseteq V$. In the following we will occassionally write U instead of $(U, \mathfrak{T}^*(U), \mathfrak{B}^*(U))$, when the meaning is clear.

Lemma 4 *Let $u, v \in \mathscr{D}'$ be convolvable and $\Phi \in \mathfrak{B}(\mathscr{D})$. There exist $\tilde{u} \in |\{u\}|^{\curlyvee}_{\mathfrak{B}}$, $\tilde{v} \in |\{v\}|^{\curlyvee}_{\mathfrak{B}}$ and $\Psi \in \mathfrak{B}(\mathscr{D})$ such that $|u|_\Phi * |v|_\Phi \leq |\tilde{u} * \tilde{v}|_\Psi$.*

Proof Using (2.4c), (2.12) and (2.4a) one obtains

$$|u|_\Phi * |v|_\Phi = |\overline{\mathrm{T}}_{2Q}(|u|_\Phi * |v|_\Phi)|_\Psi \leq |\overline{\mathrm{T}}_Q(|u|_\Phi) * \overline{\mathrm{T}}_Q(|v|_\Phi)|_\Psi = |\tilde{u} * \tilde{v}|_\Psi \tag{6.5}$$

for $\tilde{u} := |u|_{\mathrm{T}_Q\Phi}$ and $\tilde{v} := |v|_{\mathrm{T}_Q\Phi}$ with $Q = [-1, 1]^d$ and $\Psi := \{\psi\}$ with $\psi \in \mathscr{D}_Q$ such that $\int \psi(x)\,\mathrm{d}x = 1$. □

Lemma 5 *Let V, W be convolution perfect distribution spaces with $V \subseteq W$ and let $u \in \mathscr{O}'_C(V, W)$, where V, W are endowed with the topologies from (6.1a). The continuous extension $C_u\colon V \to W$ of the mapping $\mathscr{D} \ni \phi \mapsto u * \phi \in W$ is given by convolution of distributions $C_u(v) = u * v$.*

Proof Let $u \in \mathscr{D}'$ such that $\mathscr{D} \ni \phi \mapsto u * \phi \in W$ is $\mathfrak{T}^*(V)$-$\mathfrak{T}^*(W)$-continuous. Then $v \mapsto C_u(v)$ is continuous as a mapping $\mathscr{E}' \to \mathscr{D}'$ by Remarks 5, 6 and 7. Because $\mathscr{D}$ is dense in $\mathscr{E}'$ and convolution of distributions is hypocontinuous as a mapping $\mathscr{D}' \times \mathscr{E}' \to \mathscr{D}'$ it follows that $C_u(v) = u * v$ for all $v \in \mathscr{E}'$.

Now, let (ϕ_n) be an approximate unit and $v \in V$. Proposition 1 and Lebesgue's theorem of dominated convergence imply that $(\phi_n v)$ converges in V. Thus, $C_u(\phi_n v) = u * (\phi_n v)$ is a $\mathfrak{T}^*(W)$-Cauchy-filter that converges in $\mathscr{D}'$ by continuity, completeness and Remarks 5 and 6. Then [39, Thm. 7.1] implies that (u, v) is convolvable and that $\lim_{n\to\infty} C_u(\phi_n v) = u * v$ within $\mathscr{D}'$. According to Remark 8, this implies $C_u(v) = u * v$ for all $v \in V$. □

Theorem 9 *Let $V, W \subseteq \mathscr{D}'$ be convolution perfect with $V \subseteq W$. Then*

$$(U, \mathfrak{T}^*(U), \mathfrak{B}^*(U)) \to \mathscr{O}'_C\big((V, \mathfrak{T}^*(V), \mathfrak{B}^*(V)), (W, \mathfrak{T}^*(W), \mathfrak{B}^*(W))\big) \quad (6.6a)$$

with the arrow "$\to$" indicating a bounded and continuous inclusion and with the convolution perfect distribution space U specified as

$$U = (|W|^{\curlyvee}_{\mathfrak{B}} \not\ast_T |V|^{\curlyvee}_{\mathfrak{B}})^{\preceq}_{\mathscr{D}'}. \quad (6.6b)$$

The space U is characterized as the largest regularization-solid space of distributions that is contained in $\mathscr{O}'_C(V, W)$.

Proof Proposition 4 and Corollary 4 allow to apply Equation (1.6) to Equation (6.6b) to obtain $|U|^{\curlyvee}_{\mathfrak{B}} * |V|^{\curlyvee}_{\mathfrak{B}} \subseteq |W|^{\curlyvee}_{\mathfrak{B}}$. Then Theorem 5 yields that (U, V) is convolvable with $U * V \subseteq W$ and Theorem 8 yields the continuous and bounded inclusion $U \to \mathscr{O}'_C(V, W)$ from (6.6a).

Now, let $v \in V$, $\Phi \in \mathfrak{B}(\mathscr{D})$ and $u \in \mathscr{D}'$ with $|\{u\}|^{\curlyvee}_{\mathfrak{B}} \subseteq \mathscr{O}'_C(V, W)$. Lemma 4 yields $\tilde{u} \in |\{u\}|^{\curlyvee}_{\mathfrak{B}}$, $\tilde{v} \in V$ and $\Psi \in \mathfrak{B}(\mathscr{D})$ with $\Phi \subseteq \Psi$ and $|\tilde{u} * \tilde{v}|_\Psi \geq |u|_\Phi * |v|_\Phi$. By Lemma 5 $(\tilde{u}, \tilde{v})$ is convolvable and $\tilde{u} * \tilde{v} \in W$. Thus, Theorem 2 yields

$$|u|_\Phi * |v|_\Phi * |w'|_\Phi \leq |\tilde{u} * \tilde{v}|_\Psi * |w'|_\Psi \in \mathscr{I}^+_{\mathrm{lb}} \qquad \text{for all } w' \in (W)^*_{\mathscr{D}'}. \quad (6.7)$$

Finally, Equation (6.7) and Proposition 8 imply

$$|u|_\Phi \in (|V|^{\curlyvee}_{\mathfrak{B}} * (|W|^{\curlyvee}_{\mathfrak{B}})^*_{\mathrm{T}})^*_{\mathrm{T}} = |W|^{\curlyvee}_{\mathfrak{B}} \not\ast_{\mathrm{T}} |V|^{\curlyvee}_{\mathfrak{B}}. \quad (6.8)$$

□

Example 9 The Hilbert transform $H : u \mapsto \mathrm{PV}(1/x) * u$ is a convolutor of $\mathscr{D}'_{L^p}(\mathbb{R})$ for all $1 < p < \infty$ [42, p. 356], [29]. One calculates that

$$|\{\mathrm{PV}(1/x)\}|^{\curlyvee}_{\mathfrak{B}} = (\{w_{-1}\})^{\curlyvee} \not\subseteq \mathscr{I}^+_{\mathrm{i},1}, \quad (6.9)$$

with the weights w_p and $w_{p;k}$ as defined in Example 6. Using Proposition 11, Proposition 13 and Equations (4.17) one obtains

$$(\mathrm{PV}(1/x))^{*\mathrm{M}}_{\mathscr{D}'} = \left\{u \in \mathscr{D}'(\mathbb{R});\, \forall k \in \mathbb{N}, \phi \in \mathscr{D} : \int \frac{|(u * \phi)(x)|}{w_{1;-k}(x)}\, \mathrm{d}x < \infty\right\}. \quad (6.10)$$

Note, that $(\mathrm{PV}(1/x))^{*}_{\mathscr{D}'} \supsetneq (\mathrm{PV}(1/x))^{*\mathrm{M}}_{\mathscr{D}'}$, as observed in [20, Remark (3)].

According to Equations (6.9) and (3.18) the distribution $\mathrm{PV}(1/x)$ is not contained in the largest regularization-solid subspace of the space of convolutors $\mathscr{O}'_C(\mathscr{D}'_{L^p})$, compare Theorem 9. On the other hand, (6.10) entails

$$(\mathrm{PV}(1/x))^{*\mathrm{M}}_{\mathscr{D}'} \supseteq \bigcup_{p>1} \mathscr{D}'_{L^p}. \tag{6.11}$$

Theorem 7 guarantees that $H(Hf) = -\pi^2 f$ holds for all $f \in (\mathrm{PV}(1/x))^{*\mathrm{M}}_{\mathscr{D}'}$.

7 Distributional causal fractional calculus on the real line

The machinery developed so far is now applied to the causal fractional integration and differentiation operators. We determine their largest natural domains, generalize the index laws and determine spaces on which semigroups of these operators operate continuously and linearly. Following Schwartz [51] the fractional integrals I^{α}_{+} and derivatives D^{α}_{+} with $\alpha \in \mathbb{C}$ are defined as convolution operators [21, Sec.9]

$$\mathrm{I}^{\alpha}_{+}\, u := Y_{\alpha} * u \qquad \text{for } u \in \mathscr{D}'_{+}, \tag{7.1a}$$

$$\mathrm{D}^{\alpha}_{+}\, u := Y_{-\alpha} * u \quad \text{for } u \in \mathscr{D}'_{+}, \tag{7.1b}$$

with kernel

$$Y_{\alpha}(t) := \frac{(t)^{\alpha-1}_{+}}{\Gamma(\alpha)} = \begin{cases} t^{\alpha-1}/\Gamma(\alpha) & \text{for } t > 0, \\ 0 & \text{for } t \leq 0, \end{cases} \tag{7.2a}$$

$$Y_{\alpha} := \mathrm{D}^{m}\, Y_{\alpha+m} \tag{7.2b}$$

for $\Re\alpha > -m$ and $m \in \mathbb{N}_0$. The distributions $\{Y_{\alpha}\,;\ \alpha \in \mathbb{C}\}$ form a convolution group, more precisely

$$Y_{\alpha} * Y_{\beta} = Y_{\alpha+\beta} \qquad \text{for all } \alpha, \beta \in \mathbb{C}. \tag{7.3}$$

The operators I^{α}_{+} and D^{α}_{+} are continuous and linear on the space of causal distributions $\mathscr{D}'_{+}$. Equation (7.3) entails the index law $\mathrm{I}^{\alpha}_{+}(\mathrm{I}^{\beta}_{+}\, u) = \mathrm{I}^{\alpha+\beta}_{+}\, u$ for all $u \in \mathscr{D}'_{+}$ and $\alpha, \beta \in \mathbb{C}$. Enlarged domains to be obtained now are described using the spaces $P^{\mu;k}_{+}$ from Example 6 on page 18.

7.1 Largest distributional domains and index laws

Calculating the generalized absolute values of the distributions Y_{α} one obtains

$$|Y_{\alpha}|^{\curlyvee}_{\mathfrak{B}} = \begin{cases} P^{\Re\alpha-1}_{+} & \text{if } \alpha \in \mathbb{C} \setminus -\mathbb{N}_0, \\ \mathscr{I}^{+}_{\mathrm{c}} & \text{if } m \in \mathbb{N}_0. \end{cases} \tag{7.4a}$$

Using the convolution dual operator $(-)^{*}_{\mathscr{D}'}$ we specify the domain of I^{α}_{+} as

$$\operatorname{Dom} \mathrm{I}^{\alpha}_{+} := (Y_\alpha)^{*}_{\mathscr{D}'} = \begin{cases} ((P_{+}^{\Re\alpha-1})^{*}_{\mathrm{T}})^{\preceq}_{\mathscr{D}'} & \text{if } \alpha \in \mathbb{C} \setminus -\mathbb{N}_0, \\ \mathscr{D}' & \text{if } \alpha \in -\mathbb{N}_0. \end{cases} \tag{7.4b}$$

Using Remark 2 the domain of I^{α}_{+} with $\alpha \in \mathbb{C} \setminus -\mathbb{N}_0$ can also be written as

$$\operatorname{Dom} \mathrm{I}^{\alpha}_{+} = \left\{ u \in \mathscr{D}' \, ; \, \forall \phi \in \mathscr{D} : \int_{-\infty}^{-1} |(\phi * u)(t)| \cdot |t|^{\Re\alpha - 1} \, \mathrm{d}t < \infty \right\}. \tag{7.5}$$

Remark 7 and Theorem 8 guarantee that I^{α}_{+} coincides with the unique continuous extension from $\mathscr{D}$ to $\operatorname{Dom} \mathrm{I}^{\alpha}_{+}$.

Remark 9 Equation (7.4) clearly extends Schwartz' definition of I^{α}_{+}. Further, its extension contains exponentially weighted spaces of distributions [35] and the test function space $\mathfrak{G}_{+}$ from [49, p. 146-147]. For $\Re\alpha < 0$ the domain (7.4) contains the distributional analogues for differentiable functions suitable for Marchaud's approach of fractional differentiation [49, p. 109].

Theorem 10 *Let $\alpha, \beta \in \mathbb{H} := \{z \in \mathbb{C} \, ; \, \Re z > 0\}$. The following equations hold true with all expressions well defined in the sense of (7.4)*

$$\mathrm{I}^{\alpha}_{+}(\mathrm{I}^{\beta}_{+} u) = \mathrm{I}^{\beta}_{+}(\mathrm{I}^{\alpha}_{+} u) = \mathrm{I}^{\alpha+\beta}_{+} u \qquad \textit{for all } u \in \operatorname{Dom} \mathrm{I}^{\alpha+\beta}_{+}, \tag{7.6a}$$

$$\mathrm{D}^{\alpha}_{+}(\mathrm{D}^{\beta}_{+} u) = \mathrm{D}^{\beta}_{+}(\mathrm{D}^{\alpha}_{+} u) = \mathrm{D}^{\alpha+\beta}_{+} u \qquad \textit{for all } u \in \operatorname{Dom} \mathrm{D}^{\alpha}_{+} \cap \operatorname{Dom} \mathrm{D}^{\beta}_{+}, \tag{7.6b}$$

$$\mathrm{I}^{\alpha}_{+}(\mathrm{D}^{\beta}_{+} u) = \mathrm{D}^{\beta}_{+}(\mathrm{I}^{\alpha}_{+} u) = \mathrm{I}^{\alpha-\beta}_{+} u \qquad \textit{for all } u \in \operatorname{Dom} \mathrm{I}^{\alpha}_{+}. \tag{7.6c}$$

This includes the special case

$$\mathrm{I}^{\alpha}_{+}(\mathrm{D}^{m} u) = \mathrm{D}^{m}(\mathrm{I}^{\alpha}_{+} u) = \mathrm{I}^{\alpha-m}_{+} u \qquad \textit{for all } \alpha \in \mathbb{C}, \, m \in \mathbb{N}_0, \, u \in \operatorname{Dom} \mathrm{I}^{\alpha}_{+}. \tag{7.6d}$$

Proof The proof is an application of Theorem 6. Equation (7.6a) follows using $U := (P_{+}^{\Re\alpha-1})^{\preceq}_{\mathscr{D}'}$ and $V := (P_{+}^{\Re\beta-1})^{\preceq}_{\mathscr{D}'}$ which results in $(U)^{*}_{\mathscr{D}'} = \operatorname{Dom} \mathrm{I}^{\alpha}_{+}$, $(V)^{*}_{\mathscr{D}'} = \operatorname{Dom} \mathrm{I}^{\beta}_{+}$ and $(U * V)^{*}_{\mathscr{D}'} = \operatorname{Dom} \mathrm{I}^{\alpha+\beta}_{+}$ according to equations (7.4), (4.17a) and Theorem 5. The proofs for the remaining equations are similar. □

Remark 10 The conditions on u in Theorem 10 have a similar form as the index laws for fractional powers of generators of semigroups [16, Thm. 5.32], [32, Prop. 5.2]. See [7,21,38] for more references on this topic. Using the theory developed in the present work has the advantage that one obtains simple and explicit formulas for the domains.

Remark 11 Interpreting I^{α}_{+} as an improper integral leads to different sufficient conditions for the index law, see Theorem 1.3 and Remark 1.4 in [37].

7.2 Largest distributional domains for endomorphic operation

In order to realize the operators I_+^α as convolution endomorphisms of distribution spaces we apply the operator $(-)^{*\mathrm{M}}_{\mathscr{D}'}$. Using Equations (4.17) one obtains

$$\operatorname{Endom} \mathrm{I}_+^\alpha := (Y_\alpha)^{*\mathrm{M}}_{\mathscr{D}'} = \begin{cases} (P_+)^*_{\mathscr{D}'} & \text{if } \alpha \in \mathbb{H}, \\ (R_+)^*_{\mathscr{D}'} & \text{if } \alpha \in \mathrm{i}\mathbb{R} \setminus \{0\}, \\ (P_+^{\Re\alpha-1})^*_{\mathscr{D}'} & \text{if } \alpha \in -(\mathbb{H} \setminus \mathbb{N}_0), \\ \mathscr{D}' & \text{if } \alpha \in -\mathbb{N}_0. \end{cases} \tag{7.7}$$

One observes, that $\operatorname{Dom} \mathrm{D}_+^\alpha = \operatorname{Endom} \mathrm{D}_+^\alpha$ iff $\alpha \in \{0\} \cup \mathbb{H}$. Applying the operator $(-)^{*\mathrm{A}}_{\mathscr{D}'}$ induces natural semigroups of operators D_+^α as described in the following: Let I denote the closure operator associated to the closure system

$$\mathfrak{I}(\mathbb{C}) := \{\emptyset\} \cup \{(\overline{\mathbb{H}}+p) \cup \mathbb{N}_0\,;\ p \geq 0\} \cup \{(\mathbb{H}+p) \cup \mathbb{N}_0\,;\ p \geq 0\} \cup \{\mathbb{C}\}. \tag{7.8a}$$

Using Equations (4.17) we calculate that

$$(\{Y_{-\alpha}\,;\ \alpha \in A\})^{*\mathrm{A}}_{\mathscr{D}'} \cap \{Y_{-\alpha}\,;\ \alpha \in \mathrm{I}(A)\} = \{Y_{-\alpha}\,;\ \alpha \in \mathrm{I}(A)\} \tag{7.8b}$$

for all $A \subseteq \mathbb{C}$. This means that the closure operator I on $\mathbb{C}$ generates those subsemigroups of $\{Y_\alpha\,;\ \alpha \in \mathbb{C}\}$, that are maximal with respect to *total* convolvability on some joint domain space from the class of convolution perfect distribution spaces.

Theorem 11 *Let $p > 0$. The convolution semigroup X operates continuously and linearly on the distribution space Y by convolution of distributions for*

$$X = \{Y_{-\alpha}\,;\ \alpha \in \overline{\mathbb{H}} + p\}, \qquad Y = (P_+^{-p-1})^*_{\mathscr{D}'}, \tag{7.9a}$$

$$X = \{Y_{-\alpha}\,;\ \alpha \in \mathbb{H} + p\}, \qquad Y = (P_+^{<-p-1})^*_{\mathscr{D}'}, \tag{7.9b}$$

$$X = \{Y_{-\alpha}\,;\ \alpha \in \overline{\mathbb{H}}\}, \qquad Y = (R_+)^*_{\mathscr{D}'}, \tag{7.9c}$$

$$X = \{Y_{-\alpha}\,;\ \alpha \in \mathbb{H}\}, \qquad Y = (Q_+)^*_{\mathscr{D}'}. \tag{7.9d}$$

The convolution group X operates bijectively and continuously on Y for

$$X = \{Y_\alpha\,;\ \alpha \in \mathbb{C}\}, \qquad Y = (P_+)^*_{\mathscr{D}'}, \tag{7.9e}$$

$$X = \{Y_\alpha\,;\ \alpha \in i\mathbb{R}\}, \qquad Y = (R_+)^*_{\mathscr{D}'}. \tag{7.9f}$$

In all cases compact sets of indices α map to equicontinuous sets of operators.

Proof This is proved similar to Theorem 10 using (7.8), Theorems 7 and 8 . □

Example 10 The convolution dual $(\{\exp(\mathrm{i}x^2)\})^*_{\mathscr{D}'} = \mathscr{S}'$ was calculated in [56, Satz 4]. Due to $Y_\alpha \in \mathscr{S}'$ this implies the existence of $\mathrm{I}_+^\alpha \exp(\mathrm{i}x^2)$ for all $\alpha \in \mathbb{C}$. Theorem 11 yields in addition, that $\mathrm{I}_+^\alpha \exp(\mathrm{i}x^2) \in \operatorname{Dom} \mathrm{I}_+^\beta$ and $\mathrm{I}_+^{\alpha+\beta} \exp(\mathrm{i}x^2) = \mathrm{I}_+^\beta(\mathrm{I}_+^\alpha \exp(\mathrm{i}x^2))$ for all $\alpha, \beta \in \mathbb{C}$ because $\exp(\mathrm{i}x^2) \in \mathscr{O}'_C \subseteq (P_+)^*_{\mathscr{D}'}$.

Remark 12 The above results can be placed into the context of the *desiderata* for fractional calculus proposed in [23].

a) For the group $\{Y_\alpha\,;\,\alpha\in\mathbb{C}\}$ the domain $(P_+)^*_{\mathscr{D}'}=\mathsf{G}_{\,(\mathrm{b})}=\mathsf{G}_{\,(\mathrm{c})}$ satisfies the *desiderata* (a)-(c) by virtue of Theorem 11. Theorem 8 yields that the *desiderata* (d)-(f) are satisfied as well with $\mathsf{G}_{\,(\mathrm{d})}=(P_+)^*_{\mathscr{D}'}$.
b) For the semigroup cases (7.9a)-(7.9d) Theorem 11 shows that *desideratum* (b) in [23] is satisfied for fractional derivatives instead of integrals. However, apart from *desideratum* (c) also *desideratum* (d) can never hold for the semigroup case $(Q_+)^*_{\mathscr{D}'}$ in (7.9d), not even if $\mathbb{H}$ were replaced with $\mathbb{H}\cup\{0\}$. The reason is, that $1\in(Q_+)^*_{\mathscr{D}'}$ and Remark (3) in [43, p. 327] implies $Y_{-\alpha}*1=0$ for all $\Re\alpha>0$.
c) The case (7.9f) in Theorem 11 establishes a kind of (purely imaginary) "fractional calculus of order zero" in the sense that $\Re\alpha=0$ for all operators in the *desiderata* of [23].

8 Distributional domains for fractional laplacians

In the following we consider the fractional (negative) Laplacian $(-\Delta)^{\alpha/2}$ on $\mathbb{R}^d$ with general $\alpha\in\mathbb{C}$. The operator $(-\Delta)^{\alpha/2}$ can be defined as

$$(-\Delta)^{\alpha/2}\colon\ \mathrm{Dom}(-\Delta)^{\alpha/2}\to\mathscr{D}',\qquad u\mapsto R_{-\alpha}*u, \tag{8.1}$$

with the domain $\mathrm{Dom}(-\Delta)^{\alpha/2}:=(R_{-\alpha})^*_{\mathscr{D}'}$ and the Riesz kernel R_α [42, p. 369], [34], which is given by the density

$$R_\alpha=\frac{\Gamma((d-\alpha)/2)}{\pi^{d/2}2^\alpha\Gamma(\alpha/2)}\cdot|x|^{\alpha-d}\qquad\text{for all }\alpha\in\mathbb{H}\setminus(d+2\mathbb{N}_0). \tag{8.2}$$

The mapping $\alpha\mapsto R_\alpha$ is extended to all of $\mathbb{C}$ by taking the finite part of the meromorphic extension of (8.2). Using [42, p. 369] the generalized absolute values of R_α are calculated as

$$|R_\alpha|^{\curlyvee}_{\mathfrak{B}}=\begin{cases}P^{\Re\alpha-d} & \text{for all }\alpha\in\mathbb{C}\setminus(-2\mathbb{N}_0\cup(d+2\mathbb{N}_0)),\\ P^{\Re\alpha-d,1} & \text{for all }\alpha\in d+2\mathbb{N}_0,\\ \mathscr{I}^+_{\mathrm{c}} & \text{for all }\alpha\in-2\mathbb{N}_0,\end{cases} \tag{8.3}$$

with the notations from Equation (4.13a). Because R_α and R_β are convolvable if and only if $\Re(\alpha+\beta)<d$, the convolution kernel R_α gives rise to an endomorphism on a regularization-solid distribution space if and only if $\Re\alpha\le 0$. For such α, Equations (4.17b) and (4.17c) furnish

$$\mathrm{Endom}(-\Delta)^{\alpha/2}:=(R_{-\alpha})^{*\mathrm{M}}_{\mathscr{D}'}=\begin{cases}\mathrm{Dom}(-\Delta)^{\alpha/2} & \text{if }\alpha\in\mathbb{H}\setminus 2\mathbb{N}_0,\\ (R)^*_{\mathscr{D}'} & \text{if }\alpha\in\mathrm{i}(\mathbb{R}\setminus\{0\}),\\ \mathscr{D}' & \text{if }\alpha\in 2\mathbb{N}_0.\end{cases} \tag{8.4}$$

Remark 13 Consider the special case $d = 1$.

a) Then, if $\Re\alpha > -1$, the operator $(-\Delta)^{\alpha/2}$ can be factorized into causal and anti-causal fractional derivatives $R_{-\alpha} = Y_{-\alpha/2} * \check{Y}_{-\alpha/2}$ due to the differentiation rule from [43] and Equation (1.5.7) from [45] (see also [49, §12]). Further, using Theorem 6 and the relation $P_+^q \check{*} P_-^q = P^q$ for $q < -1$ one finds, for $\Re\alpha > 0$, the factorization theorem

$$(-\Delta)^{2\alpha} u = \mathrm{D}_+^\alpha (\mathrm{D}_-^\alpha u) = \mathrm{D}_-^\alpha (\mathrm{D}_+^\alpha u) \tag{8.5}$$

for all $u \in \operatorname{Dom} \mathrm{D}_+^\alpha \cap \operatorname{Dom} \mathrm{D}_-^\alpha$, where $\operatorname{Dom} \mathrm{D}_+^\alpha \cap \operatorname{Dom} \mathrm{D}_-^\alpha = \operatorname{Dom}(-\Delta)^\alpha$ if $\alpha \notin \mathbb{Z}$. Thus, the factorization (8.5) is valid only on a strict subdomain of the fractional Laplacian. This domain contains the joint domain $\bigcap\{\mathscr{D}'_\alpha\,;\, \Re\alpha > 0\} = (Q)^*_{\mathscr{D}'}$ of all fractional Laplacians with $\Re\alpha > 0$. In particular $(Q)^*_{\mathscr{D}'} \supsetneq \mathscr{B}' = \mathscr{D}'_{L^\infty}$.

b) The set of distributions $F := \{Y_\alpha, \check{Y}_\alpha, R_\alpha, \mathrm{PV}(1/x)\,;\, \Re\alpha \leq 0\}$ is totally convolvable according to Example 2, Equations (7.4a), (8.3), (4.17b) and (4.17c). The algebra of convolution operators $(F)^{*\mathrm{a}}_{\mathscr{D}'}$ generated by this set contains Riesz-Feller derivatives and generalized Hilbert transforms [6,40]. According to Theorem 7 and eqs. (7.7), (6.10) and (8.4) this algebra operates on $(R)^*_{\mathscr{D}'}$ by convolution. In contrast to the above $(R)^*_{\mathscr{D}'} \subsetneq \mathscr{B}'$.

Remark 14 For $\alpha > 0$, $\alpha \notin 2\mathbb{N}$ the endomorphic domain of $(-\Delta)^{\alpha/2}$ reads

$$\begin{aligned}\operatorname{Endom}(-\Delta)^{\alpha/2} = \mathscr{D}'_\alpha &:= (P^{-d-\alpha})^*_{\mathscr{D}'}\\ &= \left\{u \in \mathscr{D}'\,;\, \forall\phi \in \mathscr{D} : \int \frac{|(\phi * u)(x)|}{w_{d+\alpha}(x)}\,\mathrm{d}x < \infty\right\}\end{aligned} \tag{8.6a}$$

with the weight

$$w_{d+\alpha}(x) = (1 + x^2)^{d+\alpha} \qquad \text{for } x \in \mathbb{R}^d. \tag{8.6b}$$

Note, that $\mathscr{D}'_\alpha = \mathscr{D}'_{L^1,\alpha}$. Thus, $(-\Delta)^{\alpha/2}$ is a continuous linear endomorphism of $(\mathscr{D}'_\alpha, \mathfrak{T}^*(\mathscr{D}'_\alpha))$. The space $\mathscr{D}'_\alpha$ was envisaged in [34,53] as the dual space of

$$\mathscr{D}_\alpha := \left\{f \in \mathscr{E}\,;\, \forall\beta \in \mathbb{N}_0^d : \sup \left\|(\partial^\beta f) \cdot w_{d+\alpha}\right\|_\infty < \infty\right\} \tag{8.7}$$

endowed with the seminorms $f \mapsto \|(\partial^\beta f) \cdot w_{d+\alpha}\|_\infty$ and $\beta \in \mathbb{N}_0^d$. The dual of $\mathscr{D}_\alpha$ is not contained in $\mathscr{D}'$, contrary to the statements in [34] and [53, Sect. 2.1], because $\mathscr{D}$ is not dense in $\mathscr{D}_\alpha$. However, the oversight can be repaired by using the set of seminorms $f \mapsto \|(\partial^\beta f) \cdot w_{d+\alpha} \cdot g\|_\infty$ with $g \in \mathscr{C}_0$ and $\beta \in \mathbb{N}_0^d$ instead, similar to [44, Prop. 1.3.1.]. With this topology the dual space of $\mathscr{D}_\alpha$ is indeed given by $\mathscr{D}'_\alpha$ as defined in (8.6a). For $\alpha \in (0, 2)$ the same argument as in [34] proves that $(-\Delta)^{\alpha/2}$ defines a continuous linear endomorphism of $\mathscr{D}_\alpha$ with respect to the modified topology.

Remark 15 Using Theorem 6 and (4.17c), as in the proof of Theorem 10, it follows that the convolution operators $(-\Delta)^{\alpha/2}$ and $(-\Delta)^{-\alpha/2}$, $\Re\alpha \geq 0$, can be composed both ways when restricted to $u \in \operatorname{Dom}(-\Delta)^{-\alpha/2}$, and then obey

$$(-\Delta)^{\alpha/2}\left[(-\Delta)^{-\alpha/2}u\right] = (-\Delta)^{-\alpha/2}\left[(-\Delta)^{\alpha/2}u\right] = u. \tag{8.8}$$

Thus, one obtains an inversion theorem for Riesz potentials free from the restriction $\alpha < d$ that applies for L^p-spaces, compare [49, Thm. 26.3].

Remark 16 The Riesz potentials R_α with $\alpha \in \mathbb{H} \setminus (d + 2\mathbb{N}_0)$ can be turned into a group with $\alpha \in \mathbb{C}$, if they are defined as operators on the dual space Φ' of the Lizorkin space [49, (25.16)]

$$\Phi := \left\{ f \in \mathscr{S}(\mathbb{R}^d)\,;\ \forall \beta \in \mathbb{N}_0^d : \left(\partial^\beta \mathscr{F} f\right)(0) = 0 \right\} \tag{8.9}$$

where $\mathscr{F}$ denotes the Fourier transform on the Schwartz space. However, the elements of Φ' can not be interpreted as distributions because $\Phi \cap \mathscr{D} = \{0\}$. This follows from the Wiener-Paley Theorem and the Taylor formula for multivariate power series.

Remarks 15 and 16 combined with $\mathscr{D}'_\alpha \subseteq \mathscr{S}'$ lead us to conjecture that there exist random variables $w \in \mathscr{D}'_\alpha$ and a measure on $\mathscr{D}'_\alpha$ such that for all $g \in \mathscr{S}$ the dual pairing $\langle (-\Delta)^{-\alpha/2} w, g \rangle$ is a centered Gaussian with variance $\|g\|^2_{L^2(\mathbb{R}^d)}$. If true, the conjecture would render the traditional construction of multidimensional fractional Brownian fields more direct.

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