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RESEARCH PAPER

WEYL INTEGRALS ON WEIGHTED SPACES

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Abstract

Weighted spaces of continuous functions are introduced such that Weyl fractional integrals with orders from any finite nonnegative interval define equicontinuous sets of continuous linear endomorphisms for which the semigroup law of fractional orders is valid. The result is obtained from studying continuity and boundedness of convolution as a bilinear operation on general weighted spaces of continuous functions and measures.

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1. Introduction

Although Weyl fractional integrals [21] are well known to be linear endomorphisms on smooth functions with sufficiently rapid decay [14] their continuity properties, the topological structure of their domains, or extensions to less regular functions remain largely unexplored. Domains and ranges for an extension to families of weighted locally convex spaces of continuous functions are introduced and analyzed here following the spirit of recent *desiderata* for fractional integrals advanced in [9].

Mostly topological properties of fractional Weyl integrals will be studied in this work. As a starting point consider Radon measures on a locally compact group G with finite norm $|||\mu|||_w := \int^\bullet w(x)\,\mathrm{d}|\mu|(x)$ where $\int^\bullet$ denotes the essential upper integral of w with respect to the absolute value

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$|\mu|$ of μ. In [5, Chapter 6] and [7, Section 4] submultiplicativity of a lower semicontinuous weight $w\colon G \to \mathbb{R}^+ := [0, \infty^+)$, defined as

$$w(xy) \leqslant w(x)w(y) \qquad \text{for all } x, y \in G, \tag{1.1}$$

is shown to be equivalent to the weighted norm inequality

$$|||\mu * \nu|||_w \leqslant |||\mu|||_w |||\nu|||_w \qquad \text{for } \mu, \nu \in \mathfrak{M}(w), \tag{1.2}$$

where $\mathfrak{M}(w)$ is the set of measures on G. Our objective here is to generalize (1.1) and (1.2) to multiple weights. Recall from [5, 7], that $(\mathfrak{M}(w), |||\cdot|||_w)$ is a Banach convolution algebra of measures, if either (1.2) or (1.1) holds. Equicontinuity of fractional Weyl integrals is obtained from the general framework when G is fixed to $G = \mathbb{R}$. Mathematically interesting is the central role played by supremal convolution, a type of convolution in which integration is replaced with supremum formation.

Define, for two arbitrary weight functions $w, v\colon G \to \mathbb{R}^{+\infty}$, their multiplicative *supremal convolution* as

$$(w \vartriangle v)(z) := \sup\{w(x)v(y) : z, y \in G,\ xy = z\} \tag{1.3}$$

and denote their multiplicative *infimal convolution* by

$$w \triangledown v := 1/((1/w) \vartriangle (1/v)) \tag{1.4}$$

where the convention $0 \cdot \infty = \infty \cdot 0 = 0$ is used. Expressing (1.1) in terms of infimal convolution as

$$w \leqslant w \triangledown w \tag{1.5}$$

exhibits the fundamental role played by supremal and/or infimal convolution for weighted convolution inequalities. Infimal convolution has an additive variant that is known as "inf-convolution" or "epi-addition" in the literature on convex analysis [15, 19].

Given the equivalence (1.2) $\iff$ (1.1) $\iff$ (1.5) our objective here is to admit triples of weights $(w, v; u)$ instead of a single weight w. Let $w[\mathfrak{M}]$ denote the subset of measures in $\mathfrak{M}(w)$ obeying $\int^\bullet (1/w(x))\,\mathrm{d}|\mu|(x) \leqslant 1$. Our generalization to three locally bounded weights $w, v, u\colon G \to \mathbb{R}^+$ states that $w \vartriangle v \leqslant u$ is equivalent to $w[\mathfrak{M}] * v[\mathfrak{M}] \subseteq u[\mathfrak{M}]$ (see Theorem 7.3). Regarding existence of $w[\mathfrak{M}] * v[\mathfrak{M}]$, this is found to be equivalent to local boundedness of $w \vartriangle v$. Indeed, we have

$$w \vartriangle v \leqslant u \iff w[\mathfrak{M}] * v[\mathfrak{M}] \subseteq u[\mathfrak{M}] \iff w[\mathfrak{M}] * \mathcal{C}_{\mathrm{v}}[u] \subseteq \mathcal{C}_{\mathrm{v}}[v], \tag{1.6}$$

where $\mathcal{C}_{\mathrm{v}}[w]$ denotes the space of continuous functions $f\colon G \to \mathbb{C}$ with $w|f| \leqslant 1$ and such that $w|f|$ vanishes at infinity. Assumptions for (1.6) require that the weights are upper semicontinuous and locally bounded away from zero. More importantly, Theorem 8.1, which follows from Theorem

7.3, characterizes continuity and boundedness of convolution as a bilinear operation in locally convex and bornological spaces.

After Sections 2 and 3 have defined symbols and supremal image formation, Section 4 introduces supremal convolution and its properties precisely. An inverse-like deconvolution operation is studied in Section 5, which is a residual division in the sense of [20]. Next, Section 6 summarizes definitions for integration on locally compact spaces and introduces a slight generalization of the classical theory from [2] to allow functions with divergent values. Theorem 7.1 and 7.2 in Section 7 establish an equivalence analogous to the first one in (1.6) for weighted balls. Theorem 7.3 then establishes the equivalences (1.6) and Theorem 8.1 characterizes continuity and boundedness of convolution. Maximal tightness for ideals of weights is introduced and studied in Section 9, while convolution endomorphisms of weighted spaces of continuous functions are discussed in Section 10. Finally, the general framework is applied to Weyl fractional calculus in Section 11. Because of page limitations proofs are omitted. They can be found in [12].

2. Some notation and conventions

The symbol $\mathbb{K}$ will be used for either $\mathbb{R}$ or $\mathbb{C}$, with $\mathbb{K}^\infty := \mathbb{K} \cup \{\infty\}$ and $\mathbb{K}^\times := \mathbb{K} \backslash \{0\}$. Here ∞ means "divergent", "undefined" or "infinite" in a generic unsigned sense. For $\mathbb{K} = \mathbb{R}$ define $\mathbb{R}^+ := [0, \infty^+)$ and $\mathbb{R}^* := (0, \infty^+)$, $\mathbb{R}^{\pm\infty} := \mathbb{R} \cup \{\infty^+\} \cup \{\infty^-\}$ and $\mathbb{R}^{+\infty} := \mathbb{R}^+ \cup \{\infty^+\}$. The symbols ∞^+ and ∞^- denote "divergent" or "infinite" in a signed sense distinguishing positive and negative variants. Note that $\mathbb{R}^\infty \neq \mathbb{R}^{\pm\infty}$. The extensions $\mathbb{R}^\infty \supseteq \mathbb{R}$ and $\mathbb{C}^\infty \supseteq \mathbb{C}$ are considered as topological extensions, namely Alexandroff compactifications. The extensions $\mathbb{R}^{\pm\infty} \supseteq \mathbb{R}$ and $\mathbb{R}^{+\infty} \supseteq \mathbb{R}^+$ are considered as ordering theoretic extensions, where ∞^+ (and ∞^-) are adjoined as greatest (and smallest) elements.

Addition, subtraction, multiplication and division are extended as

$$\infty + x = x + \infty = \infty, \qquad \infty \cdot x = x \cdot \infty = \begin{cases} \infty & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases} \tag{2.1a}$$

$$x - y := x + (-y), \qquad x/y := x \cdot y^{-1} \tag{2.1b}$$

to $\mathbb{K}^\infty$ for all $x, y \in \mathbb{K}^\infty$. In addition $-\infty := \infty$, $\infty^{-1} := 0$, and $0^{-1} := \infty$. Associativity and commutativity hold for both, addition and multiplication, but the distributive law can fail, because $(1-1)\cdot\infty \neq 1\cdot\infty + 1\cdot\infty$. Real and imaginary part of ∞ are $\operatorname{Re}\infty := \infty$ and $\operatorname{Im}\infty := \infty$. The absolute value of ∞ is defined as $|\infty| := \infty^+$. The complex conjugate of ∞ is $\overline{\infty} := \infty$.

An extension of $+$, $\cdot$ and $(\cdot)^{-1}$ to $\mathbb{R}^{+\infty}$ is obtained by replacing ∞ with ∞^+ in (2.1) and setting $(\infty^+)^{-1} := 0$, and $0^{-1} := \infty^+$. The extended multiplication, denoted as $\hat{\cdot}$, is referred to as *supremal multiplication*. The distributive law holds in $\mathbb{R}^{+\infty}$. Subtraction and "negative" elements are not defined.

On $\mathbb{R}^{\pm\infty}$ the *positive/negative part of a number* $x \in \mathbb{R}^{\pm\infty}$ is defined as $(x)^{\pm} := \max\{0, \pm x\}$. The four arithmetic operations are extended by replacing eqs. (2.1a) in eq. (2.1) with

$$\infty^{\pm} + x = x + \infty^{\pm} = \begin{cases} \infty^{\pm}, & \text{if } x \neq \infty^{-}, \\ \infty^{-} & \text{if } x = \infty^{-}, \end{cases}$$

$$\infty^{\pm} \cdot x = x \cdot \infty^{\pm} = \begin{cases} \infty^{\pm} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\infty^{\pm} & \text{if } x < 0, \end{cases} \tag{2.2}$$

for all $x \in \mathbb{R}^{\pm\infty}$ with $(\infty^{\pm})^{-1} := 0$, $0^{-1} := \infty^{-}$, $-\infty^{+} := \infty^{-}$ and $-\infty^{-} = \infty^{+}$. The extended addition is called *supremal addition* and denoted as $\overset{\wedge}{+}$.

The set of functions $S \to \mathbb{K}$ on any set S is denoted by $\mathcal{F}(S)$. The symbol $\mathcal{F}$ is replaced by $\mathcal{F}^{\infty}$, $\mathcal{F}^{\times}$, $\mathcal{F}^{+}$, $\mathcal{F}^{*}$, $\mathcal{F}^{+\infty}$ or $\mathcal{F}^{\pm\infty}$ to denote functions with values in $\mathbb{K}^{\infty}, \mathbb{K}^{\times}, \mathbb{R}^{+}, \mathbb{R}^{*}, \mathbb{R}^{+\infty}$ or $\mathbb{R}^{\pm\infty}$ instead of $\mathbb{K}$.

A topological space S is called locally compact, if every of its points has a compact neighborhood. All locally compact spaces are assumed to be Hausdorff. The set of $\mathbb{K}$-*valued continuous functions* on S is denoted by $\mathcal{C}(S)$. A function $f \in \mathcal{F}^{\pm\infty}(S)$ is called *upper* (resp. *lower*) *semicontinuous* if $\{f \geqslant a\}$ (resp. $\{f \leqslant a\}$) is closed for every $a \in \mathbb{R}^{\pm\infty}$. The sets of $\mathbb{R}^{+\infty}$-*valued lower* respectively *upper semicontinuous functions* are denoted by $\mathcal{L}^{+\infty}(S)$ respectively $\mathcal{U}^{+\infty}(S)$. The symbol

$$\lceil w \rceil := \inf\{u \in \mathcal{U}^{+\infty}(S) : w \leqslant u\}. \tag{2.3}$$

stands for *upper semicontinuous envelope* of $w \in \mathcal{F}^{+\infty}(S)$. The set $\mathcal{U}^{+\infty}(S)$ is known to be closed with respect to pointwise formation of suprema of arbitrary subsets, i.e. it is a closure system [4, Definition 2.33] in $\mathcal{F}^{+\infty}(S)$.

Subscripts $_{\mathrm{lb}}$,$_{\mathrm{b}}$,$_{\mathrm{c}}$,$_{\mathrm{v}}$ added to any set of functions (as in $\mathcal{F}_{\mathrm{lb}}$, $\mathcal{C}_{\mathrm{b}}$, $\mathcal{L}^{+\infty}_{\mathrm{c}}$, $\mathcal{U}^{+\infty}_{\mathrm{v}}$ etc.) signify locally bounded (lb), uniformly bounded (b), compactly supported functions (c), or functions vanishing at infinity (v). A function $f\colon S \to \mathbb{K}^{\infty}$ "vanishes at infinity" if and only if for every $\varepsilon > 0$ there is a compact set $K_{\varepsilon} \subset S$ such that $|f(x)| \leqslant \varepsilon$ for all $x \in S \backslash K_{\varepsilon}$.

3. Supremal image functions and upper semicontinuity

For $\Phi\colon S \to T$ a general mapping and $w\colon T \to \mathbb{R}^{+\infty}$ an arbitrary function the *pullback* or *inverse image function* $\Phi^{-1}w$ of w under Φ can always be defined as $\Phi^{-1}w := w \circ \Phi\colon S \to \mathbb{R}^{+\infty}$. Here $\circ$ means composition of mappings. There is no natural notion for a "pushforward" or an "image function" of a function $w \in \mathcal{F}^{+\infty}(S)$ under Φ, because Φ is in general neither bijective nor invertible. However, the following definition provides a construction, that resembles the "pushforward" of a function $w\colon S \to \mathbb{R}^{+\infty}$ under an arbitrary mapping $\Phi\colon S \to T$. It arises from considering $\mathbb{R}^{+\infty}$ as an ordered set $(\mathbb{R}^{+\infty}, \leqslant)$, where $\leqslant$ is the canonical ordering.

Definition 3.1. The *supremal image function* $\widehat{\Phi}w$ of w under Φ is defined as

$$(\widehat{\Phi}w)(t) := \sup\{w(s) : s \in \Phi^{-1}(t)\}, \tag{3.1}$$

for $t \in T$. A supremal image function $\widehat{\Phi}w$ is called *exact* if the supremum in (3.1) is a maximum for all $t \in \Phi(S)$.

Supremal image formation is analogous to "epi-composition" in [19, Eq. 1(17)]. Note, that $\widehat{\Phi}w$ coincides with the pullback under the inverse mapping Φ^{-1}, i.e. $\widehat{\Phi}w = (\Phi^{-1})^{-1}w$, whenever Φ is bijective.

Supremal image functions can be characterized in two ways. Firstly, one associates the strict hypograph $H_w := \{(s, \alpha) \in S\times\mathbb{R}^* : \alpha < w(s)\} \subseteq S\times\mathbb{R}^*$ to a function $w \in \mathcal{F}^{+\infty}(S)$. The image of the hypograph H_w under the product mapping $\Phi \times \mathrm{id}\colon S \times \mathbb{R}^* \to T \times \mathbb{R}^*$ results in a set $H_{\Phi,w} \subseteq T \times \mathbb{R}^*$ that is precisely the strict hypograph associated to $\widehat{\Phi}w$.

Secondly, the mapping $\widehat{\Phi}\colon \mathcal{F}^{+\infty}(S) \to \mathcal{F}^{+\infty}(T)$ associated to any mapping $\Phi\colon S \to T$ by Definition 3.1 can be seen as the (lower) adjoint to the inverse image operator $\Phi^{-1}\colon \mathcal{F}^{+\infty}(T) \to \mathcal{F}^{+\infty}(S)$ in the sense of ordering theory [4, Definition 7.23], where $\mathcal{F}^{+\infty}(S)$ and $\mathcal{F}^{+\infty}(T)$ are endowed with the canonical pointwise ordering. This fact is formulated as

Proposition 3.1. *The two mappings $\widehat{\Phi}\colon \mathcal{F}^{+\infty}(S) \to \mathcal{F}^{+\infty}(T)$ and $\Phi^{-1}\colon \mathcal{F}^{+\infty}(T) \to \mathcal{F}^{+\infty}(S)$ are isotone. The pair of mappings $(\widehat{\Phi}, \Phi^{-1})$ is a Galois connection (or adjoint pair) in the sense that the two statements*

$$\widehat{\Phi}w \leqslant v \iff w \leqslant \Phi^{-1}v \qquad \text{for all } w \in \mathcal{F}^{+\infty}(S), v \in \mathcal{F}^{+\infty}(T) \tag{3.2a}$$

and

$$w \leqslant \Phi^{-1}(\widehat{\Phi}w) \text{ for } w \in \mathcal{F}^{+\infty}(S), \quad \widehat{\Phi}(\Phi^{-1}v) \leqslant v \text{ for } v \in \mathcal{F}^{+\infty}(T) \tag{3.2b}$$

hold true.

Let now $\Phi\colon S \to T$ be continuous, and let $\lceil\cdot\rceil$ denote upper semicontinuous envelopes. The following two results are stated here for later use.

Corollary 3.1. *Proposition 3.1 holds also when $\mathcal{F}^{+\infty}(T)$ is replaced by $\mathcal{U}^{+\infty}(T)$ and $\widehat{\Phi}$ is replaced by the assignment $w \mapsto \left\lceil \widehat{\Phi} w \right\rceil$.*

Proposition 3.2. *Assume that T is locally compact. If $w\colon S \to \mathbb{R}^{+\infty}$ is upper semicontinuous and the restriction of w to the preimage $\Phi^{-1}(C)$ is vanishing at infinity for any compact $C \subseteq T$, then $\widehat{\Phi} w\colon T \to \mathbb{R}^{+\infty}$ is upper semicontinuous as well. Further, the supremal image function $\widehat{\Phi} w$ is exact.*

4. Supremal convolution on locally compact groups

Let G be a locally compact group. Its multiplication $\Gamma\colon G \times G \to G$ is written as $xy = \Gamma(x,y)$ for $x, y \in G$ and assumed to be continuous.

Definition 4.1. Let $w, v \in \mathcal{F}^{+\infty}(G)$. The *(multiplicative) supremal convolution* of w and v, denoted as $w \vartriangle v$, is defined by

$$(w \vartriangle v)(z) := \sup \{ w(x) \mathbin{\dot{\wedge}} v(y) : x, y \in G,\ xy = z \}, \qquad z \in G. \tag{4.1}$$

Supremal convolutes $w \vartriangle v$ are called *exact*, if the supremum in (4.1) is a maximum for all $z \in G$.

Supremal convolution is an associative binary operation $\vartriangle$ on $\mathcal{F}^{+\infty}(G)$ [19, 15]. It can be decomposed as

$$\vartriangle : \mathcal{F}^{+\infty}(G) \times \mathcal{F}^{+\infty}(G) \xrightarrow{\widehat{\otimes}} \mathcal{F}^{+\infty}(G \times G) \xrightarrow{\widehat{\Gamma}} \mathcal{F}^{+\infty}(G) \tag{4.2}$$

into the supremal image $\widehat{\Gamma}$ under the group multiplication Γ and the supremal tensor product $\widehat{\otimes}$. The *supremal tensor product* is defined as

$$(w \mathbin{\widehat{\otimes}} v)(s,t) = w(s) \mathbin{\dot{\wedge}} v(t) \tag{4.3}$$

whenever $s \in S$, $t \in T$, $w \in \mathcal{F}^{+\infty}(S)$, $v \in \mathcal{F}^{+\infty}(T)$ and S, T are sets. For finite-valued functions it coincides with the usual tensor product $\otimes$ of functions.

The adjective "supremal" characterizes supremal operations (convolution, multiplication or tensor product) as those unique extensions of the finite-valued case, that preserve suprema of arbitrary subsets in each argument. This means that

$$\sup(A \bigcirc B) = \sup A \bigcirc \sup B \tag{4.4}$$

holds for $\bigcirc = \dot{\wedge}, \widehat{\otimes}, \vartriangle$. Here $A, B \subseteq \mathbb{R}^{+\infty}$ or $A, B \subseteq \mathcal{F}^{+\infty}(G)$ for $\bigcirc = \dot{\wedge}$, while $A, B \subseteq \mathcal{F}^{+\infty}(G)$ for $\bigcirc = \widehat{\otimes}, \vartriangle$.

Remark 4.1. Note that the lattice $(\mathcal{F}^{+\infty}(G), \leqslant)$ with the binary operation $\bigcirc = \hat{\cdot}$ or $\bigcirc = \triangle$ forms a quantale, i.e. an order theoretic algebraic structure [20]. Such quantales have always a residual division that will be useful for studying maximal tightness in Section 9 below. For $\bigcirc = \triangle$ the residual division is called deconvolution in Section 5.

Proposition 4.1. *Let $w, v \in \mathcal{U}^+(G)$ such that $w \otimes v$ is vanishing at infinity on $\Gamma^{-1}(C) = \{(x, y) \in G \times G : xy \in C\}$ for all compact $C \subseteq G$. Then $w \triangle v$ is upper semicontinuous and exact.*

Corollary 4.1. *The following inclusions hold:*

$$\mathcal{U}^+_{\mathrm{v}}(G) \triangle \mathcal{U}^+_{\mathrm{b}}(G) \subseteq \mathcal{U}^+_{\mathrm{b}}(G), \tag{4.5a}$$

$$\mathcal{U}^+(G) \triangle \mathcal{U}^+_{\mathrm{c}}(G) \subseteq \mathcal{U}^+(G). \tag{4.5b}$$

Proposition 4.2. *Let $w, v, u \in \mathcal{U}^+(G)$ such that $w \triangle v$ and $v \triangle u$ are locally bounded. Then the following associative law holds:*

$$\lceil \lceil w \triangle v \rceil \triangle u \rceil = \lceil w \triangle v \triangle u \rceil = \lceil w \triangle \lceil v \triangle u \rceil \rceil \tag{4.6}$$

5. Deconvolution and maximally tight triples of weights

A triple $(w, v; u)$ of functions $G \to \mathbb{R}^{+\infty}$ obeying $w \triangle v \leqslant u$ can be "tightened" by finding smaller u or larger w, v. The choice $u = \lceil w \triangle v \rceil$ is optimal for u in the sense that it is the smallest upper semicontinuous $u \colon G \to \mathbb{R}^{+\infty}$ fulfilling the inequality $w \triangle v \leqslant u$ for prescribed $w, v \in \mathcal{F}^{+\infty}(G)$. Deconvolution yields optimal w and v explicitly. Before discussing deconvolution and maximally tight triples the same phenomenon is illustrated for supremal image formation.

Proposition 5.1. *Let $(w, v) \in \mathcal{F}^{+\infty}(S) \times \mathcal{U}^{+\infty}(T)$ be a pair fulfilling $\widehat{\Phi} w \leqslant v$. Then there exists another pair $(w', v') \in \mathcal{F}^{+\infty}(S) \times \mathcal{U}^{+\infty}(T)$ fulfilling $\widehat{\Phi} w' \leqslant v'$, such that $w \leqslant w'$ and $v' \leqslant v$ hold and that the pair (w', v') is maximal in this sense. A pair (w', v') is maximal if and only if $w' = v' \circ \Phi$ and $v' = \widehat{\Phi} w'$. In particular, w' is an upper semicontinuous function whenever (w', v') is maximal.*

The proposition holds also for pairs $(w, v) \in \mathcal{F}^{+\infty}(S) \times \mathcal{F}^{+\infty}(T)$ or $(w, v) \in \mathcal{U}^{+\infty}(S) \times \mathcal{U}^{+\infty}(T)$. In the latter case the supremal image $\widehat{\Phi}$ is replaced by the assignment $w \mapsto \left\lceil \widehat{\Phi} w \right\rceil$. Similar results will be obtained for $\triangle$-triples.

Definition 5.1. A triple $(w, v; u)$ with $w, v, u \in \mathcal{F}^{+\infty}(G)$ (resp. $w, v, u \in \mathcal{U}^{+\infty}(G)$) is called $\dot\triangle$*-triple of functions* (resp. *upper semicontinuous functions*) if

$$w \mathbin{\dot\triangle} v \leqslant u \tag{5.1a}$$

holds. The *tightness ordering* $\sqsubseteq$ on $\dot\triangle$-triples of (upper semicontinuous) functions is explained by

$$(w, v; u) \sqsubseteq (w', v'; u') \qquad \Longleftrightarrow \qquad w \leqslant w',\ v \leqslant v',\ u' \leqslant u. \tag{5.1b}$$

A $\dot\triangle$-triple of (upper semicontinuous) functions is called *maximally tight* if it is maximal with respect to $\sqsubseteq$.

Definition 5.2. Let $w, v, u \in \mathcal{F}^{+\infty}(G)$. *Multiplicative supremal left/right deconvolution* is defined as

$$(w \mathbin{\triangle\!\!\!\!\diagdown} u)(z) := \inf\left\{(1/w(x^{-1})) \curlyvee u(y) : x, y \in G, xy = z\right\}, \quad z \in G, \tag{5.2a}$$

$$(u \mathbin{\triangle\!\!\!\!\diagup} v)(z) := \inf\left\{u(x) \curlyvee (1/v(y^{-1})) : x, y \in G, xy = z\right\}, \quad z \in G. \tag{5.2b}$$

Here "$\curlyvee$" is *infimal multiplication* defined such that $0 \curlyvee \infty^+ = \infty^+ \curlyvee 0 = \infty^+$. The definition is analogous to the *deconvolution* from [13, Introduction].

Proposition 5.2. *Let $w, v, u \in \mathcal{F}^{+\infty}(G)$.*

(a) *The following equivalences hold:*

$$w \mathbin{\dot\triangle} v \leqslant u \qquad \Longleftrightarrow \qquad v \leqslant w \mathbin{\triangle\!\!\!\!\diagdown} u \qquad \Longleftrightarrow \qquad w \leqslant u \mathbin{\triangle\!\!\!\!\diagup} v \tag{5.3}$$

(b) *The following inequalities hold:*

$$v \leqslant w \mathbin{\triangle\!\!\!\!\diagdown} (w \mathbin{\dot\triangle} v), \qquad w \mathbin{\dot\triangle} (w \mathbin{\triangle\!\!\!\!\diagdown} u) \leqslant u, \tag{5.4a}$$

$$w \leqslant (w \mathbin{\dot\triangle} v) \mathbin{\triangle\!\!\!\!\diagup} v, \qquad (u \mathbin{\triangle\!\!\!\!\diagup} v) \mathbin{\dot\triangle} v \leqslant u, \tag{5.4b}$$

$$w \leqslant u \mathbin{\triangle\!\!\!\!\diagup} (w \mathbin{\triangle\!\!\!\!\diagdown} u), \qquad v \leqslant (u \mathbin{\triangle\!\!\!\!\diagup} v) \mathbin{\triangle\!\!\!\!\diagdown} u. \tag{5.4c}$$

(c) *If u is upper semicontinuous then $w \mathbin{\triangle\!\!\!\!\diagdown} u$ and $u \mathbin{\triangle\!\!\!\!\diagup} v$ are upper semicontinuous as well.*

Proposition 5.3. *Let $(w, v; u)$ be a $\dot\triangle$-triple of (upper semicontinuous) functions on G.*

(a) *The $\dot\triangle$-triple of functions $(w, v; u)$ is maximally tight if and only if*

$$u = w \mathbin{\dot\triangle} v, \qquad v = w \mathbin{\triangle\!\!\!\!\diagdown} u, \qquad w = u \mathbin{\triangle\!\!\!\!\diagup} v. \tag{5.5}$$

For the upper semicontinuous case replace "$u = w \mathbin{\dot\triangle} v$" by "$u = \lceil w \mathbin{\dot\triangle} v \rceil$".

(b) *There exists a maximally tight $\dot\triangle$-triple of (upper semicontinuous) functions $(w', v'; u')$ such that $(w, v; u) \sqsubseteq (w', v'; u')$.*

(c) If u is upper semicontinuous and $(w, v; u)$ is maximally tight, then w and v are upper semicontinuous as well.

6. Integration of $\mathbb{K}^\infty$-valued measurable functions

An extension of essential integration is introduced in this section (Equations (6.2) and (6.4)). It is required by the extended arithmetic from Section 2 and allows integration of arbitrary $\mathbb{K}^\infty$-valued measurable functions with respect to any $\mathbb{K}$-valued Radon measure. As a result of this extension the class of universal measurable $\mathbb{K}^\infty$-valued functions on G is proved to be preserved under left convolution with moderated Radon measures on G (Theorem 6.2). The result follows from Theorem 6.1 below.

From here on all measures will be Radon measures. The specification "Radon" will therefore be left out.

Let S be a locally compact space. The set of $\mathbb{K}$-valued continuous functions with compact support is denoted by $\mathcal{K}(S)$. The set of $\mathbb{K}$-*valued measures* on S is $\mathfrak{M}(S)$, the set of *positive measures* is $\mathfrak{M}^+(S)$. The *real part*, *imaginary part* and *absolute value* of a measure $\mu \in \mathfrak{M}(S)$ is denoted $\operatorname{Re}\mu$, $\operatorname{Im}\mu$ and $|\mu|$, respectively, and for a real-valued measure its *positive/negative part* is denoted by $\mu^\pm$ [2, Ch. III, §1, Nos. 3, 5, 6]. The notation

$$\mu^*(f) = \int^* f(s)\,\mathrm{d}\mu(s) \qquad \text{respectively} \qquad \mu^\bullet(f) = \int^\bullet f(s)\,\mathrm{d}\mu(s) \tag{6.1}$$

will be used for the *upper integral* [2, Ch. IV, §1, No. 1, Def. 1] respectively the *essential upper integral* [2, Ch. V, §1, No. 1, Def. 1] of $f \in \mathcal{F}^{+\infty}(S)$ with respect to $\mu \in \mathfrak{M}^+(S)$.

For given and fixed μ, the set $\mathcal{I}^\infty(\mu)$ of *essentially μ-integrable $\mathbb{K}^\infty$-valued functions* is defined as the set of all functions f from $\mathcal{F}^\infty(S)$ such that $I_f := \{s \in S : f(s) = \infty\}$ is a locally μ-negligible set and f coincides on $S\backslash I_f$ with some $\mathbb{K}$-*valued μ-integrable function* f' [2, Ch. IV, §4, No. 1, Def. 1]. This provides a useful extension of the essentially μ-integrable functions from [2, Ch. V, §1, No. 3, Def. 3] to $\mathbb{K}^\infty$. For $f \in \mathcal{I}^\infty(\mu)$ its *essential integral with respect to* μ is then defined as

$$\mu(f) = \int f(s)\,\mathrm{d}\mu(s) := \mu(f') \tag{6.2}$$

where $\mu(f')$ is the integral of f' as defined in [2, Ch. IV, §4, No. 1, Def. 1]. This definition of the essential integral $\mu(f)$ is analogous to the one in [2, Ch. V, §1, No. 3, Def. 3]. The set $\mathcal{M}^\infty(\mu)$ of *μ-measurable $\mathbb{K}^\infty$-valued functions* is defined as in [2, Ch. IV, §5, No. 1, Def. 1] where $\mathbb{K}^\infty$ is endowed with the topology it obtains when considered as an Alexandroff compactification of $\mathbb{K}$. The set of *universally measurable* $\mathbb{K}^\infty$*-valued functions* on S is defined as $\bigcap_{\mu\in\mathfrak{M}^+(S)} \mathcal{M}^\infty(\mu)$ as in [2, Ch. V, §3, No. 4, Def. 2].

Proposition 6.1. *The sets $\mathcal{M}^\infty(\mu)$ with $\mu \in \mathfrak{M}(S)$ and $\mathcal{M}^\infty(S)$ are closed under the extended addition, multiplication and $\mathbb{K}^\infty$-scalar multiplication. Further, they are closed under additive and multiplicative inverses, the formation of real/imaginary parts, positive/negative parts (if $\mathbb{K} = \mathbb{R}$) and the absolute value.*

The equivalence

$$f \in \mathcal{I}^\infty(\mu) \qquad \Longleftrightarrow \qquad |\mu|^\bullet(|f|) < \infty^+ \text{ and } f \in \mathcal{M}^\infty(\mu) \tag{6.3}$$

is obtained from an application of [2, Ch. V, §1, No. 3, Prop. 9] to the definition of the essential μ-integral in (6.2). Extending the essential μ-integral by

$$\mu(f) := \infty \qquad \text{whenever} \qquad |\mu|^\bullet(|f|) = \infty^+ \tag{6.4}$$

defines the *extended essential μ-integral* $\mu(f)$ of $f \in \mathcal{M}^\infty(\mu)$ with respect to $\mu \in \mathfrak{M}(S)$. This extension is motivated by the equivalence (6.3), because of which one obtains a well-defined mapping $\mu\colon \mathcal{M}^\infty(\mu) \to \mathbb{K}^\infty$.

Proposition 6.2. *The extended essential integral obeys*

$$\mu(\alpha f + \alpha g) = \alpha\mu(f) + \beta\mu(g) \tag{6.5a}$$

for $f, g \in \mathcal{M}^\infty(\mu)$, $\alpha, \beta \in \mathbb{K}$, $\mu \in \mathfrak{M}(S)$ if and only if

$$|\mu|(|f|) < \infty^+ \quad \text{or} \quad |\mu|(|g|) < \infty^+ \quad \text{or} \quad |\mu|(|\alpha f + \beta g|) = \infty^+. \tag{6.5b}$$

Similarly,

$$(\alpha\mu + \beta\nu)(f) = \alpha\mu(f) + \beta\nu(f) \tag{6.6a}$$

holds for $f \in \mathcal{M}^\infty(\mu) \cap \mathcal{M}^\infty(\nu)$, $\alpha, \beta \in \mathbb{K}$, $\mu, \nu \in \mathfrak{M}(S)$ if and only if

$$|\mu|(|f|) < \infty^+ \quad \text{or} \quad |\nu|(|g|) < \infty^+ \quad \text{or} \quad |\alpha\mu + \beta\nu|(|f|) = \infty^+. \tag{6.6b}$$

But,

$$\infty \cdot \mu(f) = \mu(\infty \cdot f) \tag{6.7a}$$

for $f \in \mathcal{M}^\infty(\mu)$, $\mu \in \mathfrak{M}(\mu)$ if and only if

$$\mu(f) \neq 0 \quad \text{or} \quad |\mu|(|f|) = 0. \tag{6.7b}$$

The standard estimate

$$|\mu(f)| \leqslant |\mu|(|f|) \tag{6.8}$$

holds for all $f \in \mathcal{M}^\infty(\mu)$, $\mu \in \mathfrak{M}(S)$. *When* ∞ *and* ∞^+ *are identified with each other, one has*

$$\mu(f) = \mu^\bullet(f) \tag{6.9}$$

for all $f \in \mathcal{M}^{+\infty}(S)$, $\mu \in \mathfrak{M}^+(S)$.

Corollary 6.1. *The following decomposition formula holds:*

$$\mu(f) = \sum_{k,l=0}^{3} \mathrm{i}^{k+l}(\mathrm{P}_k\mu)(\mathrm{P}_l f) \tag{6.10}$$

for all $f \in \mathcal{M}^\infty(\mu)$ *and* $\mu \in \mathfrak{M}(S)$, *where* $\mathrm{P}_0 := (\mathrm{Re}\,\cdot)^+$, $\mathrm{P}_1 := (\mathrm{Im}\,\cdot)^+$, $\mathrm{P}_2 := (\mathrm{Re}\,\cdot)^-$ *and* $\mathrm{P}_3 := (\mathrm{Im}\,\cdot)^-$.

The *product measure* $\mu \otimes \nu$ of $\mu \in \mathfrak{M}(S)$ and $\nu \in \mathfrak{M}(T)$ is defined for $f \in \mathcal{K}(S \times T)$ as [2, Ch. III, §4, No. 1]

$$(\mu \otimes \nu)(f) = \int \left(\int f(s,t)\,\mathrm{d}\mu(s) \right) \mathrm{d}\nu(t) = \int \left(\int f(s,t)\,\mathrm{d}\nu(t) \right) \mathrm{d}\mu(s) \tag{6.11a}$$

by virtue of the fact that the assignments

$$t \mapsto \mu(f)(t) := \int f(s',t)\,\mathrm{d}\mu(s'), \quad s \mapsto \nu(f)(s) := \int f(s,t')\,\mathrm{d}\nu(t'). \tag{6.11b}$$

define continuous functions of compact support.

A $\mathbb{K}$-valued measure μ on S is called *moderated*, if S is a countable union of μ-integrable sets [2, Ch. V, §1, No. 2, Def. 2]. In this case the support $\operatorname{supp}\mu$ of μ is a countable union of compact subsets of S [1, Lemma 1]. The set of moderated measures on S will be denoted by $\mathfrak{M}_\sigma(S)$.

Theorem 6.1. *Let* $\mu \in \mathfrak{M}_\sigma(S)$, $\nu \in \mathfrak{M}_\sigma(T)$ *and* $f \in \mathcal{M}^\infty(S \times T)$. *Then the functions* $\mu(f)$ *and* $\nu(f)$ *in* (6.11b) *are well-defined, in a point-wise sense, as extended essential integrals of* $\mathbb{K}^\infty$*-valued measurable functions and they fulfill* $\mu(f) \in \mathcal{M}^\infty(T)$ *and* $\nu(f) \in \mathcal{M}^\infty(S)$. *In addition formula* (6.11a) *holds, if* $|\mu \otimes \nu|(|f|) < \infty^+$ *or* μ, ν *and* f *are positive.*

Convolution of two measures is defined as the image of their product measure under the multiplication Γ. The *image* of a measure $\mu \in \mathfrak{M}(S)$ under a continuous mapping $\Phi\colon S \to T$ between locally compact spaces is defined by $(\Phi\mu)(f) := \mu(f \circ \Phi)$, $f \in \mathcal{K}(T)$ whenever $|\mu|(f \circ \Gamma) < \infty^+$ for all $f \in \mathcal{K}^+(S)$ [2, Ch. V, §6, No. 4, Def. 2]. The *convolution* of two measures

$\mu, \nu \in \mathfrak{M}(G)$ on a locally compact group G is defined as

$$\mu * \nu := \Gamma[\mu \otimes \nu] \tag{6.12}$$

whenever μ and ν are *convolvable*, i.e. whenever $|\mu \otimes \nu|(f \circ \Gamma) < \infty^+$ for all $f \in \mathcal{K}^+(S)$ [2, Ch. VIII, §3, No. 1]. For a function $f \in \mathcal{M}^\infty(G)$ its *left convolution* of with a measure $\mu \in \mathfrak{M}(G)$ is defined as

$$(\mu * f)(x) := \int f(y^{-1}x)\,\mathrm{d}\mu(y), \tag{6.13}$$

with $x \in G$. This defines a binary operation $*\colon \mathfrak{M}(G) \times \mathcal{M}^\infty(G) \to \mathcal{F}^\infty(G)$. Linearity holds with restrictions similar to those for the binary and bilinear operation $(\mu, f) \mapsto \mu(f)$ of extended essential integration in Proposition 6.2 considered as the mapping $\mathfrak{M}(G) \times \mathcal{M}^\infty(G) \to \mathbb{K}^\infty$.

Theorem 6.2. *Left convolution* (6.13) *is well-defined as a binary operation*

$$*\colon \mathfrak{M}_\sigma(G) \times \mathcal{M}^\infty(G) \to \mathcal{M}^\infty(G). \tag{6.14}$$

The transposition law

$$(\mu * \nu)(f) = \nu(\mu * f) \tag{6.15a}$$

holds for $f \in \mathcal{M}^\infty(G)$ *and* $\mu, \nu \in \mathfrak{M}_\sigma(G)$ *whenever* μ *and* ν *are convolvable and one of the following two expressions is finite:*

$$(|\mu| * |\nu|)(|f|), \qquad\qquad |\nu|(|\mu| * |f|). \tag{6.15b}$$

Theorem 6.2 stands out, because (6.14) does not need a condition similar to convolvability on the pairs of measures and functions involved. This is in stark contrast to the approach in [3, Ch. VIII, §4, No. 1] to convolution of measures and functions, where the functions are treated as densities of measures that are continuous with respect to another measure.

For a sequence of positive numbers $x_n \in \mathbb{R}^{+\infty}, n \in \mathbb{N}$, declare the value of their sum to be

$$\sum_{n\in\mathbb{N}} x_n = \sup_m \sum_{n=1}^m x_n \tag{6.16}$$

and for a sequence $x_n \in \mathbb{K}^\infty$

$$\sum_{n\in\mathbb{N}} x_n = \begin{cases} \lim\limits_{m\to\infty} \sum\limits_{n=1}^m x_n, & \text{if } \sum_{n\in\mathbb{N}} |x_n| < \infty^+, \\ \infty, & \text{if } \sum_{n\in\mathbb{N}} |x_n| = \infty^+. \end{cases} \tag{6.17}$$

Then linearity $\sum_{n\in\mathbb{N}}(\alpha x_n + \beta y_n) = \alpha \sum_{n\in\mathbb{N}} x_n + \beta \sum_{n\in\mathbb{N}} y_n$ holds always for positive sequences $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ from $\mathbb{R}^{+\infty}$ and $\alpha, \beta \in \mathbb{R}^{+\infty}$. For $\alpha, \beta \in \mathbb{K}$

and $(x_n)_{n\in\mathbb{N}}$, $(y_n)_{n\in\mathbb{N}}$ from $\mathbb{K}^\infty$ linearity holds if and only if $\sum_{n\in\mathbb{N}}|x_n| < \infty^+$, or $\sum_{n\in\mathbb{N}}|y_n| < \infty^+$, or $\sum_{n\in\mathbb{N}}|\alpha x_n + \beta y_n| = \infty^+$. The relation $\sum_{n\in\mathbb{N}} \infty \cdot x_n = \infty \cdot \sum_{n\in\mathbb{N}} x_n$ holds if and only if $\sum_{n\in\mathbb{N}} x_n \neq 0$, or $x_n = 0$ for all $n \in \mathbb{N}$. The inequality $|\sum_{n\in\mathbb{N}} x_n| \leqslant \sum_{n\in\mathbb{N}}|x_n|$ holds for all sequences $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{K}^\infty$.

7. Operations on weighted balls characterized by weights

In this section S and T will denote locally compact spaces, $\Phi\colon S \to T$ is a continuous mapping and G is a locally compact group with multiplication mapping $\Gamma\colon G \times G \to G$. The image, product and convolution operations on weighted balls of measures and functions are characterized now in terms of supremal operations on the weight functions.

Definition 7.1. For $w \in \mathcal{F}^{+\infty}(S)$ the *w-ball of measures* is defined as

$$w[\mathfrak{M}] := \{\mu \in \mathfrak{M}(S) : |\mu|^\bullet(1/w) \leqslant 1\}. \tag{7.1}$$

Remark 7.1. Let $w, v \in \mathcal{F}^{+\infty}(S)$. If w is locally bounded, then the ball $w[\mathfrak{M}]$ consists of moderated measures and the inequality $|\mu|^\bullet(1/w) \leqslant 1$ in (7.1) may be replaced by $|\mu|^*(1/w) \leqslant 1$.

Definition 7.2. Let $w \in \mathcal{F}^{+\infty}(S)$. The *$w$-weighted ball of $\mathbb{K}^\infty$-valued universally measurable functions* is defined as

$$\mathcal{M}_{\mathrm{b}}^\infty[w] := \{f \in \mathcal{M}^\infty(S) : \|f\|_w \leqslant 1\} \tag{7.2a}$$

where the *w-supremum norm* is

$$\mathcal{F}^\infty(S) \ni f \mapsto \|f\|_w := \sup\{w(s)|f(s)| : s \in S\} \in \mathbb{R}^{+\infty}. \tag{7.2b}$$

The *w-weighted ball of w-continuous functions* is defined as

$$\mathcal{C}_{\mathrm{v}}^\sim[w] := \overline{\mathcal{M}_{\mathrm{b}}^\infty[w] \cap \mathcal{K}(S)}^{\|\cdot\|_w}, \tag{7.2c}$$

where the formula for $\mathcal{C}_{\mathrm{v}}^\sim[w]$ means the closure of $\mathcal{M}_{\mathrm{b}}^\infty[w] \cap \mathcal{K}(S)$ in $\mathcal{M}_{\mathrm{b}}^\infty[w]$ with respect to the pseudo-metric $(f, g) \mapsto \|f - g\|_w$ on $\mathcal{M}_{\mathrm{b}}^\infty[w]$.

Remark 7.2. Let $w, v \in \mathcal{F}^{+\infty}(S)$. The set $\mathcal{C}_{\mathrm{v}}^\sim[w]$ consists of $\mathbb{K}^\infty$-valued universally measurable functions that are $\mathbb{K}$-valued on the set $\{w > 0\}$, such that its restrictions to the sets $\{w \geqslant \lambda\}$, $\lambda > 0$ are continuous and $w \cdot f \in \mathcal{F}_{\mathrm{v}}(S)$.

Theorems 7.1, 7.2 are now stated in preparation for the main results in Theorem 7.3.

Theorem 7.1. *Let $w \in \mathcal{F}^+_{\mathrm{lb}}(S)$, $v \in \mathcal{F}^+_{\mathrm{lb}}(T)$. The images $\Phi\mu$, $\mu \in w[\mathfrak{M}]$ exist if and only if $\widehat{\Phi}w$ is locally bounded. The following are equivalent:*

(a) The inequality $\widehat{\Phi}w \leqslant v$ holds.

(b) The inclusion $\Phi(w[\mathfrak{M}]) \subseteq v[\mathfrak{M}]$ holds.

Theorem 7.2. *Let $w \in \mathcal{F}^+_{\mathrm{lb}}(S)$, $v \in \mathcal{F}^+_{\mathrm{lb}}(T)$ and $u \in \mathcal{F}^+_{\mathrm{lb}}(S \times T)$. The following are equivalent:*

(a) The inequality $w \otimes v \leqslant u$ holds.

(b) The inclusion $w[\mathfrak{M}] \otimes v[\mathfrak{M}] \subseteq u[\mathfrak{M}]$ holds.

Theorem 7.3. *Let $w, v, u \in \mathcal{F}^+_{\mathrm{lb}}(G)$. All pairs $(\mu, \nu) \in w[\mathfrak{M}] \times v[\mathfrak{M}]$ are convolvable if and only if $w \triangle v$ is locally bounded. The following statements are equivalent:*

(a) The inequality $w \triangle v \leqslant u$ holds.

(b) The inclusion $w[\mathfrak{M}] * v[\mathfrak{M}] \subseteq u[\mathfrak{M}]$ holds.

(c) The inclusion $\breve{w}[\mathfrak{M}] * \mathcal{M}^\infty_{\mathrm{b}}[u] \subseteq \mathcal{M}^\infty_{\mathrm{b}}[v]$ holds.

In addition the following statements are equivalent to (a)–(c) if u is upper semicontinuous:

(a′) The inequality $\lceil w \triangle v \rceil \leqslant u$ holds.

(c′) The inclusion $\breve{w}[\mathfrak{M}] * \mathcal{C}^\sim_{\mathrm{v}}[u] \subseteq \mathcal{C}^\sim_{\mathrm{v}}[v]$ holds.

8. Convolution as a bounded bilinear operation

In this section the space S and the group G are locally compact.

Definition 8.1. The elements of $\mathcal{U}^+(S)$ will be called *weights* on S. Those sets W of weights on S such that for all $u \in \mathcal{U}^+(S)$, $w, v \in W$, $\lambda \in \mathbb{R}^+$

$$u \leqslant \lambda \sup\{w, v\} \quad \Rightarrow \quad u \in W \tag{8.1a}$$

are called *cone ideals of weights* on S (or *cone ideals* on S). The *cone ideal generated by a set V of weights* on S is denoted by $\langle V \rangle$ and given by

$$\langle V \rangle = \big\{ u \in \mathcal{U}^+(S) \,|\, \exists n \in \mathbb{N}, \lambda \in \mathbb{R}^+, v_1, \ldots, v_n \in V : u \leqslant \lambda \sup\{v_1, \ldots, v_n\} \big\} \tag{8.1b}$$

for non-empty V, while $\langle \varnothing \rangle = \{0\}$.

Definition 8.2. Let W be a set of weights on S. The linear space of *W-vanishing-at-infinity continuous functions* [16, Section 22] is

$$\mathcal{C}_{\mathrm{v}}(W) := \big\{ f \in \mathcal{C}(S) \;\big|\; \forall\, w \in W : w|f| \in \mathcal{F}^+_{\mathrm{v}}(S) \big\}\,. \tag{8.2a}$$

The space of *W-vanishing-at-infinity W-continuous functions* is introduced as

$$\mathcal{C}_{\mathrm{v}}^{\sim}(W) := \bigcap_{w \in W} \mathcal{C}_{\mathrm{v}}^{\sim}(w), \qquad \mathcal{C}_{\mathrm{v}}^{\sim}(w) := \overline{\mathcal{M}_{\mathrm{b}}^{\infty}(w) \cap \mathcal{K}(S)}^{\|\cdot\|_w} \tag{8.2b}$$

where $\mathcal{M}_{\mathrm{b}}^{\infty}(w) := \{f \in \mathcal{M}^{\infty}(S) : \|f\|_w < \infty^+\}$, $w \in \mathcal{U}^+(S)$ and the formula for $\mathcal{C}_{\mathrm{v}}^{\sim}(w)$ means the closure of $\mathcal{M}_{\mathrm{b}}^{\infty}(w) \cap \mathcal{K}(S)$ in $\mathcal{M}_{\mathrm{b}}^{\infty}(w)$ with respect to the pseudo-metric $(f, g) \mapsto \|f - g\|_w$ on $\mathcal{M}_{\mathrm{b}}^{\infty}(w)$. The *$W$-weighted topology* $\mathscr{T}_W$ on $\mathcal{C}_{\mathrm{v}}(W)$ or $\mathcal{C}_{\mathrm{v}}^{\sim}(W)$ is generated by the pseudo-metric

$$(e, f) \mapsto \|e - f\|_w, \quad w \in W. \tag{8.2c}$$

The relation $\mathcal{C}_{\mathrm{v}}(W) = \mathcal{C}_{\mathrm{v}}^{\sim}(W) \cap \mathcal{C}(S)$ holds, $\mathscr{T}_W$ is the subspace topology derived from $\mathscr{T}_W^{\sim}$ and Remark 7.2 implies that $\mathcal{C}_{\mathrm{v}}(W) = \mathcal{C}_{\mathrm{v}}^{\sim}(W)$ whenever $\chi_K \in W$ for all compact $K \subseteq G$. Introducing the set $\mathcal{C}_{\mathrm{v}}^{\sim}(W)$ allows a homogeneous formulation of Theorem 8.1 below. The functions $\|\cdot\|_w$, $w \in W$ are seminorms on $\mathcal{C}_{\mathrm{v}}(W)$ making $(\mathcal{C}_{\mathrm{v}}(W), \mathscr{T}_W)$ a locally convex space. In general $\mathcal{C}_{\mathrm{v}}^{\sim}(W)$ is not even linear, but the natural Hausdorff quotients of $(\mathcal{C}_{\mathrm{v}}(W), \mathscr{T}_W)$ and $(\mathcal{C}_{\mathrm{v}}^{\sim}(W), \mathscr{T}_W^{\sim})$ are isomorphic as locally convex spaces.

For any set W of weights on S the cone ideal $\langle W \rangle$ is the largest set V of weights on S such that $\mathscr{T}_V = \mathscr{T}_W$. It follows that the assignment $W \mapsto (\mathcal{C}_{\mathrm{v}}(W), \mathscr{T}_W)$ defines a bijection between cone ideals W and the locally convex spaces of W-vanishing continuous functions $(\mathcal{C}_{\mathrm{v}}(W), \mathscr{T}_W)$. In particular, this means that the set system of cone ideals is a closure system on the power set of $\mathcal{U}^+(S)$ which is naturally ordered by inclusion. Cone ideals are also maximal Nachbin families [16, Definition 1, Section 22] in the sense that for any Nachbin family W the cone ideal $\langle W \rangle$ is the largest Nachbin family V that obeys $\mathscr{T}_V = \mathscr{T}_W$. Introducing the notion of cone ideals allows one to work with a closure system. The set of all Nachbin families does not constitute a closure system.

Definition 8.3. Let W be a cone ideal on S. The *linear space of measures with W-finite measure* is defined as

$$W(\mathfrak{M}) := \left\{\mu \in \mathfrak{M}(S) \;\middle|\; \exists w \in W : |\mu|^{\bullet}(1/w) < \infty^+\right\}. \tag{8.3a}$$

The bornology $\mathscr{K}_W$ is defined as

$$\mathscr{K}_W := \{M \subseteq W(\mathfrak{M}) \mid \exists w \in W : M \subseteq w[\mathfrak{M}]\}. \tag{8.3b}$$

The space $W(\mathfrak{M})$ is the topological dual of $(\mathcal{C}_{\mathrm{v}}(W), \mathscr{T}_W)$ as shown in [17, p. 152] (see also [22, Theorem 3.1]). Here the pairing $\langle\cdot,\cdot\rangle_W : \mathcal{C}_{\mathrm{v}}(W) \times$

$W(\mathfrak{M}) \to \mathbb{K}$ is given by integration

$$(f,\mu) \mapsto \langle f,\mu\rangle_W := \mu(f) = \int f(s)\,\mathrm{d}\mu(s). \tag{8.4}$$

The definitions imply that $w[\mathfrak{M}]$ is the polar set of $\mathcal{C}_{\mathrm{v}}(W) \cap \mathcal{C}_{\mathrm{v}}[w]$ whenever $w \in W$. The polar set is the unit ball with respect to $\|\cdot\|_w$ in $\mathcal{C}_{\mathrm{v}}(W)$. Conversely, $\mathcal{C}_{\mathrm{v}}(W) \cap \mathcal{C}_{\mathrm{v}}[w]$ is the polar set of $w[\mathfrak{M}]$. Therefore, $\mathscr{K}_W$ is the equicontinuous compactology on $W(\mathfrak{M})$ associated to $\mathscr{T}_W$ [11] (see also [22, Section 4]). The space $(W(\mathfrak{M}), \mathscr{K}_W)$ by itself is a linear space with convex vector bornology [11, Def. 1.1].

By Remark 7.1 the elements of $W(\mathfrak{M})$ are moderated measures, i.e. $W(\mathfrak{M}) \subseteq \mathfrak{M}_\sigma(G)$. Conversely, if $W = \mathcal{U}^+(S)$, then $W(\mathfrak{M})$ consists of all moderated measures on S. Therefore $W(\mathfrak{M}) = \mathfrak{M}(G)$ can be achieved if and only if G is σ-compact.

Theorem 8.1. *Let W, V, U be cone ideals on G. The following statements are equivalent:*

(a) The following inclusion holds:

$$\lceil W \triangle V \rceil \subseteq U. \tag{8.5}$$

(b) The following two conditions hold:

$$*\colon W(\mathfrak{M}) \times V(\mathfrak{M}) \to U(\mathfrak{M}) \text{ is well defined,} \tag{8.6a}$$

$$\mathscr{K}_W * \mathscr{K}_V \subseteq \mathscr{K}_U. \tag{8.6b}$$

(c) The mapping

$$\mathrm{L} : \breve{W}(\mathfrak{M}) \to \{\mathcal{C}^\sim_{\mathrm{v}}(U) \to \mathcal{C}^\sim_{\mathrm{v}}(V)\} \tag{8.7a}$$

$$\mu \mapsto \mathrm{L}_\mu \colon \mathcal{C}^\sim_{\mathrm{v}}(U) \to \mathcal{C}^\sim_{\mathrm{v}}(V) \tag{8.7b}$$

$$f \mapsto \mathrm{L}_\mu f = \mu * f \tag{8.7c}$$

is well defined. Here $\mathrm{L}_\mu f = \mu * f$ denotes the left convolution of f with μ as in eq. (6.13). Images of bounded subsets of $(\breve{W}(\mathfrak{M}), \mathscr{K}_W)$ under L are equicontinuous sets of continuous mappings

$$(\mathcal{C}^\sim_{\mathrm{v}}(U), \mathscr{T}_U) \to (\mathcal{C}^\sim_{\mathrm{v}}(V), \mathscr{T}_V). \tag{8.7d}$$

If the assumption $\chi_K \in V$ for all compact $K \subseteq G$ holds in addition, then the sets $\mathcal{C}^\sim_{\mathrm{v}}(U)$ and $\mathcal{C}^\sim_{\mathrm{v}}(V)$ in statement (c) of Theorem 8.1 can be replaced by the sets $\mathcal{C}_{\mathrm{v}}(U)$ and $\mathcal{C}_{\mathrm{v}}(V)$. Then L with domain $\breve{W}(\mathfrak{M})$ is a linear mapping with continuous linear operators between locally convex spaces as co-domain.

Section 5	Section 9
ordering relation $\leqslant$	ordering relation $\subseteq$
$\mathcal{F}^{+\infty}(G)$	power set of $\mathcal{U}^{+}(G)$
$\mathcal{U}^{+\infty}(G)$	set of cone ideals on G
$(w, v) \mapsto w \mathbin{\dot{\triangle}} v$	$(W, V) \mapsto \lceil W \mathbin{\dot{\triangle}} V \rceil$
$(w, v) \mapsto \lceil w \mathbin{\dot{\triangle}} v \rceil$	$(W, V) \mapsto \langle \lceil W \mathbin{\dot{\triangle}} V \rceil \rangle$
$(w; u) \mapsto w \mathbin{\not{\dot{\triangle}}} u$	$(W; U) \mapsto \mathcal{U}^{+}(W, \bullet; U)$
$(v; u) \mapsto u \mathbin{\not{\dot{\triangle}}} v$	$(V; U) \mapsto \mathcal{U}^{+}(\bullet, V; U)$

Table 1. Order theoretic analogy between Section 5 and Section 9

The implication "(b) $\Rightarrow$ (a)" in Theorem 8.1 fails if the condition (8.6b) is dropped. The reason is that the assignment $W \mapsto \mathscr{K}_W$, with W a cone ideal, is injective, but the assignment $W \mapsto W(\mathfrak{M})$ is not. For example, one obtains $W(\mathfrak{M}) = V(\mathfrak{M}) = U(\mathfrak{M}) = \mathfrak{M}_{\mathrm{f}}(G)$ for $W = V = \mathcal{U}^{+}_{\mathrm{b}}(G)$, $U = \mathcal{U}^{+}_{\mathrm{v}}(G)$, but $W \neq U$ if G is non-compact. This shows that (8.6a) by itself does not imply (8.5), because $\lceil \mathcal{U}^{+}_{\mathrm{b}}(G) \mathbin{\dot{\triangle}} \mathcal{U}^{+}_{\mathrm{b}}(G) \rceil \nsubseteq \mathcal{U}^{+}_{\mathrm{v}}(G)$ contradicts (8.5), but convolution $*$ is known to be a well defined internal operation on $\mathfrak{M}_{\mathrm{f}}(G)$ in agreement with (8.6a).

The equivalences "(a) $\Longleftrightarrow$ (b)" in each of Theorems 7.3 and 8.1 generalize Section 4 of [7, p. 335]. To see this, let $W = V = U = \langle w \rangle$ for some $w \in \mathcal{U}^{+}(G)$. The assignment

$$|||\cdot|||_w \colon \mathfrak{M}(G) \to \mathbb{R}^{+\infty}, \quad \mu \mapsto |||\mu|||_w := |\mu|^{\bullet}(1/w), \tag{8.8}$$

defines a seminorm on $W(\mathfrak{M}) = \mathfrak{M}(w) := \{\mu \in \mathfrak{M}(G) : |||\mu|||_w < \infty^{+}\}$ which has $w[\mathfrak{M}] = \mathfrak{M}[w] := \{\mu \in \mathfrak{M}(G) : |||\mu|||_w \leqslant 1\}$ as its unit ball. By virtue of Remark 7.1 there is no difference to the definition in [7] which used upper integrals instead of essential upper integrals. This establishes the connection between Theorem 8.1 and the result from [7]. Theorem 7.3 shows that this result holds equally for $w \in \mathcal{F}^{+}_{\mathrm{lb}}(G)$, and hence the assumptions (i), (ii) and (iii) for w preceding equation (17) in [7, p. 335] are unnecessarily strong.

9. Maximally tight triples of ideals of weights

In this section it will be seen that the order theoretic structural results for individual functions from Section 5 carry over one by one to sets of weights using the correspondences listed in Table 1. Indeed Proposition 9.1 can be viewed as an application of the theory of quantales [20] (see Remark 4.1). Specifically, Proposition 9.1 follows from example (10) in [20, p.18].

Definition 9.1. Let $W, V, U \subseteq \mathcal{U}^+(G)$ be sets of weights. Define the sets

$$\mathcal{U}^+(\bullet, V; U) := \left\{ w \in \mathcal{U}^+(G) \;\middle|\; \forall\, v \in V : \lceil\, w \mathbin{\triangle} v \,\rceil \in U \right\}, \tag{9.1a}$$

$$\mathcal{U}^+(W, \bullet; U) := \left\{ v \in \mathcal{U}^+(G) \;\middle|\; \forall\, w \in W : \lceil\, w \mathbin{\triangle} v \,\rceil \in U \right\}. \tag{9.1b}$$

Proposition 9.1. *Let $W, V, U \subseteq \mathcal{U}^+(G)$ be sets of weights.*
(a) The following equivalences hold:

$$\lceil\, W \mathbin{\triangle} V \,\rceil \subseteq U \iff W \subseteq \mathcal{U}^+(\bullet, V; U) \iff V \subseteq \mathcal{U}^+(W, \bullet; U). \tag{9.2}$$

(b) The following inclusions hold:

$$V \subseteq \mathcal{U}^+(W, \bullet; \lceil\, W \mathbin{\triangle} V \,\rceil), \qquad \left\lceil W \mathbin{\triangle} \mathcal{U}^+(W, \bullet; U) \right\rceil \subseteq U, \tag{9.3a}$$

$$W \subseteq \mathcal{U}^+(\bullet, V; \lceil\, W \mathbin{\triangle} V \,\rceil), \qquad \left\lceil \mathcal{U}^+(\bullet, V; U) \mathbin{\triangle} V \right\rceil \subseteq U, \tag{9.3b}$$

$$W \subseteq \mathcal{U}^+(\bullet, \mathcal{U}^+(W, \bullet; U); U), \qquad V \subseteq \mathcal{U}^+(\mathcal{U}^+(\bullet, V; U), \bullet; U). \tag{9.3c}$$

(c) If U is a cone ideal, then $\mathcal{U}^+(\bullet, V; U)$ and $\mathcal{U}^+(W, \bullet; U)$ are also cone ideals on G.

Definition 9.2. A triple $(W, V; U)$ of sets of weights on G (resp. of cone ideals) is called a $\triangle$*-triple of sets of weights* (resp. $\triangle$*-triple of cone ideals*) if it obeys

$$\lceil\, W \mathbin{\triangle} V \,\rceil \subseteq U. \tag{9.4}$$

The *tightness ordering* is defined on $\triangle$-triples of sets of weights (resp. $\triangle$-triples of cone ideals) on G as

$$(W, V; U) \sqsubseteq (W', V'; U') \quad :\Longleftrightarrow \quad W \subseteq W',\, V \subseteq V',\, U' \subseteq U. \tag{9.5}$$

A $\triangle$-triple of sets of weights (resp. $\triangle$-triple of cone ideals) will be called *maximally tight* if it is maximal with respect to $\sqsubseteq$.

Proposition 9.2. *Let $(W, V; U)$ be a triple of sets of weights.*
(a) The triple $(W, V; U)$ is maximally tight if and only if

$$U = \lceil\, W \mathbin{\triangle} V \,\rceil, \quad W = \mathcal{U}^+(\bullet, V; U), \quad V = \mathcal{U}^+(W, \bullet; U). \tag{9.6}$$

For the case of cone ideals replace $U = \lceil\, W \mathbin{\triangle} V \,\rceil$ with $U = \langle \lceil\, W \mathbin{\triangle} V \,\rceil \rangle$.
(b) There exists a maximally tight triple tighter than $(W, V; U)$.
(c) If U is a cone ideal and $(W, V; U)$ is maximally tight, then W and V are cone ideals as well.

The last proposition in this section establishes the relation to the case of single weights that was treated in Section 5.

PROPOSITION 9.3. *Let $w, v, u \in \mathcal{U}^+(G)$ be weights on G and assume that $w \mathbin{\dot\triangle} v$ is locally bounded. Then $\langle \lceil w \mathbin{\dot\triangle} v \rceil \rangle = \lceil \langle w \rangle \mathbin{\dot\triangle} \langle v \rangle \rceil$, $\langle w \mathbin{\not\triangle} u \rangle = \mathcal{U}^+(\langle w \rangle, \bullet; \langle u \rangle)$ and $\langle u \mathbin{\not\triangle} v \rangle = \mathcal{U}^+(\bullet, \langle v \rangle; \langle u \rangle)$.*

10. **Invariance for ideals of weights**

Proposition 10.1 and Proposition 10.2 below discuss two constructions, that furnish two cone ideals W and V on G that fulfill $\lceil W \mathbin{\dot\triangle} W \rceil \subseteq W$ and $\lceil W \mathbin{\dot\triangle} V \rceil \subseteq V$. In other words, W is closed with respect to supremal convolution and V is left-W-invariant with respect to supremal convolution. Left-W-invariance for the cases $W := \langle \{\chi_x : x \in G\} \rangle$ and $W := \langle \{\chi_K : K \subseteq G\} \rangle$ are characterized in Proposition 10.3.

PROPOSITION 10.1. *Let V be a cone ideal on G such that $\chi_K \in V$ for all compact $K \subseteq G$. Then*

$$W := \mathcal{U}^+(\bullet, V; V), \tag{10.1a}$$

defines a cone ideal such that

$$\lceil W \mathbin{\dot\triangle} W \rceil \subseteq W. \tag{10.1b}$$

In addition $\chi_0 \in W$ and $W \subseteq V$.

PROPOSITION 10.2. *Let W be a cone ideal on G such that*

$$\lceil W \mathbin{\dot\triangle} W \rceil \subseteq W. \tag{10.2a}$$

Then

$$V := \mathcal{U}^+(W, \bullet; \mathcal{U}^+(G)) \tag{10.2b}$$

is a cone ideal that fulfills

$$\lceil W \mathbin{\dot\triangle} V \rceil \subseteq V. \tag{10.2c}$$

In addition $\chi_K \in V$ and $\chi_K \mathbin{\dot\triangle} V \subseteq V$ for all compact $K \subseteq G$.

Starting with a cone ideal W obeying $\lceil W \mathbin{\dot\triangle} W \rceil \subseteq W$ one can apply Proposition 10.2 and then Proposition 10.1 to obtain a maximally tight triple $(W', V'; V')$ with $\lceil W' \mathbin{\dot\triangle} W' \rceil \subseteq W'$ and $W \subseteq W'$. Similarly one can start with a cone ideal V that fulfills $\chi_K \in V$ for all compact $K \subseteq G$ and apply Proposition 10.1, then Proposition 10.2 and again Proposition 10.1

to obtain a maximally tight triple $(W', V'; V')$ with $\lceil W' \vartriangle W' \rceil \subseteq W'$ and $V \subseteq V'$.

In both constructions $\chi_K \in W'$ and $\chi_K \in V'$ hold for all compact $K \subseteq G$ which can be concluded from Corollary 4.1. And thus, also $W' \vartriangle \chi_K \subseteq W'$, $\chi_K \vartriangle W' \subseteq W'$ and $\chi_K \vartriangle V' \subseteq V'$. This renders W' as left and right translation invariant and V' as left translation invariant. Further, another maximally tight triple $(W', W'; W')$ is obtained, because $\chi_0 \in W'$. In addition W' is the smallest non-zero cone ideal U such that $\lceil W' \vartriangle U \rceil \subseteq U$. By definition V' is the largest one with this property.

Theorem 8.1 implies that $(W'(\mathfrak{M}), \mathscr{K}_{W'})$ is a bornological algebra that operates on the locally convex spaces $\mathcal{C}_{\mathrm{v}}(V')$ or $\mathcal{C}_{\mathrm{v}}(W')$ by left convolution. The equivalence in Theorem 8.1 implies that W' is the largest cone ideal with this property. As a special case of Theorem 8.1 one obtains the following proposition.

Proposition 10.3. *Let W be a cone ideal on G.*
The following statements are equivalent:
- *(a) The inclusion $\chi_x \vartriangle W \subseteq W$ holds for all $x \in G$.*
- *(b) The compactology $\mathscr{K}_W$ is invariant under left translations.*
- *(c) Left translations L_x, $x \in G$ are continuous endomorphisms of $(\mathcal{C}_{\mathrm{v}}^{\sim}(W), \mathscr{T}_W)$.*

The following statements are equivalent:
- *(a') The inclusion $\chi_K \vartriangle W \subseteq W$ holds for all compact $K \subseteq G$.*
- *(b') The compactology $\mathscr{K}_W$ is invariant under left convolution with compactly supported measures.*
- *(c') The assignment $x \mapsto \mathrm{L}_x$ maps compact subsets of G to equicontinuous sets of continuous endomorphisms of $(\mathcal{C}_{\mathrm{v}}^{\sim}(W), \mathscr{T}_W)$.*

Let W be a cone ideal on G. Then W is called *left translation invariant* whenever $\chi_x \vartriangle W \subseteq W$ for all $x \in G$ and of *left translation type* whenever $\chi_K \vartriangle W \subseteq W$ for all compact $K \subseteq G$.

Left translation invariance is a natural generalization of the term "left moderated" that has been introduced for single weights [7]. This is seen from Proposition 9.3, Proposition 5.2 and the fact that $\chi_x \vartriangle w \leqslant C_x w$ with $C_x \in \mathbb{R}^+$ for all $x \in G$ is equivalent to $w \not\vartriangle w \in \mathcal{U}^*(G)$. By Proposition 5.2(a) the latter is equivalent to the existence of some $v \in \mathcal{U}^*(G)$ that fulfills $v \vartriangle w \leqslant w$, for example $v := w \not\vartriangle w$. The function $w \not\vartriangle w$ corresponds to the function M in [7]. If $w \in \mathcal{U}^*(G)$ fulfills $w \not\vartriangle w \in \mathcal{U}^*(G)$, then $\langle w \rangle$ is of left translation type [6, Satz 2.7]. Thus, left translation invariance and left translation type coincide in this case. The second stronger invariance condition is a useful replacement for the first one whenever general cone ideals are considered. For single weights it has been introduced in [18]. A weight function is of left translation type in the sense of [18, VI,19.(iii)] if and only if $\langle w\rangle$ is of left translation type.

11. **Application to fractional Weyl integrals**

The left sided fractional Weyl integrals I_α, $\alpha \in (0,\infty^+)$ are defined pointwise as [8, 10, 14]

$$(\mathrm{I}_\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty^+} (y-x)^{\alpha-1} f(y)\,\mathrm{d}y \tag{11.1}$$

and $(\mathrm{I}_0 f)(x) := f(x)$ for $x \in \mathbb{R}$. Here $f \in \mathcal{C}(\mathbb{R})$ is such that $x^n f(x) \to 0$ when $x \to \infty^+$ for all $n \in \mathbb{N}$. The functions f with this property are exactly those in $\mathcal{C}_\mathrm{v}(P_+)$, where P_+ denotes the cone ideal of those weights $w(x)$ that increase at most like a power and vanish for all x below a finite threshold.

Applying results of previous sections establishes basic functional analytic properties of the operators I_α, once $\mathcal{C}_\mathrm{v}(P_+)$ is endowed with the topology $\mathscr{T}_{P_+}$. It will be shown that $\{\mathrm{I}_\alpha : \alpha \in [0,\beta]\}$ with $\beta \in [0,\infty^+)$ defines an equicontinuous set of continuous linear endomorphisms of $(\mathcal{C}_\mathrm{v}(P_+), \mathscr{T}_{P_+})$ and that the index law $\mathrm{I}_\alpha \circ \mathrm{I}_\beta = \mathrm{I}_{\alpha+\beta}$, $\alpha,\beta \in [0,\infty^+)$ holds. To do this, the operators I_α, $\alpha \in [0,\infty^+)$ are rewritten as

$$(\mathrm{I}_\alpha f)(x) = (\check{\mu}_\alpha * f)(x), \qquad x \in \mathbb{R},\ f \in \mathcal{C}_\mathrm{v}(P_+), \tag{11.2}$$

where the measures μ_α, $\alpha \in (0,\infty^+)$ on $\mathbb{R}$ have the Lebesgue density

$$\lambda_\alpha(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} \chi_{(0,\infty^+)}(x), \qquad x \in \mathbb{R}, \tag{11.3}$$

and $\mu_0 = \delta_0$ for $\alpha = 0$. The measures obey

$$\mu_\alpha * \mu_\beta = \mu_{\alpha+\beta}, \qquad \alpha,\beta \in [0,\infty^+). \tag{11.4}$$

Define (or choose) the one-parameter family of weights $\{w_\alpha : \alpha \geqslant 0\}$ as

$$w_\alpha(x) := \left(\max\{0,x\}/\alpha\right)^\alpha, \qquad x \in \mathbb{R}, \tag{11.5}$$

for $\alpha > 0$ and set $w_0 := \chi_{[0,\infty^+)}$. Then

$$w_\alpha \mathbin{\triangle\!\!\!\cdot}\ w_\beta = w_{\alpha+\beta}, \qquad \alpha,\beta \in [0,\infty^+). \tag{11.6}$$

To prove (11.6) compute the maximum of $y \mapsto x^\alpha(y-x)^\beta$ for $x > 0$, $0 < y < x$ and $\alpha,\beta > 0$ using calculus. In the case $\alpha = 0$ or $\beta = 0$, Equation (11.6) follows from the fact that

$$\chi_{[0,\infty^+)} \mathbin{\triangle\!\!\!\cdot}\ f = f \mathbin{\triangle\!\!\!\cdot}\ \chi_{[0,\infty^+)} = f \tag{11.7}$$

holds whenever f is an increasing function $\mathbb{R} \to \mathbb{R}^+$.

Shifting the weight w_β one can arrange that

$$\mu_\alpha(1/(\mathrm{L}_{-\varepsilon}w_\beta)) = \int_0^{\infty^+} \frac{x^{\alpha-1}}{\Gamma(\alpha)}\,\frac{1}{(x+\varepsilon)^\beta}\,\mathrm{d}x < \infty^+ \tag{11.8}$$

for $\beta > \alpha > 0$ and $\varepsilon > 0$. The function $\alpha \mapsto \mu_\alpha(1/(\mathrm{L}_{-\varepsilon}w_\beta))$ is continuous for $\alpha \in (0,\beta)$ when $\beta > 0$ and $\varepsilon > 0$ are fixed and for $\alpha \to 0$ one obtains $\mu_\alpha(1/(\mathrm{L}_{-\varepsilon}w_\beta)) \to \varepsilon^{-\beta}$. From this, one obtains that the inclusion

$$\{\mu_\alpha : \alpha \in [0,\beta]\} \subseteq \lambda_{\gamma,\beta,\varepsilon}\mathrm{L}_{-\varepsilon}w_\gamma[\mathfrak{M}] \tag{11.9}$$

holds for some $\lambda_{\gamma,\beta,\varepsilon} \in \mathbb{R}^+$ whenever $\gamma > \beta > 0$ and $\varepsilon > 0$.

One can show, that P_+ is the smallest cone ideal that contains the set $\{w_\alpha : \alpha \in [0,\infty^+)\}$ and is translation invariant. Using Proposition 4.1 and eq. (11.6) one can show that

$$\lceil P_+ \mathbin{\triangle\!\!\!\cdot}\, P_+ \rceil = P_+ \mathbin{\triangle\!\!\!\cdot}\, P_+ \subseteq P_+. \tag{11.10}$$

By Theorem 8.1, the space

$$(P_+(\mathfrak{M}), \mathscr{K}_{P_+}) \tag{11.11}$$

is a bornological convolution algebra. Now, Theorem 8.1, (11.10) and (11.9) imply that the set $\{\mathrm{I}_\alpha : \alpha \in [0,\beta]\}$ with $\beta \in [0,\infty)$ is an equicontinuous set of continuous linear endomorphisms of $(\mathcal{C}_\mathrm{v}(P_+), \mathscr{T}_{P_+})$. The index law $\mathrm{I}_\alpha \circ \mathrm{I}_\beta = \mathrm{I}_{\alpha+\beta}$ follows from eq. (11.4) and Fubini's theorem.

By virtue of the discussion following Proposition 10.2 the smallest nonzero cone ideal W on $\mathbb{R}$ such that $(P_+(\mathfrak{M}), \mathscr{K}_{P_+})$ operates on $(\mathcal{C}_\mathrm{v}(W), \mathscr{T}_W)$ in the sense of Theorem 8.1 is given by P_+. Proposition 10.2 implies that the largest such cone ideal is given by $(P_+)^* := \mathcal{U}^+(P_+, \bullet; \mathcal{U}^+(\mathbb{R}))$ which is calculated to be the set of weights on $\mathbb{R}$ that decrease faster than any power for $x \to +\infty$. The set $\mathcal{C}_\mathrm{v}((P_+)^*)$ consists of all continuous functions f that fulfill $|f| \in P_+$. The two triples $(P_+, P_+; P_+)$ and $(P_+, (P_+)^*; (P_+)^*)$ of cone ideals are both maximally tight.

Let $\langle \mathrm{e}_\lambda \rangle$, $\lambda \in \mathbb{R}$ be the cone ideal generated by the exponential weight $\mathrm{e}_\lambda(x) := \exp(\lambda x)$, $x \in \mathbb{R}$. The identity

$$\mathrm{e}_\lambda \mathbin{\triangle\!\!\!\cdot}\, w = c(w,\lambda)\,\mathrm{e}_\lambda \tag{11.12}$$

holds for all $w \in \mathcal{F}^{+\infty}(\mathbb{R})$ with some $c(w,\lambda) \in \mathbb{R}^{+\infty}$. The constant $c(w,\lambda)$ is finite if and only if $\mathrm{e}_\lambda \mathbin{\triangle\!\!\!\cdot}\, w$ is locally bounded. The supremal convolute $\mathrm{e}_\lambda \mathbin{\triangle\!\!\!\cdot}\, w_\alpha$ is locally bounded whenever $\lambda > 0$ and $\alpha \in [0,\infty^+)$. Using (11.12) yields

$$P_+ \mathbin{\triangle\!\!\!\cdot}\, \langle \mathrm{e}_\lambda \rangle \subseteq \langle \mathrm{e}_\lambda \rangle. \tag{11.13}$$

By virtue of Theorem 8.1 we obtain a second space $(\mathcal{C}_{\mathrm{v}}(\langle \mathrm{e}_\lambda \rangle), \mathscr{T}_{\langle \mathrm{e}_\lambda \rangle})$ on which the operators $\{\mathrm{I}_\alpha : \alpha \in [0, \beta]\}$ with $\beta \in [0, \infty^+)$ form an equicontinuous set of continuous linear endomorphisms that obeys the index law.

The triple $(P_+, \langle \mathrm{e}_\lambda \rangle ; \langle \mathrm{e}_\lambda \rangle)$ is not maximally tight. A tighter maximally tight triple is found to be $(\langle \mathrm{e}_\lambda \rangle, \langle \mathrm{e}_\lambda \rangle ; \langle \mathrm{e}_\lambda \rangle)$. This triple is uniquely determined with this property.

References

[1] S. Berberian, On the convolution of a measure and a function. *Proc. Amer. Math. Soc.* **53** (1975), 399–403.

[2] N. Bourbaki, *Elements of Mathematics: Integration I.* Springer, Berlin (2004).

[3] N. Bourbaki, *Elements of Mathematics: Integration II.* Springer, Berlin (2004).

[4] B. Davey, H. Priestley, *Introduction to Lattices and Order*, 2 Ed. Cambridge University Press, Cambridge (2002).

[5] R. Edwards, The stability of weighted Lebesgue spaces. *Trans. Amer. Math. Soc.* **93** (1959), 369–394.

[6] H. Feichtinger, Gewichtsfunktionen auf lokalkompakten Gruppen. *Sitzungsberichte der Österreichischen Akademie der Wissenschaften Mathem.-naturw. Klasse, Abt.II* **188** (1979), 451–471.

[7] G. Gaudry, Multipliers of weighted Lebesgue and measure spaces. *Proc. London Math. Soc.* **19** (1969), 327–340.

[8] R. Hilfer, *Applications of Fractional Calculus in Physics.* World Scientific Publ. Co., Singapore (2000); DOI: 10.1142/3779; https://www.worldscientific.com/worldscibooks/10.1142/3779.

[9] R. Hilfer, Y. Luchko, *Desiderata* for fractional derivatives and integrals. *Mathematics* **7** (2019), ID 149; DOI: 10.3390/math7020149; https://www.mdpi.com/2227-7390/7/2/149.

[10] E. Hille, R. Phillips, *Functional Analysis and Semi-Groups.* American Mathematical Society, Providence (1957).

[11] H. Hogbe-Nlend, *Bornologies and Functional Analysis.* North-Holland Publ. Co., Amsterdam (1977).

[12] T. Kleiner, R. Hilfer, Convolution operators on weighted spaces of continuous functions and supremal convolution. *Annali di Matematica Pura ed Applicata* **198** (2019), Accepted.

[13] M.L. Mazure, Equations de convolution et formes quadratiques. *Annali di Matematica Pura ed Applicata* **CLVIII** (1991), 75–97.

[14] K. Miller, The Weyl fractional calculus. In: B. Ross (Ed.) *Fractional Calculus and its Applications*, *Lecture Notes in Math.* **457** (1975), 80–91, Springer–Verlag.

[15] J. Moreau, Fonctionelles convexes. *Seminaire Jean Leray* **2** (1967), 1–108.

[16] L. Nachbin, *Elements of Approximation Theory.* Instituto de Matematica Pura e Aplicada de Conselho Nacional de Pesquisas, Rio de Janeiro (1965).

[17] J. Prolla, Weighted spaces of vector-valued continuous functions. *Annali di Matematica Pura ed Applicata* **89** (1971), 145–157.

[18] H. Reiter, Über den Satz von Weil-Cartier. *Monatshefte für Mathematik* **86** (1978), 13–62.

[19] H. Rockafellar, R. Wets, *Variational Analysis.* Grundlehren der mathematischen Wissenschaften **317**, Springer, Berlin (2009).

[20] K. Rosenthal, *Quantales and Their Applications.* Longman Group, London (1990).

[21] S. Samko, A. Kilbas, O. Marichev, *Fractional Integrals and Derivatives.* Gordon and Breach, Berlin (1993).

[22] W. Summers, Dual spaces of weighted spaces. Trans. of Amer. Math. Soc. **151** (1970), 323–333.

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