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ORIGINAL ARTICLE

Convolution on distribution spaces characterized by regularization

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Abstract
Locally convex convolutor spaces are studied which consist of those distributions that define a continuous convolution operator mapping from the space of test functions into a given locally convex lattice of measures. The convolutor spaces are endowed with the topology of uniform convergence on bounded sets. Their locally convex structure is characterized via regularization and function-valued seminorms under mild structural assumptions on the space of measures. Many recent generalizations of classical distribution spaces turn out to be special cases of the general convolutor spaces introduced here. Recent topological characterizations of convolutor spaces via regularization are extended and improved. A valuable property of the convolutor spaces in applications is that convolution of distributions inherits continuity properties from those of bilinear convolution mappings between the locally convex lattices of measures.

KEYWORDS
convolution, distribution spaces

1 | INTRODUCTION

A number of recent publications [2, 3, 9, 11, 12, 18, 34] has given significant spur and impetus toward the generalization and unification of classical results in the theory of convolution on locally convex distribution spaces. The present study is concerned with locally convex distribution spaces that are associated with a so-called solid regularization-invariant space, as defined below. This vastly extends and further unifies several results from the cited works on the structure of locally convex distribution spaces and their compatibility with convolution.

Motivated by the characterization of convolvability of distributions via regularization [20, 30] and topological characterizations of certain classes of locally convex distribution spaces [2, 3, 9, 11, 12], we consider the convolutor spaces

$$\mathscr{O}'_C(\mathscr{D},E) := \{f \in \mathscr{D}' \,;\, (\phi \mapsto \phi * f) \in \mathscr{L}(\mathscr{D},E)\} \tag{1.1}$$

on $\mathbb{R}^d$, $d \in \mathbb{N}$ endowed with the topology of uniform convergence on the bounded subsets of the space of test functions $\mathscr{D} = \mathscr{D}(\mathbb{R}^d)$. As usual $*$ is convolution and $\mathscr{L}(\mathscr{D},E)$ denotes the continuous linear mappings $\mathscr{D} \to E$. In this work, E is a *solid regularization-invariant space*. This means that E is a locally convex lattice and a solid subspace of the space $\mathscr{M}$ of Radon measures on $\mathbb{R}^d$, that continuously includes the space $\mathscr{K}$ of continuous functions of compact support and on which regularization mappings $f \mapsto \phi * f$ with $\phi \in \mathscr{D}$ operate continuously (see also Section 3.2). Every solid translation-invariant Banach space of distributions [9] is a solid regularization-invariant space.

Dividing our presentation into four parts, we prove equivalent, but simpler, characterizations of the locally convex structure of the spaces $\mathscr{O}'_C(\mathscr{D}, E)$ via regularization and function-valued seminorms in the first part. In the second part, the structure of the spaces in Equation (1.1) is investigated more deeply when E is a weighted space, in particular when $E = L^1_W$, generalizing results from [11]. In the third part, identities with several classes of distribution spaces are established, that were studied previously [9, 16, 28], but introduced in a different way. Finally, it is shown how the introduced distribution spaces are perfectly adapted to establish continuity for convolution of distributions. An outline of each part is following.

1.1 | Characterizing the locally convex structure of convolutor spaces

Given certain structural assumptions on E, it is a natural question whether classical regularization properties hold for spaces of the form $F = \mathscr{O}'_C(\mathscr{D}, E)$. For example: are bounded sets (respectively relatively compact sets) of distributions $H \subseteq F$ characterized by the property that $\phi * H$ is bounded (respectively relatively compact) in E for all $\phi \in \mathscr{D}$? Does a sequence $(f_n)_{n\in\mathbb{N}}$ converge in F if and only if $\phi * f_n$ converges in E for all $\phi \in \mathscr{D}$? Schwartz proved results of this kind for the spaces $\mathscr{D}'_{L^p}, 1 \le p \le \infty$ [28]. These were extended only recently [2, Prop. 17, 19] to normal complete distribution spaces E and F when E has a complete web and F is ultrabornological. Similar results were obtained for distribution spaces $\mathscr{D}'_{E'}$ associated with translation-invariant Banach spaces of distributions E [12, Thm. 3, Cor. 4, 5].

In Section 3 we establish several characterizations of the locally convex spaces $F = \mathscr{O}'_C(\mathscr{D}, E)$ in terms of regularization under the sole assumption that E is a *solid regularization-invariant space*. In our Theorem 3.11 (on p. 7), the locally convex structure of F can be characterized via linear regularization operators and function-valued seminorms that map into E. For example, F is characterized by the projective spectrum given by the mappings $F \to E$, $f \mapsto \phi * f$ with $\phi \in \mathscr{D}$. This characterization of F entails the regularization properties mentioned above. Comparing to the earlier results, a new insight is that the closed graph theorem plays no role here and that the restriction to sequences can be dropped.

Another characterization of the space $F = \mathscr{O}'_C(\mathscr{D}, E)$ uses the *generalized absolute values* that we introduce as

$$|f|_\Phi := \sup\{|\phi * f|\,;\, \phi \in \Phi\} \qquad \text{for } f \in \mathscr{D}',\ \Phi \in \mathfrak{B}(\mathscr{D}), \tag{1.2}$$

where $\mathfrak{B}(\mathscr{D})$ denotes the bounded subsets of $\mathscr{D}$. The operators $|-|_\Phi$ can be interpreted as function-valued seminorms. The topology of F is proved in Theorem 3.11 to be generated by the seminorms $f \mapsto p(|f|_\Phi)$, where p ranges over the continuous seminorms on E and Φ ranges over $\mathfrak{B}(\mathscr{D})$.

The characterizations are based on Theorem 3.2 (p. 5): given $\Phi \in \mathfrak{B}(\mathscr{D})$ one always has an inclusion

$$\Phi \subseteq \Psi * \theta_1 + \dots + \Psi * \theta_{2^d} \qquad \text{for suitable } \Psi \in \mathfrak{B}(\mathscr{D}) \text{ and } \theta_1, \dots, \theta_{2^d} \in \mathscr{D}. \tag{1.3}$$

Equation (1.3) is a stronger formulation than the more common weak factorization property $\phi = \psi_1 * \theta_1 + \dots + \psi_{2^d} * \theta_{2^d}$ of $\mathscr{D}$ that was proved in [13, 25]. For more information on and applications of factorization theorems similar to Equation (1.3) we also refer to [10], and references therein. Following the terminology in [33], Equation (1.3) may be referred to as the *bounded weak factorization property* of the topological algebra $(\mathscr{D}, *)$. Equation (1.3) is obtained by reexamining the proof of Theorem 3 from [13]. In connection with the result (1.3), the operators (1.2) become a useful and intuitive notation.

1.2 | The locally convex structure, duals and preduals of weighted distribution spaces

An important construction to obtain new locally convex spaces from old ones is to introduce weights. In Section 4, weighted spaces E_W are studied where E is a solid regularization-invariant space and W is a moderated cone ideal of the space $\mathscr{I}_{+\mathrm{lb}}$ of non-negative locally bounded lower semicontinuous functions. The definition of E_W is analogous to weighted L^p-spaces L^p_W. A *moderated cone ideal* is a lower set (that is $W \ni w \ge v \in \mathscr{I}_{+\mathrm{lb}}$ implies $v \in W$) that obeys $\mathscr{K}^+ \subseteq W$, $W + W \subseteq W$ and $\overline{\mathrm{T}}_K W \subseteq W$ for all $K \in \mathfrak{K} := \{K \subseteq \mathbb{R}^d\,;\, K \text{ compact}\}$ where $\overline{\mathrm{T}}_K$ denotes the *K-translation shell* defined as

$$\overline{\mathrm{T}}_K w := \sup\{\mathrm{T}_x w\,;\, x \in K\} \qquad \text{for all } w \in L^{\infty,+}_{\mathrm{loc}},\, K \in \mathfrak{K}\,. \tag{1.4}$$

We prove that the space $\mathscr{O}'_C(\mathscr{D}, E_W)$ can be described equivalently in terms of $\mathscr{O}'_C(\mathscr{D}, E)$ and smooth weight functions associated with W via regularization.

In a recent article [11], the spaces $\mathscr{O}'_C(\mathscr{D}, E)$, as defined in (1.1) with $E = L^1_W$, were studied in detail. The locally convex space L^1_W consists of the measurable functions f on $\mathbb{R}^d$ satisfying $\|w \cdot f\|_1 < \infty$ and is endowed with the seminorms $f \mapsto \|w \cdot f\|_1$, $w \in W$. The system of weights W was assumed to be a sequence $(w_n)_{n\in\mathbb{N}}$ of positive continuous weights with the property

$$\forall n \in \mathbb{N}\; \exists m \in \mathbb{N} :\; \sup_{x\in\mathbb{R}^d} \frac{w_n(x+\cdot)}{w_m(x)} \in L^\infty_{\mathrm{loc}}. \tag{1.5}$$

Clearly, the assumption $\overline{\mathrm{T}}_K W \subseteq W$ from above generalizes the condition (1.5) and therefore these spaces are a special case of the spaces E_W.

The case of general weighted L^1-spaces will be studied in Sections 4.2 and 4.3. The description of the dual space and the predual from [11] is generalized in Theorems 4.10 and 4.15 (p. 13, 15). Instead of the short-time Fourier transform, as in [11], we use the characterization of $\mathscr{O}'_C(\mathscr{D}, E)$ in terms of the generalized absolute values (1.2) for the proof. Further, the identity of sets $\mathscr{O}'_C(\mathscr{D}, L^1_W) = \{f \in \mathscr{D}'\,;\, \forall\phi \in \mathscr{D} : \phi * f \in L^1_W\}$ was derived from the closed graph theorem in [11], which applies when L^1_W is a Fréchet-space. Theorem 3.11 implies this relation for a general moderated cone ideal W and does not require the closed graph theorem. Normality and completeness of $\mathscr{O}'_C(\mathscr{D}, L^1_W)$ were also proved in [11, Lem. 4.10, Cor. 4.3] and we obtain these results from inheritance properties of the mapping $E \mapsto \mathscr{O}'_C(\mathscr{D}, E)$ given in Section 3.

1.3 | Reproducing the classical definitions for (DF)-type convolutor spaces

Because most classical distribution spaces, such as $\mathscr{E}'$, $\mathscr{S}'$, and $\mathscr{D}'_{L^p}$ with $1 \le p \le \infty$, were introduced as the strong dual of a Fréchet space of smooth functions [28], it is a natural question whether the definition (1.1) can reproduce these spaces via suitable choices for the space E. Related results were obtained already by Schwartz [28], who proved that $\mathscr{D}'_{L^p}$ has the same bounded sets and convergent sequences as $\mathscr{O}'_C(\mathscr{D}, L_p)$, and Grothendieck [14], the results of whom imply that the strong topology of $\mathscr{O}'_C$ is equal to the one induced by the embedding $\mathscr{O}'_C \ni f \mapsto (\phi \mapsto (\phi * f)) \in \mathscr{L}_{\mathrm{b}}(\mathscr{S}, \mathscr{S})$ [11, p. 829].

However, to the best of our knowledge, a positive answer to the above question was obtained for a certain class of weighted $\mathscr{D}'_{L^1}$-spaces just recently in [11, Thm. 1.1, p. 830]. More specifically, it was obtained that the identity of locally convex spaces

$$\mathscr{O}'_C(\mathscr{D}, L^1_W) = \mathscr{D}'_{L^1,W} \tag{1.6}$$

holds if and only if W satisfies the condition (Ω) [11, Equation (1.2)]. Here, W is a weight system of the form (1.5) above. The space $\mathscr{D}'_{L^1,W}$ is defined as the strong dual of the inductive limit of the spaces $\dot{\mathscr{B}}_{v_n}$ with $v_n := 1/w_n$, $n \in \mathbb{N}$ and $(w_n)_{n\in\mathbb{N}}$ the same weight system as in Equation (1.5). The space $\dot{\mathscr{B}}_{v_n}$ is the space of smooth functions h such that $v_n \cdot \partial^\alpha h$ vanishes at infinity, endowed with the seminorms $h \mapsto \|v_n \cdot \partial^\alpha h\|_\infty$, $\alpha \in \mathbb{N}_0^d$.

Another class of spaces of the form $\mathscr{D}'_{E'}$ will be studied in Section 5. The space $\mathscr{D}'_{E'}$ is defined as the strong dual $(\mathscr{D}_E)'_{\mathrm{b}}$ of the space $\mathscr{D}_E$ of smooth functions with all derivatives contained in E, endowed with the topology generated by the seminorms $h \mapsto p(\partial^\alpha h)$ with $\alpha \in \mathbb{N}_0^d$ and p a continuous seminorm on E. Many classical distribution spaces are of this form [28] and this construction was generalized in [12] from classical function spaces such as $E = L^p$ to translation-invariant Banach spaces of tempered distributions. Generalizing the identity $\mathscr{D}'_{L^1,w} = \mathscr{O}'_C(\mathscr{D}, L^1_w)$ in a direction different from [11, Thm. 1.1] we obtain in Theorem 5.6 (p. 17) that the spaces $(\mathscr{D}_E)'_{\mathrm{b}}$ and $\mathscr{O}'_C(\mathscr{D}, E')$ are equal as locally convex spaces when E is a solid regularization-invariant Fréchet space continuously included in L^1_{loc} and having $\mathscr{K}$ as a dense subset. As a byproduct, we obtain the identity of locally convex spaces $\mathscr{O}'_C(\mathscr{D}, L^p_w) = \mathscr{D}'_{L^p,w}$ for all $1 < p \le \infty$ and any positive moderated weight w, complementing the result for $p = 1$ from [11, Thm. 5.1, p. 853]. Corresponding identities for the spaces $\mathscr{E}'$ and $\mathscr{S}'$ are given in Examples 5.7 and 5.8.

The topology of uniform convergence on the compact subsets of the canonical predual space was recently characterized for the space $\mathscr{D}'_{L^p}$ with $1 \leq p \leq \infty$ using function seminorms [3, Props. 2.5, 3.2]. In order to establish the "dual" results, we prove the identity of locally convex spaces $\mathscr{D}'_{L^p,\mathrm{c}} = \mathscr{O}'_C(\mathscr{D}, (L^p)_{\mathrm{str}})$. Here, the subindex "c" in $\mathscr{D}'_{L^p,\mathrm{c}}$ indicates, for $p \neq 1$, the topology $\kappa(\mathscr{D}'_{L^p}, \mathscr{D}_{L^q})$ with $1/p + 1/q = 1$ and, for $p = 1$, the topology $\kappa(\mathscr{D}'_{L^1}, \dot{\mathscr{B}})$. Further, $(L^p)_{\mathrm{str}}$ denotes the space L^p with the strict topology given by the weighted seminorms $f \mapsto \|f \cdot w\|_p$ with $w \in \mathscr{C}_0^+$.

1.4 | Inherited properties of bilinear convolution mappings between convolutor spaces

In the concluding section, we study the convolution of distributions [22, 32] on the spaces $\mathscr{O}'_C(\mathscr{D}, E)$. Notice the following corollary of Equation (1.3): given $\Phi \in \mathfrak{B}(\mathscr{D})$ one can always find $\Psi \in \mathfrak{B}(\mathscr{D})$ such that Φ is contained in the absolute convex hull of $\Psi * \Psi$. From this, we will derive the *generalized absolute value inequality for convolution of distributions*

$$|f * g|_\Phi \leq |f|_\Psi * |g|_\Psi \qquad \text{for all convolvable } (f, g) \in \mathscr{D}' \times \mathscr{D}'. \tag{1.7}$$

Now, let E, F, G be solid regularization-invariant spaces and assume that convolution is well-defined, continuous, separately continuous, compactly hypocontinuous or boundedly hypocontinuous as a mapping $* : E \times F \to G$. Using Equation (1.7) and the regularization properties of the spaces $\mathscr{O}'_C(\mathscr{D}, E)$ obtained in Theorem 3.11 we prove that any of these properties of $* : E \times F \to G$ implies the corresponding property for convolution of distributions $* : \mathscr{O}'_C(\mathscr{D}, E) \times \mathscr{O}'_C(\mathscr{D}, F) \to \mathscr{O}'_C(\mathscr{D}, G)$.

2 | SOME NOTATIONS

Some general notations are summarized. Notations for function-valued seminorms and locally convex lattices are found in Sections 3.1 and 3.2.

Let F, G be a locally convex spaces. The set of continuous seminorms on F is denoted by $\operatorname{csn} F$, as in [9]. The continuous linear operators $F \to G$ are denoted by $\mathscr{L}(F, G)$. The subindices in $\mathscr{L}_{\mathrm{s}}$ and $\mathscr{L}_{\mathrm{b}}$ indicate the topologies of simple convergence and uniform convergence on the bounded sets, respectively. The bounded resp. compact subsets of F are be denoted by $\mathfrak{B}(F)$ resp. $\mathfrak{K}(F)$ and $\operatorname{acx} A$ denotes the absolute convex hull of a set $A \subseteq F$.

We will use the usual notations for distribution spaces from [16, p. 440] or [28]. All distribution spaces have the domain $\mathbb{R}^d$ if nothing is stated to the contrary. Translation and reflection of a distribution f will be denoted by $\mathrm{T}_x f$ and $\check{f}$. A sequence $(\phi_n)_{n\in\mathbb{N}} \subseteq \mathscr{D}$ will be called an *approximate unit* if $\phi_n \to 1$ within $\mathscr{E}$ and $\{\phi_n\,;\, n \in \mathbb{N}\}$ is a bounded set in $\mathscr{B}$. Note that the latter is equivalent to $\phi_n \to 1$ for $n \to \infty$ within the *strict topology of* $\mathscr{B}$. This is the finest topology on $\mathscr{B}$ that induces the same topology as $\mathscr{E}$ on every bounded subset of $\mathscr{B}$ (see [21, p. 11]). The strict topology on $\mathscr{B}$ will be indicated by $\mathscr{B}_{\mathrm{c}}$.

Let F, G be distribution spaces on $\mathbb{R}^d$ with the property that the continuous inclusions $\mathscr{D} \subseteq F, G \subseteq \mathscr{D}'$ hold. Here, we use the convention that $T \subseteq S$ continuously for topological spaces T, S means that T is a subset of S such that the canonical inclusion $T \to S$ is continuous. As usual, the notation $\mathscr{O}'_C(F, G)$ denotes the space of convolutors $F \to G$ for normal F. That is, the space of distributions $k \in \mathscr{D}'$ with the property that the convolution operator $\phi \mapsto \phi * k, \mathscr{D} \to G$ continuously extends to F. The space $\mathscr{O}'_C(F, G)$ is endowed with the subspace topology induced by $\mathscr{L}_{\mathrm{b}}(F, G)$ if nothing is stated to the contrary.

3 | CHARACTERIZATION OF CONVOLUTOR SPACES VIA REGULARIZATION

The purpose of this section is to characterize the locally convex structure of the convolutor spaces $\mathscr{O}'_C(\mathscr{D}, E)$ introduced in Equation (1.1) and to prove some inheritance properties for the mapping $E \mapsto \mathscr{O}'_C(\mathscr{D}, E)$.

3.1 | Function-valued seminorms defined by regularization

In the following, generalized absolute values and translation shells are introduced and important properties summarized. Proposition 3.6 will be fundamental for the characterization in Theorem 3.11.

Some notations for the concept of function-valued seminorms are required. The symbol $\mathscr{I}_+$ will denote the lower semicontinuous functions $f : \mathbb{R}^d \to \overline{\mathbb{R}}_+$, where $\overline{\mathbb{R}}_+ = [0, +\infty]$. The functions from $\mathscr{I}_+$ that are finite-valued and locally bounded are denoted by $\mathscr{I}_{+\mathrm{lb}}$. A non-empty lower set $W \subseteq \mathscr{I}_{+\mathrm{lb}}$ (i.e., $W \ni w \geq v \in \mathscr{I}_{+\mathrm{lb}}$ implies $v \in W$) with the property $W + W \subseteq W$ will be called *cone ideal*. The set of cone ideals constitutes a *closure system* on $\mathscr{I}_{+\mathrm{lb}}$. That is, a set $\mathfrak{C}$ of subsets of $\mathscr{I}_{+\mathrm{lb}}$ that contains $\mathscr{I}_{+\mathrm{lb}}$ and is closed with respect to intersection of non-empty families of sets from $\mathfrak{C}$ (such set systems $\mathfrak{C}$ are called "Moore family" in [5, p. 111] or "topped $\cap$-structure" in [8, p. 145]). An absolutely homogeneous, subadditive mapping $F \to \mathscr{I}_{+\mathrm{lb}}$ on a linear space F will be called $\mathscr{I}_{+\mathrm{lb}}$*-valued seminorm on* F. Two sets P, Q of $\mathscr{I}_{+\mathrm{lb}}$-valued seminorms on F will be called *equivalent* if they generate the same cone ideal, that is, if and only if

$$\forall p \in P,\, q \in Q\, \exists P' \subseteq P,\, Q' \subseteq Q \text{ finite} : q \leq \sup P',\; p \leq \sup Q'. \tag{3.1}$$

Definition 3.1. Let $f \in \mathscr{D}'$ and $\Phi \in \mathfrak{B}(\mathscr{D})$. The Φ-*absolute value* of f is defined as

$$|f|_\Phi := \sup\{|\phi * f|\,;\, \phi \in \Phi\}. \tag{3.2}$$

Let $w \in L^{\infty,+}_{\mathrm{loc}}$ and $K \in \mathfrak{K}$. The K-*translation shell* of w is defined as

$$\overline{\mathrm{T}}_K w := \sup\{\mathrm{T}_y w\,;\, y \in K\}. \tag{3.3}$$

The suprema in Equations (3.2) and (3.3) are formed within the order complete vector lattice of real-valued measurable functions that are essentially locally bounded. The supremum in Equation (3.2) can equivalently be understood as a point-wise supremum.

Let us summarize some readily verified properties. The Φ-absolute value $|f|_\Phi$ of a distribution f, where $\Phi \in \mathfrak{B}(\mathscr{D})$, is always a locally Lipschitz continuous function $\mathbb{R}^d \to \mathbb{R}_+$. In particular, one has $|f|_\Phi \in \mathscr{I}_{+\mathrm{lb}}$ and $|f|_\Phi$ is a regular distribution. Due to the relation $|(\phi * f)(x)| = |\langle f, \mathrm{T}_x \check{\phi}\rangle|$, the operators $|-|_\Phi$ are $\mathscr{I}_{+\mathrm{lb}}$-valued seminorms on $\mathscr{D}'$. It can also be proved that they are locally Lipschitz continuous. The K-translation shell operates endomorphically on the spaces $\mathscr{C}^+$, $\mathscr{I}_{+\mathrm{lb}}$, and $L^{\infty,+}_{\mathrm{loc}}$ and can be defined point-wise on $\mathscr{I}_{+\mathrm{lb}}$. Note, that $\overline{\mathrm{T}}_K$ can be interpreted as a "supremal convolution operator" with the indicator function 1_K as kernel [17].

The following relations hold:

$$\forall f \in \mathscr{D}',\, K \in \mathfrak{K},\, \Phi \in \mathfrak{B}(\mathscr{D}) : |f|_{\mathrm{T}_K\Phi} = \overline{\mathrm{T}}_K |f|_\Phi, \tag{3.4a}$$

$$\forall w \in L^{\infty,+}_{\mathrm{loc}},\, \Phi \in \mathfrak{B}(\mathscr{D}) : |w|_\Phi \leq \sup\{\|\phi\|_1; \phi \in \Phi\}\overline{\mathrm{T}}_{(\bigcup \operatorname{supp} \Phi)} w. \tag{3.4b}$$

Moreover, given $K \in \mathfrak{K}$ with non-empty interior, one finds $\phi \in \mathscr{D}_{-K}$ such that

$$w \leq |\overline{\mathrm{T}}_K w|_{\{\phi\}} \qquad \text{for all } w \in L^{\infty,+}_{\mathrm{loc}}. \tag{3.4c}$$

In connection with these inequalities, we will frequently use that

$$\mathrm{T}_K \Phi = \{\mathrm{T}_x \phi\,;\, \phi \in \mathscr{D}, x \in K\} \in \mathfrak{B}(\mathscr{D}) \qquad \text{for all } \Phi \in \mathfrak{B}(\mathscr{D}), K \in \mathfrak{K}. \tag{3.5}$$

The operators $|-|_\Phi$ preserve supports up to compact sets, more precisely

$$\operatorname{supp} |f|_\Phi \subseteq \operatorname{supp} f + K_\Phi \qquad \text{for all } f \in \mathscr{D}',\; \Phi \in \mathfrak{B}(\mathscr{D}), \tag{3.6a}$$

where $K_\Phi := \overline{\bigcup \operatorname{supp} \Phi}$. If $h \in \mathscr{E}$ is such that $\{h = 1\} \supseteq \operatorname{supp} f + (\{0\} \cup K_\Phi)$, then

$$|f|_\Phi = |h| \cdot |f|_\Phi = |h \cdot f|_\Phi. \tag{3.6b}$$

Because integration is a monotone operation, interchanging the order of translation shells and convolutions yields

$$\overline{\mathrm{T}}_K (w * v) \leq (\overline{\mathrm{T}}_K w) * v \qquad \text{for all } K \in \mathfrak{K},\; w \in L^{\infty,+}_{\mathfrak{K}},\; v \in L^{\infty,+}_{\mathrm{loc}}, \tag{3.7}$$

where $L^{\infty,+}_{\mathfrak{K}}$ denotes the compactly supported elements of $L^{\infty,+}_{\mathrm{loc}}$. Equation (3.7) also holds for $K \subseteq \mathbb{R}^d$ and $w, v \in \mathscr{I}_+$.

The following semifactorization theorem has important consequences for the introduced function-valued seminorms. In particular, Corollary 3.3 will be useful to derive the generalized absolute value inequality for convolution of distributions in the last section of this article.

Theorem 3.2. *Let $\Phi \in \mathfrak{B}(\mathscr{D})$. There exist $\Psi \in \mathfrak{B}(\mathscr{D})$ and $\theta_1, \dots, \theta_{2^d} \in \mathscr{D}$ such that*

$$\Phi \subseteq \Psi * \theta_1 + \dots + \Psi * \theta_{2^d} \,. \tag{3.8}$$

Corollary 3.3. *Let $\Phi \in \mathfrak{B}(\mathscr{D})$. There exists $\Psi \in \mathfrak{B}(\mathscr{D})$ such that*

$$\Phi \subseteq \operatorname{acx}(\Psi * \Psi). \tag{3.9}$$

Corollary 3.4. *Let $\phi \in \mathscr{D}$. There exist $\psi_1, \theta_1, \dots, \psi_{2^d}, \theta_{2^d} \in \mathscr{D}$ such that*

$$\phi = \psi_1 * \theta_1 + \dots + \psi_{2^d} * \theta_{2^d}. \tag{3.10}$$

A proof for Theorem 3.2 is obtained by examining the proof of Théorème 3.1 from [13]. For the convenience of the reader, we present the adapted proof for the case $G = \mathbb{R}^d$ on the following lines. The proof is based on the following lemma which is contained in [13, Lemme 2.5] and can be reused without change:

Lemma 3.5. *Let $b_k > 0$, $k \in \mathbb{N}_0$. There exist $a_k > 0$, $k \in \mathbb{N}_0$ and test functions $\theta, \chi \in \mathscr{D}(\mathbb{R})$ such that $a_k \le b_k$ for all $k \in \mathbb{N}_0$ and*

$$\sum_{k=0}^{n} (-1)^k a_k \theta^{(2k)} \overset{n\to\infty}{\longrightarrow} \delta + \chi \textit{ within } \mathscr{E}'(\mathbb{R}). \tag{3.11}$$

Proof of Theorem 3.2. The boundedness of Φ guarantees the finiteness of

$$M_{\alpha,e,k} := \sup\{\|\partial^\alpha \partial_e^{2k}\phi\|_\infty \,;\, \phi \in \Phi\} \quad \text{for all } \alpha \in \mathbb{N}_0^d,\ e = 1, \dots, d,\ k \in \mathbb{N}, \tag{3.12}$$

where ∂_e denotes the partial derivative in the e-the coordinate. From Lemma 3.5, we obtain $a_k > 0$, $k \in \mathbb{N}_0$ and $\theta, \chi \in \mathscr{D}(\mathbb{R})$ such that Equation (3.11) holds and

$$\sum_{k=0}^{\infty} a_k M_{\alpha,e,k} < \infty \qquad \text{for all } \alpha \in \mathbb{N}_0^d,\ e = 1, \dots, d. \tag{3.13}$$

From Equation (3.11), we derive that

$$\theta_e * \sum_{k=0}^{n} (-1)^k a_k \partial_e^{2k}\phi = \left(\sum_{k=0}^{n} (-1)^k a_k \partial_e^{2k}\theta_e \right) * \phi \overset{n\to\infty}{\longrightarrow} \phi + \chi_e * \phi, \tag{3.14}$$

where $\theta_e, \chi_e \in \mathscr{E}'$ are the image measures of θ, χ under the canonical injection $\mathbb{R} \to \mathbb{R}^d$ of the eth coordinate. From Equations (3.13) and (3.14), we conclude that

$$\theta_e * \psi(\phi) = \phi + \chi_e * \phi \qquad \text{with } \psi(\phi) := \sum_{k=0}^{\infty} (-1)^k a_k \partial_e^{2k}\phi \qquad \text{for all } \phi \in \Phi. \tag{3.15}$$

The series that defines the function $\psi(\phi)$ converges within $\mathscr{D}$ according to Equations (3.12) and (3.13). Equation (3.15) yields

$$\theta_e * \tilde{\Phi} + \chi_e * \tilde{\Phi} \supseteq \Phi \qquad \text{with } \tilde{\Phi} := \{\psi(\phi) \,;\, \phi \in \Phi\} \cup -\Phi, \tag{3.16}$$

Equations (3.12) and (3.13) also imply that the set $\tilde{\Phi}$ is bounded in $\mathscr{D}$. Any of the functions $\eta_1 * \dots * \eta_d = \eta_1 \otimes \dots \otimes \eta_d$ with $\eta_e \in \{\theta_e, \chi_e\}$, $e = 1, \dots, d$ belongs to $\mathscr{D}$. If we denote these by $\theta_1, \dots, \theta_{2^d}$ and apply Equation (3.16) successively for each dimension $e = 1, \dots, d$ we obtain Equation (3.8) for some $\Psi \in \mathfrak{B}(\mathscr{D})$. □

We can now use Theorem 3.2 to prove

Proposition 3.6. *The following three sets of mappings*

$$\{f \mapsto |f|_\Phi\,;\,\Phi \in \mathfrak{B}(\mathscr{D})\}, \tag{3.17a}$$

$$\{f \mapsto 1_K * |f|_{\{\phi\}}\,;\,\phi \in \mathscr{D}, K \in \mathfrak{K}\}, \tag{3.17b}$$

$$\{f \mapsto \overline{\mathrm{T}}_K |f|_{\{\phi\}}\,;\,\phi \in \mathscr{D}, K \in \mathfrak{K}\}, \tag{3.17c}$$

define equivalent sets of $\mathscr{I}_{+\mathrm{lb}}$-valued seminorms on $\mathscr{D}'$ (in the sense of Equation (3.1)).

Proof. Let $\Phi \in \mathfrak{B}(\mathscr{D})$. Choose $\Psi \in \mathfrak{B}(\mathscr{D})$ and $\theta_1, \dots, \theta_{2^d} \in \mathscr{D}$ such that Equation (3.8) of Theorem 3.2 holds. Then, for a suitable constant $C \in \mathbb{R}_+$ and $K \in \mathfrak{K}$, one obtains the inequalities

$$\sup_{\phi\in\Phi} |\phi * f| \le \sup_{\psi\in\Psi} \left| \sum_{k=1}^{2^d} \psi * \theta_k * f \right| \le \sum_{k=1}^{2^d} \left(\sup_{\psi\in\Psi} |\psi| \right) * |f|_{\theta_k} \le C \sum_{k=1}^{2^d} \overline{\mathrm{T}}_K |f|_{\theta_k} \qquad \text{for all } f \in \mathscr{D}'. \tag{3.18}$$

The proposition follows from Equations (3.18), (3.4), and (3.5). □

Proposition 3.7. *Let (θ_n) be an approximate unit and $\Phi \in \mathfrak{B}(\mathscr{D})$. Then*

$$|(1-\theta_n) f|_\Phi \xrightarrow{n\to\infty} 0, \qquad |\theta_n f|_\Phi \xrightarrow{n\to\infty} |f|_\Phi \qquad \textit{in } L^\infty_{\mathrm{loc}} \textit{ for all } f \in \mathscr{D}', \tag{3.19a}$$

and there exists $\Psi \in \mathfrak{B}(\mathscr{D})$ such that

$$|(1-\theta_n) f|_\Phi \le |f|_\Psi, \qquad |\theta_n f|_\Phi \le |f|_\Psi \qquad \textit{for all } n \in \mathbb{N} \textit{ and } f \in \mathscr{D}'. \tag{3.19b}$$

Proof. Let $K \subseteq \mathbb{R}^d$ compact and $f \in \mathscr{D}'$. The elements of the set $(1-\theta_n)\mathrm{T}_K\check{\Phi}$ converge to zero uniformly in $\mathscr{D}$ for $n \to \infty$ due to Equation (3.5) and hypocontinuity of $\cdot : \mathscr{E} \times \mathscr{D} \to \mathscr{D}$ [16, Prop. 3.6.4, p. 360]. Thus, continuity of $f : \mathscr{D} \to \mathbb{C}$ yields

$$\sup\{|(1-\theta_n) f|_\Phi(x)\,;\, x \in K\} \le \sup\left\{ \left| \left\langle f, (1-\theta_n)\mathrm{T}_x \check{\phi} \right\rangle \right| ; \phi \in \Phi, x \in K \right\} \xrightarrow{n\to\infty} 0. \tag{3.20}$$

Applying the inverse triangle inequality for the $\mathscr{I}_{+\mathrm{lb}}$-valued seminorm $|-|_\Phi$ yields that $|\theta_n f|_\Phi \to |f|_\Phi$ in L^∞_{loc}. Because the set of functions $B := \{1 - \mathrm{T}_x\check{\theta}_n, \mathrm{T}_x\check{\theta}_n\,;\, x \in \mathbb{R}^d, n \in \mathbb{N}\}$ belongs to $\mathfrak{B}(\mathscr{B})$, the set of test functions $\Psi := B \cdot \Phi$ belongs to $\mathfrak{B}(\mathscr{D})$. Now, Equation (3.19b) follows by construction. □

3.2 | Solid regularization-invariant spaces

In order to define solid regularization-invariant spaces, we will use some notations from the theory of ordered vector spaces as presented in [1, 23] or [27, Chap. V]. A *vector lattice* (also called *Riesz space*) is a real vector space E endowed with a lattice ordering $\le$ that is invariant with respect to translations and multiplication with positive scalars [1, Def. 1.1], [23, p. 4], [27, p. 204]. In particular, absolute values are well-defined by the formula $|f| = \sup\{f, -f\}$ for $f \in E$ in any vector lattice E. A *locally convex lattice* (also called *locally convex-solid Riesz space*) is a *vector lattice* endowed with a *locally convex-solid topology* [1, Def. 2.16 and p. 59], [23, Def. 4.6], [27, p. 234]. That is, a locally convex topology with a base at zero consisting of solid sets. A subset A of a vector lattice E is called *solid* if $f \in A$, $g \in E$ and $|g| \le |f|$ imply that also $g \in A$ [1, p. 8], [23, p. 35, 102], [27, p. 209]. The symbol E^+ will denote the non-negative elements of a vector lattice E. A seminorm p is called a *lattice-seminorm* if $|f| \le |g|$ implies $p(f) \le p(g)$. The continuous lattice-seminorms on a locally convex lattice E are denoted by $\operatorname{clsn} E$. The seminorms $p \in \operatorname{clsn} E$ generate the topology of E [1, Thm. 2.25], [23, p. 105], [27, p. 235].

The space of Radon measures $\mathscr{M}$ on $\mathbb{R}^d$ will be endowed with the topology generated by the seminorms $\mu \mapsto |\mu|(\phi)$, $\phi \in \mathscr{K}^+$, where $\mathscr{K}$ denotes the compactly supported continuous functions on $\mathbb{R}^d$. This is the coarsest topology on $\mathscr{M}$

that is finer than the weak topology induced by $\mathscr{K}$ and that turns $\mathscr{M}$ into a locally convex lattice [1, Def. 2.34, Thm. 2.35], [27, p. 235].

A locally convex space E will be called *solid space of measures over* $\mathbb{R}^d$ if E is a solid linear subspace of $\mathscr{M}$ and a locally convex lattice such that the canonical inclusions $\mathscr{K} \to E \to \mathscr{M}$ are continuous.

Definition 3.8. A *solid regularization-invariant space* E is a solid space of measures over $\mathbb{R}^d$ with the property that $f \mapsto \phi * f$ defines a continuous linear endomorphism of E for all $\phi \in \mathscr{D}$, that is, E is *(continuously) regularization-invariant*.

Let $p \in [1, +\infty]$. The space of locally p-integrable functions is denoted by L^p_{loc} and $L^p_{\mathfrak{K}}$ denotes the functions from L^p with compact support in $\mathbb{R}^d$. As a locally convex space, $L^p_{\mathfrak{K}}$ is the inductive limit of the spaces $L^p_K := \{f \in L^p\,;\, \operatorname{supp} f \subseteq K\}$ with $K \in \mathfrak{K}$, where $\mathfrak{K}$ denotes the compact subsets of $\mathbb{R}^d$. The norm on L^p is denoted by $\|\cdot\|_p$. The norm on the space of integrable measures $\mathscr{M}^1$ is also denoted by $\|\cdot\|_1$.

Example 3.9. Let E be a solid space of measures over $\mathbb{R}^d$. Then, if the translation group $(T_x)_{x\in\mathbb{R}^d}$ defines a locally equicontinuous $\mathscr{C}_0$-group on E and E has the convex compactness property (compare [9, Section 3]), then E is a solid regularization-invariant space. The converse is false as our definition does not even guarantee translation invariance. Consider, for example, the linear span of L^1 and δ endowed with the trace topology induced by $\mathscr{M}^1$.

The first assertion requires a remark. First, note that $(T_x)_{x\in\mathbb{R}^d}$ is a locally equicontinuous $\mathscr{C}_0$-group on the complete space $\mathscr{D}'$. The convolution of $\mu \in E$ or $\mu \in \mathscr{D}'$ and a measure $\nu \in \mathscr{M}$ with $\operatorname{supp}\nu$ compact can be defined as a vector-valued integral

$$\nu *_T \mu = \int \mathrm{T}_x\mu \, \mathrm{d}\nu(x), \tag{3.21}$$

in the space E or $\mathscr{D}'$, respectively [26, p. 77]. Due to the continuous inclusion $E \subseteq \mathscr{D}'$, the definitions in both the spaces E and $\mathscr{D}'$ coincide if $\mu \in E$. Due to the equicontinuity of $(T_x)_{x\in\mathbb{R}^d}$ and formula (3.21), one obtains a continuous linear operator, on E and on $\mathscr{D}'$, that commutes with translations. On the space $\mathscr{D}'$ the operator $(\nu *_T -)$ also commutes with partial derivatives (compare [16, Lem. 3 on p. 397]). This implies [16, Cor. on p. 399] the existence of a unique $\tilde{\nu} \in \mathscr{E}'$ such that $\nu *_T \mu = \tilde{\nu} * \mu$ for all μ, with the right-hand defined by convolution of distributions. Testing with $\mu = \phi \in \mathscr{D}$ yields $\tilde{\nu} = \nu$. Thus, for $\mu \in E$, the definition (3.21) coincides with convolution of measures and it follows that E is continuously regularization-invariant.

Proposition 3.10. *Let E be a solid regularization-invariant space. Then, the inclusion $L^\infty_{\mathfrak{K}} \to E$ is continuous and convolution defines a separately continuous mapping $L^\infty_{\mathfrak{K}} \times E \to E$.*

Proof. Let $K \in \mathfrak{K}$ and $k \in L^\infty_K$. It holds $|k * \mu| \le \phi * |\mu|$ for a suitable $\phi \in \mathscr{D}$. Thus, solidity and continuous regularization-invariance of E imply continuity of $L^\infty_K \times E \to E$ in the right-hand argument. As L^∞_K is metrizable, consider a sequence $(\mu_n) \subseteq L^\infty_K$. Then $|\mu_n| \le \lambda_n\phi$ for some $\phi \in \mathscr{D}$ and $(\lambda_n) \subseteq \mathbb{R}_+$ with $\lambda_n \to 0$. It follows $|\mu_n * \nu| \le \lambda_n\phi * |\nu| \to 0$, and thus, continuity of $L^\infty_K \times E \to E$ in the left-hand argument. The continuity of $L^\infty_K \to E$ is established similarly. Taking the inductive limit with respect to $K \in \mathfrak{K}$ completes the proof. □

3.3 | Characterization of the locally convex structure

Throughout this section, E will denote a fixed solid regularization-invariant space, as introduced in Definition 3.8.

Theorem 3.11. *The following sets of distributions coincide*

$$F_{\mathrm{a}} := \{f \in \mathscr{D}'\,;\, \forall \phi \in \mathscr{D} : \phi * f \in E\}, \tag{3.22a}$$

$$F_{\mathrm{b}} := \mathscr{O}'_C(\mathscr{D}, E), \tag{3.22b}$$

$$F_{\mathrm{c}} := \{f \in \mathscr{D}'\,;\, \forall \phi \in \mathscr{D}, K \in \mathfrak{K} : \overline{\mathrm{T}}_K|\phi * f| \in E\}, \tag{3.22c}$$

$$F_{\mathrm{d}} := \{f \in \mathscr{D}'\,;\, \forall \Phi \in \mathfrak{B}(\mathscr{D}) : |f|_\Phi \in E\}, \tag{3.22d}$$

where $\mathscr{O}'_C(\mathscr{D},E)$ was defined in Section 2. The following topologies on the space $\mathscr{O}'_C(\mathscr{D},E)$ coincide: the initial topologies $\mathscr{T}_\mathrm{a}$, $\mathscr{T}_\mathrm{b}$, $\mathscr{T}_\mathrm{c}$ induced by each of the following mappings or sets of mappings:

$$\mathscr{O}'_C(\mathscr{D},E) \to \mathscr{L}_\mathrm{s}(\mathscr{D},E) \qquad f \mapsto (\psi \mapsto \psi * f), \tag{3.23a}$$

$$\mathscr{O}'_C(\mathscr{D},E) \to \mathscr{L}_\mathrm{b}(\mathscr{D},E) \qquad f \mapsto (\psi \mapsto \psi * f), \tag{3.23b}$$

$$\mathscr{O}'_C(\mathscr{D},E) \to E \qquad f \mapsto \phi * f \qquad \phi \in \mathscr{D}, \tag{3.23c}$$

and the topology $\mathscr{T}_\mathrm{d}$ generated by the set of seminorms

$$\mathscr{O}'_C(\mathscr{D},E) \to \mathbb{R}_+ \qquad f \mapsto p(|f|_\Phi) \qquad \Phi \in \mathfrak{B}(\mathscr{D}),\ p \in \operatorname{clsn} E. \tag{3.23d}$$

Proof. Solidity implies $F_\mathrm{c} \subseteq F_\mathrm{a}$ and from Equation (3.4a) we obtain $F_\mathrm{d} \subseteq F_\mathrm{c}$. Conversely, the inclusion $F_\mathrm{a} \subseteq F_\mathrm{d}$ follows from Proposition 3.6, solidity and the relation $L^\infty_\mathfrak{K} * E \subseteq E$ from Proposition 3.10.

Let $p \in \operatorname{clsn} E$ and $\Phi \in \mathfrak{B}(\mathscr{D})$. The inclusion $\mathscr{T}_\mathrm{b} \subseteq \mathscr{T}_\mathrm{d}$ follows from the estimate

$$\sup\{p(\phi * f)\,;\,\phi \in \Phi\} \le p(\sup\{|\phi * f|\,;\,\phi \in \Phi\}) = p(|f|_\Phi) \qquad \text{for all } f \in F_a. \tag{3.24}$$

Because $\mathscr{D}$ is bornological the finiteness of $p(|f|_\Phi)$ in Equation (3.24) implies $f \in \mathscr{O}'_C(\mathscr{D},E)$, thus $F_\mathrm{a} = F_\mathrm{b}$. Proposition 3.6, solidity, and the convolution result from Proposition 3.10 imply that for all $p \in \operatorname{clsn} E$ and $\Phi \in \mathfrak{B}(\mathscr{D})$ there exist $\phi_1, \dots, \phi_{2^d} \in \mathscr{D}$, $K \in \mathfrak{K}$ and $q \in \operatorname{clsn} E$ such that

$$p(|f|_\Phi) \le p\left(\sum_{k=1}^{2^d} 1_K * |f|_{\phi_k}\right) \le \sum_{k=1}^{2^d} q\big(|f|_{\phi_k}\big) = \sum_{k=1}^{2^d} q(\phi_k * f) \qquad \text{for all } f \in F_\mathrm{a}. \tag{3.25}$$

This implies $\mathscr{T}_\mathrm{d} \subseteq \mathscr{T}_\mathrm{a}$. The remaining inclusions are obvious, for instance, $\mathscr{T}_\mathrm{a} = \mathscr{T}_\mathrm{c}$ is immediate from the definitions. □

Remark 3.12. It is clear that $\mathscr{O}'_C(\mathscr{D},E) = \mathscr{O}'_C(\mathscr{D},E \cap \mathscr{C})$ and that all the statements of Theorem 3.11 hold for $E \cap \mathscr{C}$ as well when $E \cap \mathscr{C}$ has the subspace topology induced by E. This follows from the fact that the functions $\phi * f$, $|\phi * f|$ and $|f|_\Phi$ are continuous for all $f \in \mathscr{D}'$, $\phi \in \mathscr{D}$ and $\Phi \in \mathfrak{B}(\mathscr{D})$.

Example 3.13. It is immediate from the definition that the seminorms $f \mapsto |f|_\Phi(x)$ with $\Phi \in \mathfrak{B}(\mathscr{D})$ and $x \in \mathbb{R}^d$ define the strong topology on $\mathscr{D}'$. In contrast to this, the seminorms $f \mapsto |(\phi * f)(x)|$ with $\phi \in \mathscr{D}$ and $x \in \mathbb{R}^d$ define the weak topology on $\mathscr{D}'$. Proposition 3.6 proves that the seminorms $f \mapsto \sup\{|(\phi * f)(x)|\,;\,x \in K\}$ with $\phi \in \mathscr{D}$ and $K \in \mathfrak{K}$ generate the strong topology on $\mathscr{D}'$. The spaces $\mathscr{M}$, L^∞_loc and L^1_loc satisfy the assumptions in Theorem 3.11 (when endowed with their usual topology). From Theorem 3.11 and Proposition 3.6 it follows that $\mathscr{D}' = F_\mathrm{a} = \dots = F_\mathrm{d}$ and the strong topology on $\mathscr{D}'$ coincides with $\mathscr{T}_\mathrm{a} = \dots = \mathscr{T}_\mathrm{d}$ for $E \in \{\mathscr{M}, L^\infty_\mathrm{loc}, L^1_\mathrm{loc}\}$.

Example 3.14. Consider for E the space $\mathscr{C}$ endowed with the topology of point-wise convergence. This space is a locally convex lattice but not continuously included in $\mathscr{M}$. It holds $\mathscr{D}' = F_\mathrm{a} = \dots = F_\mathrm{d}$ and $\mathscr{T}_\mathrm{a} = \mathscr{T}_\mathrm{c}$ coincides with the weak topology on $\mathscr{D}'$ while $\mathscr{T}_\mathrm{b} = \mathscr{T}_\mathrm{d}$ coincides with the strong topology on $\mathscr{D}'$. The latter identity is immediate from the equalities

$$|f|_\Phi(x) = \sup_{\phi \in \Phi} |(\phi * f)(x)| = \sup_{\phi \in \Phi} |\langle f, \mathrm{T}_x \check{\phi} \rangle| \qquad \text{for all } \Phi \in \mathfrak{B}(\mathscr{D}),\ x \in \mathbb{R}^d,\ f \in \mathscr{D}', \tag{3.26}$$

because translation and reflection induce bijections $\mathfrak{B}(\mathscr{D}) \to \mathfrak{B}(\mathscr{D})$. Considering the space $\mathscr{E}$ of smooth functions we find that $\mathscr{D}' = F_\mathrm{a} = F_\mathrm{b}$, but $\{0\} = F_\mathrm{c} = F_\mathrm{d}$. On the other hand, for $E = \mathscr{D}'$, one obtains that $\mathscr{D}' = F_\mathrm{a} = F_\mathrm{b} = F_\mathrm{c} = F_\mathrm{d}$ and that $\mathscr{T}_\mathrm{a} = \mathscr{T}_\mathrm{b} = \mathscr{T}_\mathrm{c} = \mathscr{T}_\mathrm{d}$ due to Corollary 3.4.

Lemma 3.15. *The set $\{(\phi * f)_{\phi \in \mathscr{D}}\,;\,f \in \mathscr{D}'\}$ is contained and closed in both the product spaces $\prod_{\phi \in \mathscr{D}} \mathscr{D}'$ and $\prod_{\phi \in \mathscr{D}} \mathscr{E}$.*

Proof. According to Examples 3.13 and 3.14, the strong topology on $\mathscr{D}'$ coincides with the initial topology with respect to the mapping $f \mapsto (\phi * f)_{\phi \in \mathscr{D}}$ with the codomain $\prod_{\phi \in \mathscr{D}} \mathscr{D}'$ or $\prod_{\phi \in \mathscr{D}} \mathscr{E}$. This implies the lemma because $\mathscr{D}'$ is complete. □

Corollary 3.16. *The space $\mathscr{O}'_C(\mathscr{D}, E)$ is isomorphic to a closed subspace of $\prod_{\phi\in\mathscr{D}} E$ via the mapping $f \mapsto (\phi * f)_{\phi\in\mathscr{D}}$.*

Proof. Theorem 3.11 implies, first, that the space $\mathscr{O}'_C(\mathscr{D}, E)$ is isomorphic to the set $I := \{(\phi * f)_{\phi\in\mathscr{D}}\,;\, f \in E\}$ endowed with the subspace topology of $\prod_{\phi\in\mathscr{D}} E$, and, second, that the set I is the intersection of $\prod_{\phi\in\mathscr{D}} E$ and $\{(\phi * f)_{\phi\in\mathscr{D}}\,;\, f \in \mathscr{D}'\}$, the latter being closed in $\prod_{\phi\in\mathscr{D}} \mathscr{D}'$ by Lemma 3.15. Therefore, I is closed in $\prod_{\phi\in\mathscr{D}} E$ because $E \subseteq \mathscr{D}'$ continuously. □

Remark 3.17. According to Theorem 3.11, and because $\mathscr{D}$ is bornological, the following three locally convex spaces of distributions coincide:

1. The space $\{f \in \mathscr{D}'\,;\, \forall\phi \in \mathscr{D} : |\phi * f| \in E\}$ with the seminorms $f \mapsto p(|\phi * f|)$, $\phi \in \mathscr{D}$, $p \in \operatorname{clsn} E$.
2. The space $\{f \in \mathscr{D}'\,;\, \forall\Phi \in \mathfrak{B}(\mathscr{D}) : |\Phi * f| \in \mathfrak{B}(E)\}$ with the seminorms $f \mapsto \sup_{\phi\in\Phi} p(|\phi * f|)$, $\Phi \in \mathfrak{B}(\mathscr{D})$, $p \in \operatorname{clsn} E$.
3. The space $\{f \in \mathscr{D}'\,;\, \forall\Phi \in \mathfrak{B}(\mathscr{D}) : |f|_\Phi \in E\}$ with the seminorms $f \mapsto p(|f|_\Phi)$, $\Phi \in \mathfrak{B}(\mathscr{D})$, $p \in \operatorname{clsn} E$.

This captures, in a nutshell, the subtle differences that disappear due to the mild assumptions on E from Definition 3.8 and the factorization property Theorem 3.2.

3.4 | Conclusions from the characterization theorem

As in the previous subsection, let E be a fixed solid regularization-invariant space and let $F = \mathscr{O}'_C(\mathscr{D}, E)$ throughout the section. We will now derive conclusions from the characterizations provided in Theorem 3.11 such as topological characterizations of subsets via regularization and inheritance properties of the mapping $\mathscr{O}'_C(\mathscr{D}, -)$.

Corollary 3.18. *Let H be a subset of F.*

1. *The set H is bounded (relatively compact) in F if and only if $\phi * H$ is bounded (relatively compact) in E for all $\phi \in \mathscr{D}$.*
2. *The set H is bounded in F if and only if $|H|_\Phi$ is bounded in E for all $\Phi \in \mathfrak{B}(\mathscr{D})$.*
3. *If H is relatively compact in F, then $|H|_\Phi$ is relatively compact in E for all $\Phi \in \mathfrak{B}(\mathscr{D})$.*
4. *If the solid hull of $|H|_\Phi$ is relatively compact in E for all $\Phi \in \mathfrak{B}(\mathscr{D})$, then H is relatively compact in F.*

Proof. Corollary 3.16 and Tikhonov's theorem imply Part 1. The continuity of $F \ni f \mapsto |f|_\Phi \in E$ for all $\Phi \in \mathfrak{B}(\mathscr{D})$ implies Parts 2 and 3. Part 1 implies Part 4 because the solid hull of $|H|_\Phi$ contains $|\phi * H|$ for all $\phi \in \Phi$ and $|\cdot|$ is continuous on E. □

Remark 3.19. The third statement in Corollary 3.18 simplifies if the space E satisfies the following property: a locally convex lattice is said to satisfy the *convex-solid compactness property* if and only if the convex-solid hull of every compact subset is compact as well. The *convex-solid hull* means the smallest superset that is solid *and* convex. This is an analogue of the *convex compactness property* [35, p. 134], [9, p. 4]. Clearly, convex-solid compactness implies convex compactness. For the converse implication, a simple counter example is given by the complete space L^∞. In this space, the solid closure of the singleton set $\{1\}$ is absolute convex and bounded, but not compact.

Corollary 3.20. *The iteration property $\mathscr{O}'_C(\mathscr{D}, \mathscr{O}'_C(\mathscr{D}, E)) = \mathscr{O}'_C(\mathscr{D}, E)$ holds.*

Proof. This is immediate from Theorem 3.11 and Corollary 3.4. □

Corollary 3.21. *The space E is continuously included in F.*

Proof. This follows from Theorem 3.11 and regularization-invariance of E. □

Corollary 3.22. *The continuous inclusion $F \subseteq \mathscr{D}'$ holds.*

Proof. Example 3.13 yields $\mathscr{D}' = \mathscr{O}'_C(\mathscr{D}, \mathscr{M})$ and $\mathscr{O}'_C(\mathscr{D}, -)$ preserves continuous inclusions. □

A proof for the continuous inclusion $\mathscr{E}' \subseteq F$, that is, based on Theorem 5.6, is given in Example 5.7 on p. 18. In Proposition 3.24 below, for a space of distributions G let $G_{\mathfrak{K}} := G \cap \mathscr{E}'$ denote the elements from G with compact support and let G_0 denote the topological closure of the set $G_{\mathfrak{K}}$ in G, endowed with the subspace topology induced by G.

Lemma 3.23. *Let a sequence $(g_n) \subseteq E$ be bounded by some $g \in E_0$ and such that $g_n \to 0$ in L^∞_{loc}. Then $g_n \to 0$ in E.*

Proof. Given any $\epsilon > 0$ and $p \in \operatorname{clsn} E$, K can be chosen such that $p(1_{\mathbb{R}^d\setminus K} \cdot g_n) \leq p(1_{\mathbb{R}^d\setminus K} \cdot g) \leq \epsilon/2$. This is because $E_{\mathfrak{K}}$ is dense in E_0 and due to the inequality $|g-k| \leq |1_{\mathbb{R}^d\setminus \operatorname{supp} k} \cdot g|$ for any $k \in E_{\mathfrak{K}}$. Further, the assumption implies that $1_K \cdot g_n \to 0$ in E for any $K \in \mathfrak{K}$ because $L^\infty_{\mathfrak{K}}$ is continuously included in E. The triangle inequality implies that $p(g_n) \leq \epsilon$ for $n \in \mathbb{N}$ large enough, that is, $g_n \to 0$. □

Proposition 3.24. *The space E_0 is solid and regularization-invariant. The identities of locally convex spaces*

$$\mathscr{O}'_C(\mathscr{D}, E_0) = F_0 = \overline{\mathscr{E}'}^F = \overline{\mathscr{D}}^F \tag{3.27}$$

hold and

$$\theta_n f \to f \text{ in } F \qquad \text{for all } f \in \mathscr{O}'_C(\mathscr{D}, E_0) \text{ and all approximate units } (\theta_n). \tag{3.28}$$

In particular, if $E_{\mathfrak{K}}$ is dense in E then $\mathscr{D}$ is dense in F.

Proof. Clearly, the set $E_{\mathfrak{K}}$ is solid in E and, as E is a locally convex lattice, the closure E_0 of $E_{\mathfrak{K}}$ is solid in E as well [23, Prop. 4.8]. Further, $\mathscr{D} * E_{\mathfrak{K}} \subseteq E_{\mathfrak{K}}$ and therefore $\mathscr{D} * E_0 \subseteq E_0$ by continuity. With E_0 carrying the subspace topology of E, it follows that E is solid and regularization-invariant.

For $f \in E$, it is clear that $f \in \mathscr{E}'$ if and only if $\phi * f \in E_{\mathfrak{K}}$ for all $\phi \in \mathscr{D}$. Theorem 3.11 implies that $\mathscr{O}'_C(\mathscr{D}, E_0)$ is a closed subset of $F = \mathscr{O}'_C(\mathscr{D}, E)$ and Proposition 3.10 yields $\mathscr{E}' \subseteq F$. One concludes that $\mathscr{O}'_C(\mathscr{D}, E_0) \supseteq F_0 = \overline{\mathscr{E}'}^F = \overline{\mathscr{D}}^F$. For the converse inclusion, one shows that $\theta_n f \to f$ in F for $f \in \mathscr{O}'_C(\mathscr{D}, E_0)$ and any approximate unit (θ_n). Indeed, Proposition 3.7 and Lemma 3.23 imply that $|(1-\theta_n)f|_\Phi \to 0$ in E for all $\Phi \in \mathfrak{B}(\mathscr{D})$. Therefore, $\theta_n f \to f$ in F by Theorem 3.11. □

Proposition 3.25. *If E is (sequentially, quasi-) complete, or has the convex compactness property, then the same holds for F.*

Proof. Let f_i be a Cauchy net in F and let $\phi \in \mathscr{D}$. The net f_i has a limit f in $\mathscr{D}'$ because $F \subseteq \mathscr{D}'$ continuously. In particular, it follows from Example 3.13 that $\phi * f_i \to \phi * f$ within $\mathscr{M}$. According to Theorem 3.11, the net $\phi * f_i$ is Cauchy in E and therefore $\phi * f_i \to g$ within E if E is complete. It holds $\phi * f = g$ because limits in $\mathscr{M}$ are unique. This implies $f_i \to f$ within F. The same reasoning applies for sequential completeness. Combining this reasoning with Corollary 3.18, one obtains the inheritance of quasi-completeness and the convex compactness property. □

4 | WEIGHTED CONVOLUTOR SPACES

In this section, we consider distribution spaces of the form $\mathscr{O}'_C(\mathscr{D}, E_W)$. For the special case $E = L^1$, we characterize the dual spaces and describe a predual. The weighted space E_W is defined in terms of a solid regularization-invariant space E and a moderated cone ideal W as the linear space

$$E_W := \{\mu \in \mathscr{M} \,;\, \forall w \in W : w \cdot \mu \in E\} \tag{4.1}$$

endowed with the topology generated by the solid seminorms $\mu \mapsto p(w \cdot \mu)$ with $p \in \operatorname{clsn} E$ and $w \in W$.

Throughout this section, W denotes a fixed *moderated cone ideal*, that is, a lower set $W \subseteq \mathscr{I}_{+\mathrm{lb}}$ with the properties $\mathscr{K}^+ \subseteq W$, $W + W \subseteq W$ and $\overline{\mathrm{T}}_K W \subseteq W$ for all $K \in \mathfrak{K}$. A single weight $w \in \mathscr{C}^+$ will be called *moderated* if $0 \neq w$ and for all $K \in \mathfrak{K}$ there exists $C_K \in \mathbb{R}_+$ such that $\overline{\mathrm{T}}_K w \leq C_K \cdot w$.

4.1 Description of weighted spaces by multiplication with smooth weights

In Propositions 4.5 and 4.6, the locally convex structure of the convolutor space $\mathscr{O}'_C(\mathscr{D}, E_W)$ is described via the basic space $\mathscr{O}'_C(\mathscr{D}, E)$ and multiplication with smooth functions that are associated with the cone ideal W.

Proposition 4.1. *The space E_W is a solid regularization-invariant space.*

Proof. It is clear that E_W is solid. Let $w \in \mathscr{I}_{+\mathrm{lb}}$, $\mu \in \mathscr{M}^+$ and $K \in \mathfrak{K}$. Using $x \in x + K - K$ one obtains the estimate

$$(w \cdot (1_K * \mu))(x) = \int_{x-K} w(x)\, \mathrm{d}\mu(y) \leq \int_{x-K} \left(\sup_{z \in y+K} w(z) \right) \mathrm{d}\mu(y) = (1_K * ((\overline{\mathrm{T}}_{-K} w) \cdot \mu))(x) \qquad \text{for all } x \in \mathbb{R}^d. \tag{4.2}$$

In particular, for all $w \in W$, $\phi \in \mathscr{D}$ and $\mu \in E_W$, one has $|w \cdot (\phi * \mu)| \leq C \cdot 1_K * (\overline{\mathrm{T}}_{-K} w \cdot |\mu|)$ with $C = \|\phi\|_1$, $K = \operatorname{supp} \phi$ and therefore $w \cdot (\phi * \mu) \in E$ due to Proposition 3.10 and because W is moderated. Applying $p \in \operatorname{clsn} E$ to the last inequality one concludes that E_W is continuously regularization-invariant with the same reasoning. □

We will need some notations for spaces of smooth functions. Let $u : \mathbb{R}^d \to \overline{\mathbb{R}}_+$ be an upper semicontinuous function that is bounded away from zero, that is, $1/u \in \mathscr{I}_{+\mathrm{lb}}$. The symbol $\mathscr{B}_u$ denotes the space of smooth functions h such that $\|u \cdot \partial^\alpha h\|_\infty < \infty$ for all $\alpha \in \mathbb{N}_0^d$. The space $\mathscr{B}_u$ is endowed with the topology induced by the seminorms $h \mapsto \|u \cdot \partial^\alpha h\|_\infty$, $\alpha \in \mathbb{N}_0^d$ and the symbol $\dot{\mathscr{B}}_u$ will denote the closure of $\mathscr{D} \cap \mathscr{B}_u = \mathscr{D}_{\{u<\infty\}}$ in $\mathscr{B}_u$. In the description of their dual spaces, the normed spaces L^1_w and $\mathscr{M}^1_w$ arise, also for weights $w \in \mathscr{I}_{+\mathrm{lb}}$ that are not everywhere positive. We will always consider them as Banach spaces by passing to their Hausdorff quotients. These quotients can be interpreted as spaces of measures on $\{w > 0\}$, because the set $\{w > 0\}$ is open due to the lower semicontinuity of w.

For any $w \in \mathscr{I}_{+\mathrm{lb}}$ and $C \in \mathbb{R}_+(\mathbb{N}_0^d) := \{\mathbb{N}_0^d \to \mathbb{R}_+\}$ we introduce the notation

$$\mathrm{B}(w\,;C) := \left\{ h \in \mathscr{E}\,;\, \forall \alpha \in \mathbb{N}_0^d : |\partial^\alpha h| \leq C_\alpha \cdot w \right\}. \tag{4.3}$$

Then, for any $u : \mathbb{R}^d \to \overline{\mathbb{R}}_+$ with $1/u \in \mathscr{I}_{+\mathrm{lb}}$, one has

$$\mathscr{B}_u = \bigcup \left\{ \mathrm{B}(1/u\,;C)\,;\, C \in \mathbb{R}_+(\mathbb{N}_0^d) \right\}. \tag{4.4}$$

For $B \in \mathfrak{B}(\mathscr{B})$, $\Phi \in \mathfrak{B}(\mathscr{D})$ and $w \in \mathscr{I}_{+\mathrm{lb}}$ it is readily seen, that

$$B \cdot (\Phi * w) \subseteq \mathrm{B}\left(\overline{\mathrm{T}}_K w\,;C \right) \tag{4.5a}$$

when K is the closure of $\bigcup \operatorname{supp} \Phi$ and $C \in \mathbb{R}_+(\mathbb{N}_0^d)$ is defined by

$$C_\alpha := \sum_{\substack{\beta,\gamma \in \mathbb{N}_0^d \\ \beta+\gamma=\alpha}} \binom{\alpha}{\gamma} \sup \left\{ \|\partial^\beta b\|_\infty \cdot \|\partial^\gamma \phi\|_1\,;\, b \in B,\ \phi \in \Phi \right\} \qquad \text{for } \alpha \in \mathbb{N}_0^d. \tag{4.5b}$$

Conversely, we have the following statement:

Lemma 4.2. *Let $C \in \mathbb{R}_+(\mathbb{N}_0^d)$ and let $K \in \mathfrak{K}$ with $K^\circ \neq \emptyset$. There exist $B \in \mathfrak{B}(\mathscr{B})$ and $\psi \in \mathscr{D}$ with the following property: for all $w \in \mathscr{I}_{+\mathrm{lb}}$ there exist $w_1, \dots, w_{3^d} \in \mathscr{I}_{+\mathrm{lb}}$ such that $w_1, \dots, w_{3^d} \leq \overline{\mathrm{T}}_K w$ and such that*

$$\mathrm{B}(w\,;C) \subseteq B \cdot (\psi * w_1) + \cdots + B \cdot (\psi * w_{3^d}). \tag{4.6}$$

Proof. Without restriction one may assume that $3Q \subseteq K$ with $Q := [-1,1]^d$. Choose $\eta \in \mathscr{D}_Q$ such that the series $\sum_{z \in \mathbb{Z}^d} \mathrm{T}_z \eta$ converges to 1 absolutely in $\mathscr{B}_{\mathrm{c}}$ and let $w \in \mathscr{I}_{+\mathrm{lb}}$ and $f \in \mathrm{B}(w\,;C)$. Then, the series $\sum_{z \in \mathbb{Z}^d} (f \cdot \mathrm{T}_z \eta)/v(z)$, with the weight $v := \overline{\mathrm{T}}_Q w$, is well-defined with the convention $0/0 = 0$ and converges in $\mathscr{B}_c$. Thus, we can define

$$b_k := \sum_{z\in k+3\mathbb{Z}^d} (f \cdot \mathrm{T}_z\eta)/v(z) \qquad \text{for } k \in \{0,1,2\}^d. \tag{4.7}$$

Now, choose $\psi \in \mathscr{D}$ and plateau functions $w_k \in \mathscr{I}_{+\mathrm{lb}}$ such that

$$(\psi * w_k)(z + x) = v(z) \qquad \text{for all } x \in Q, z \in k + 3\mathbb{Z}^d, k \in \{0,1,2\}^d. \tag{4.8}$$

Such functions ψ and w_k can be constructed by taking $\psi \in \mathscr{D}_{Q/8}$ with $\int \psi(x)\,\mathrm{d}x = 1$, $\psi \geq 0$ and defining $\theta := \psi * 1_{(3/2)Q}$ and $w_k := \sum_{z\in k+3\mathbb{Z}^d} v(z)\cdot \mathrm{T}_z\theta$. The inequalities $w_k \leq \overline{\mathrm{T}}_{2Q} v = \overline{\mathrm{T}}_{3Q} w \leq \overline{\mathrm{T}}_K w$ hold, in particular, $w_k \in \mathscr{I}_{+\mathrm{lb}}$. It follows

$$f = \sum_{k=1}^{3^d}\left(\sum_{z\in k+3\mathbb{Z}^d} (f\cdot \mathrm{T}_z\eta)/v(z)\cdot v(z)\right) = \sum_{k=1}^{3^d} b_k \cdot (\psi * w_k). \tag{4.9}$$

By construction, the convergence of the series in Equation (4.7) is uniform with respect to $f \in \mathrm{B}(w\,;\,C)$ for fixed $w \in \mathscr{I}_{+\mathrm{lb}}$. Hence, the collection B of all possible functions b_k is bounded in $\mathscr{B}$. □

Lemma 4.3. *Let $C \in \mathbb{R}_+(\mathbb{N}_0^d)$ and $\Phi \in \mathfrak{B}(\mathscr{D})$. There exist $K \in \mathfrak{K}$ and $\Psi \in \mathfrak{B}(\mathscr{D})$ such that*

$$|f\cdot h|_\Phi \leq |f|_\Psi \cdot \overline{\mathrm{T}}_K w \qquad \textit{for all } h \in \mathrm{B}(w\,;\,C),\ w \in \mathscr{I}_{+\mathrm{lb}},\ f \in \mathscr{D}'. \tag{4.10}$$

Proof. Let K be the closure of $\bigcup \operatorname{supp}\Phi$. Let $h \in \mathrm{B}(w\,;\,C)$ and $w \in \mathscr{I}_{+\mathrm{lb}}$. There exist $C'_\alpha \in \mathbb{R}_+$ such that

$$\sup\{\|\partial^\alpha(\mathrm{T}_{-x}h\cdot\check\phi)\|_\infty\,;\,\phi\in\Phi\} = \sup\{|\partial_y^\alpha(h(x-y)\phi(y))|\,;\,\phi\in\Phi, y\in K\} \leq C'_\alpha\cdot\overline{\mathrm{T}}_K w(x) \quad \text{for all } x\in\mathbb{R}^d,\ \alpha\in\mathbb{N}_0^d. \tag{4.11}$$

Due to Equation (4.11) and because $\bigcup \operatorname{supp}\check\Phi$ is relatively compact one finds $\Psi \in \mathfrak{B}(\mathscr{D})$ such that $\mathrm{T}_{-x}h\cdot\check\Phi \subseteq \check\Psi\cdot\overline{\mathrm{T}}_K w(x)$ for all $x \in \mathbb{R}^d$. Then, one estimates

$$\begin{aligned}
|f\cdot h|_\Phi(x) &= \sup\left\{\left|\left\langle f\cdot h, \mathrm{T}_x\check\phi\right\rangle\right|\,;\,\phi\in\Phi\right\}\\
&= \sup\left\{\left|\left\langle f, \mathrm{T}_x(\mathrm{T}_{-x}h\cdot\check\phi)\right\rangle\right|\,;\,\phi\in\Phi\right\}\\
&\leq \sup\left\{\left|\left\langle f, \overline{\mathrm{T}}_K w(x)\cdot\mathrm{T}_x\check\psi\right\rangle\right|\,;\,\psi\in\Psi\right\} = |f|_\Psi(x)\cdot\overline{\mathrm{T}}_K w(x) \qquad \text{for all } x\in\mathbb{R}^d,
\end{aligned} \tag{4.12}$$

which is Equation (4.10). □

Lemma 4.4. *Let $\Phi \in \mathfrak{B}(\mathscr{D})$. There exist $\Psi \in \mathfrak{B}(\mathscr{D})$, $\theta \in \mathscr{D}$ and $K \in \mathfrak{K}$ with the following property: for all $w \in \mathscr{I}_{+\mathrm{lb}}$ there exist $w_1, \ldots, w_{3^d} \in \mathscr{I}_{+\mathrm{lb}}$ with $w_1, \ldots, w_{3^d} \leq \overline{\mathrm{T}}_K w$ such that*

$$|f|_\Phi \cdot w \leq |f\cdot(\theta * w_1)|_\Psi + \cdots + |f\cdot(\theta * w_{3^d})|_\Psi \qquad \textit{for all } f \in \mathscr{D}'. \tag{4.13}$$

Proof. Choose suitable $C \in \overline{\mathbb{R}}_+(\mathbb{N}_0^d)$ such that $\mathrm{T}_x\check\Phi\cdot w(x) \subseteq \mathrm{B}\left(\overline{\mathrm{T}}_{(\bigcup\operatorname{supp}\Phi)} w\,;\,C\right)$ for all $x \in \mathbb{R}^d$. Lemma 4.2 yields $K \in \mathfrak{K}$, $B \in \mathfrak{B}(\mathscr{B})$ and $\theta \in \mathscr{D}$ with the property that for all $w \in \mathscr{I}_{+\mathrm{lb}}$ one finds $w_1, \ldots, w_d$ such that $w_1, \ldots, w_{3^d} \leq \overline{\mathrm{T}}_K w$ and

$$\mathrm{T}_x\check\Phi\cdot w(x) \subseteq \mathrm{B}\left(\overline{\mathrm{T}}_{(\bigcup\operatorname{supp}\Phi)} w\,;\,C\right) \subseteq B\cdot(\theta * w_1) + \cdots + B\cdot(\theta * w_{3^d}) \qquad \text{for all } x \in \mathbb{R}^d. \tag{4.14}$$

Then, let $\chi \in \mathscr{D}$ equal to 1 on the compact set $\bigcup \operatorname{supp} \Phi \subseteq \mathbb{R}^d$ and define the set $\Psi := \{\chi \cdot \mathrm{T}_{-x}\check{B}\,;\, x \in \mathbb{R}^d\} \in \mathfrak{B}(\mathscr{D})$. Multiplying Equation (4.14) with $\mathrm{T}_x\chi$ yields

$$\mathrm{T}_x\check{\Phi} \cdot w(x) \subseteq \mathrm{T}_x\check{\Psi} \cdot (\theta * w_1) + \cdots + \mathrm{T}_x\check{\Psi} \cdot (\theta * w_{3^d}) \qquad \text{for all } x \in \mathbb{R}^d. \tag{4.15}$$

Similar to Equation (4.12) one derives Equation (4.13) from this inclusion. □

The following two propositions are now immediate from Theorem 3.11 and the two Lemmas 4.3 and 4.4.

Proposition 4.5. *Let H be one of the sets $\bigcup\{\mathscr{B}_{1/w}\,;\, w \in W\}$ or $\mathscr{D} * W$. Then, $\mathscr{O}'_C(\mathscr{D}, E_W)$ is equal to the linear space*

$$\left\{f \in \mathscr{D}'\,;\, \forall h \in H \,:\, h \cdot f \in \mathscr{O}'_C(\mathscr{D}, E)\right\} \tag{4.16}$$

endowed with the initial topology with respect to the mappings $f \mapsto h \cdot f \in \mathscr{O}'_C(\mathscr{D}, E)$ with $h \in H$.

Proposition 4.6. *Let $w \in \mathscr{C}^+$ be moderated and let $h \in \mathscr{B}_w \cap (1/\mathscr{B}_{1/w})$. The space $\mathscr{O}'_C(\mathscr{D}, E_w)$ is isomorphic to $h \cdot \mathscr{O}'_C(\mathscr{D}, E)$ where the latter space is endowed with the topology induced by the bijection $\mathscr{O}'_C(\mathscr{D}, E) \ni f \mapsto h \cdot f$.*

Remark 4.7. Let $w \in \mathscr{C}^+$ be moderated and $0 \neq \phi \in \mathscr{D}$ with $\phi \geq 0$. One readily verifies $(\phi * w)^{-1} \in \mathscr{B}_w \cap (1/\mathscr{B}_{1/w})$.

4.2 | Dual spaces of weighted L^1 convolutor spaces

The convolutor space $\mathscr{O}'_C(\mathscr{D}, L^1_W)$ is considered now, where the weighted Lebesgue space L^1_W is defined as in Equation (4.1). According to Proposition 4.1, the space L^1_W is a solid regularization-invariant space and Theorem 3.11 can be applied. It follows that the topology on $\mathscr{O}'_C(\mathscr{D}, L^1_W)$ is generated by the seminorms

$$f \mapsto \int |f|_\Phi(x) w(x)\,\mathrm{d}x \qquad \text{with } w \in W,\ \Phi \in \mathfrak{B}(\mathscr{D}), \tag{4.17}$$

and it holds $\mathscr{O}'_C(\mathscr{D}, L^1_W) = \{f \in \mathscr{D}'\,;\, \forall \phi \in \mathscr{D} \,:\, \phi * f \in L^1_W\}$. Because $\mathscr{D}$ is dense in L^1_W and L^1_W is complete, Propositions 3.24 and 3.25 yield that $\mathscr{D}$ is dense in $\mathscr{O}'_C(\mathscr{D}, L^1_W)$ and that $\mathscr{O}'_C(\mathscr{D}, L^1_W)$ is complete as well.

In Theorem 4.10, we characterize the dual space of $\mathscr{O}'_C(\mathscr{D}, L^1_W)$. The equicontinuous sets are characterized using the description of the topology by the seminorms (4.17).

Lemma 4.8. *Let $C \in \mathbb{R}_+(\mathbb{N}_0^d)$ and $K \in \mathfrak{K}$ with non-empty interior. There exists $\Phi \in \mathfrak{B}(\mathscr{D})$ such that*

$$\sup\{|\langle f, h\rangle|\,;\, h \in \mathrm{B}(w\,;\, C)\} \leq \int |f|_\Phi(x)\overline{\mathrm{T}}_K w(x)\,\mathrm{d}x \qquad \textit{for all } f \in \mathscr{E}',\ w \in \mathscr{I}_{+\mathrm{lb}}. \tag{4.18}$$

Proof. Let B, ϕ and $w_1, \dots, w_{3^d}$ be as in Lemma 4.2. Define the bounded set $\Phi := \{3^d \cdot \check{\phi}\mathrm{T}_x b\,;\, b \in B,\ x \in \mathbb{R}^d\}$, one obtains the estimate

$$|\langle f, b \cdot (\phi * w_k)\rangle| = |\langle \check{\phi} * (b \cdot f), w_k\rangle| \leq \int |b \cdot f|_{\{\check{\phi}\}}(x) \cdot w_k(x)\,\mathrm{d}x \leq \frac{1}{3^d} \int |f|_\Phi(x) \cdot w_k(x)\,\mathrm{d}x \tag{4.19}$$

for all $k = 1, \dots, 3^d$, $f \in \mathscr{E}'$ and $b \in B$. Now, Equation (4.18) follows from Equations (4.19) and (4.6). □

Lemma 4.9. *Let $\Phi \in \mathfrak{B}(\mathscr{D})$. There exists $C \in \mathbb{R}_+(\mathbb{N}_0^d)$ and $K \in \mathfrak{K}$ such that*

$$\int |f|_\Phi(x) w(x)\,\mathrm{d}x \leq \sup\left\{|\langle f, h\rangle|\,;\, h \in \mathrm{B}\left(\overline{\mathrm{T}}_K w\,;\, C\right) \cap \mathscr{D}\right\} \qquad \textit{for all } f \in \mathscr{E}',\ w \in \mathscr{I}_{+\mathrm{lb}}. \tag{4.20}$$

Proof. Let $Q := [0,1]^d$, $\Theta := \mathrm{T}_Q\Phi$ and $C'_\alpha := \sup\{\|\partial^\alpha\theta\|_\infty\,;\,\theta\in\Theta\}$. Define

$$H := \left\{\sum_{z\in\mathbb{Z}^d}\lambda_z\cdot\mathrm{T}_z\check{\theta}_z\,;\,\theta_z\in\Theta,\,\lambda_z\in\mathbb{C},\,|\lambda_z|\le\overline{\mathrm{T}}_Q w(z)\right\}.$$

The set H consists of limits of series that converge absolutely in $\mathscr{E}$. The functions $h\in H$ obey $|\partial^\alpha h|\le C_\alpha\cdot\overline{\mathrm{T}}_K w$ for all $\alpha\in\mathbb{N}_0^d$ with K the closure of $Q+\bigcup\operatorname{supp}\Theta$ and $C_\alpha := C'_\alpha\cdot N$ with $N := \#\{z\in\mathbb{Z}^d\,;\,(z+K)\cap K\ne\emptyset\}$. One estimates

$$\begin{aligned}\int|f|_\Phi(x)w(x)\,\mathrm{d}x &\le \sum_{z\in\mathbb{Z}^d}\overline{\mathrm{T}}_Q|f|_\Phi(z)\overline{\mathrm{T}}_Q w(z)\\ &= \sup_{\substack{F\subseteq\mathbb{Z}^d\\ \text{finite}}}\sum_{z\in F}\sup_{\theta_z\in\theta}\left|\left\langle f,\overline{\mathrm{T}}_Q w(z)\cdot\mathrm{T}_z\check{\theta}_z\right\rangle\right|\\ &= \sup_{\substack{F\subseteq\mathbb{Z}^d\\ \text{finite}}}\ \sup_{\substack{\theta_z\in\Theta\\ \lambda_z\in\mathbb{C}\\ |\lambda_z|\le\overline{\mathrm{T}}_Q w(z)\\ z\in F}}\left|\left\langle f,\sum_{z\in F}\lambda_z\cdot\mathrm{T}_z\check{\theta}_z\right\rangle\right|\le\sup\left\{|\langle f,h\rangle|\,;\,h\in\mathrm{B}\left(\overline{\mathrm{T}}_K w\,;\,C\right)\cap\mathscr{D}\right\}\end{aligned}$$

for all $f\in\mathscr{E}'$. ☐

Theorem 4.10. *The dual space of $\mathscr{O}'_C(\mathscr{D},L^1_W)$ is the space of smooth functions*

$$\bigcup\{\mathscr{B}_{1/w}\,;\,w\in W\}=\left\{h\in\mathscr{E}\,;\,\exists w\in W\,\forall\alpha\in\mathbb{N}_0^d\,\exists C_\alpha\in\mathbb{R}_+\,:\,|\partial^\alpha h|\le C_\alpha\cdot w\right\}. \tag{4.21}$$

The duality product satisfies the formula

$$\langle f,h\rangle=\lim_{n\to\infty}\langle f\cdot\phi_n,h\rangle=\lim_{n\to\infty}\langle f,\phi_n\cdot h\rangle\qquad\textit{for all } f\in\mathscr{O}'_C(\mathscr{D},L^1_W),\ h\in\mathscr{O}'_C(\mathscr{D},L^1_W)', \tag{4.22}$$

and any approximate unit (ϕ_n). The sets $\mathrm{B}(w\,;\,C)$ with $w\in W$ and $C\in\mathbb{R}_+(\mathbb{N}_0^d)$ define a fundamental system for the equicontinuous sets on the dual $\mathscr{O}'_C(\mathscr{D},L^1_W)'$.

Proof. Consider the subspace $(\mathscr{E}',\mathscr{T})$ of $\mathscr{O}'_C(\mathscr{D},L^1_W)$ endowed with the subspace topology $\mathscr{T}$ induced by $\mathscr{O}'_C(\mathscr{D},L^1_W)$. Proposition 3.24 implies that the continuous linear functionals of $\mathscr{O}'_C(\mathscr{D},L^1_W)$ are the unique extensions of the continuous linear functionals of $(\mathscr{E}',\mathscr{T})$. Due to Lemmas 4.8 and 4.9, the topology on $\mathscr{E}'$ is the $\mathfrak{G}$-topology with respect to the duality $\langle -,-\rangle:\mathscr{E}'\times\mathscr{E}\to\mathbb{C}$, where $\mathfrak{G}$ consists of the sets $\mathrm{B}(w\,;\,C)$ with $w\in W$ and $C\in\mathbb{R}_+(\mathbb{N}_0^d)$. The set $\mathrm{B}(w\,;\,C)$ is weakly compact in $\mathscr{E}$ by the theorem of Arzelà-Ascoli. Thus, $\mathrm{B}(w\,;\,C)$ is also compact in the weak completion of $\mathscr{E}$. Clearly, the sets $\mathrm{B}(w\,;\,C)$ are absolute convex. Thus, Theorem 7 from [15, p. 68] implies that the set in Equation (4.21) is the dual space of $\mathscr{O}'_C(\mathscr{D},L^1_W)$.

As $L^1_{\mathfrak{K}}$ is dense in L^1_W, Equation (4.22) follows from Proposition 3.24. ☐

4.3 | Predual spaces for weighted L^1 convolutor spaces

Generalizing [11, Thm. 4.6] the spaces $\mathscr{O}'_C(\mathscr{D},L^1_W)$ can also be represented as the dual of the inductive limit of the spaces $\dot{\mathscr{B}}_{1/w}$, $w\in W$ as shown in Theorem 4.15. We will require some technical results first that will also be used in the subsequent section.

We need some lemmas for spaces of smooth functions induced by a locally convex space F. We define $\mathscr{D}_F$ as the linear space

$$\{h\in\mathscr{E}\,;\,\forall\alpha\in\mathbb{N}_0^d\,:\,\partial^\alpha h\in F\} \tag{4.23}$$

endowed with the topology generated by the seminorms $h \mapsto p(\partial^\alpha h)$ with $p \in \operatorname{csn} F$ and $\alpha \in \mathbb{N}_0^d$. In order to deal with spaces defined by weight systems that are not generated by a set of everywhere positive weights we use the following notation: for any locally convex space F in which $\mathscr{D} \cap F$ is dense we denote the set

$$\mathscr{D}'\left[F'\right] := \{f \in \mathscr{D}' \,;\, \exists f' \in F' \; \forall \phi \in \mathscr{D} \cap F : \langle f, \phi \rangle = f'(\phi)\}. \tag{4.24}$$

Clearly, $\mathscr{D}'\left[F'\right] = F'$ holds if $\mathscr{D} \cap F = \mathscr{D}$, in which case $\mathscr{D} \subseteq F$ is dense.

It is straightforward to prove the following proposition and its corollary:

Proposition 4.11. *Let F be a locally convex space continuously included in $\mathscr{D}'$ and such that $\mathscr{D} \cap F$ is dense in F. Then, $\mathscr{D} \cap \mathscr{D}_F$ is a dense subset of $\mathscr{D}_F$ and one has the representation formula*

$$\mathscr{D}'\left[(\mathscr{D}_F)'\right] = \operatorname{span} \bigcup_{k \in \mathbb{N}_0} \Delta^k\left(\mathscr{D}'\left[F'\right]\right). \tag{4.25}$$

Corollary 4.12. *For all $w \in \mathscr{I}_{+\mathrm{lb}}$, one has the representation formula*

$$\mathscr{D}'\left[(\dot{\mathscr{B}}_{1/w})'\right] = \operatorname{span} \bigcup_{k \in \mathbb{N}_0} \Delta^k\left(\mathscr{D}'[\mathscr{M}_w]\right). \tag{4.26}$$

The following proposition generalizes the implication "$(iii) \Rightarrow (iv)$" of Theorem 3 from [12] from Banach spaces to metrizable spaces. The proof is adapted from [12, 28] to the more general setting.

Proposition 4.13. *Let F be a metrizable locally convex space such that $F \cap \mathscr{D}$ is dense in F and let $K \in \mathfrak{K}$ with non-empty interior. Then, all distributions $f \in \mathscr{D}'$ satisfy the implication*

$$\forall \psi \in \mathscr{D}_K : \psi * f \in \mathscr{D}'\left[F'\right] \qquad \Rightarrow \qquad \exists k \in \mathbb{N}_0 : f \in \Delta^k(\mathscr{D}'\left[F'\right]) + \mathscr{D}'\left[F'\right]. \tag{4.27}$$

Proof. Assume that $\psi * f \in \mathscr{D}'\left[F'\right]$ for all $\psi \in \mathscr{D}_K$ and let $B \subseteq F \cap \mathscr{D}$ be bounded in F. Then, $\{\langle \check{\phi} * f, \check{\psi} \rangle \,;\, \phi \in B\}$ is bounded for all $\psi \in \mathscr{D}_K$ because $\langle \check{\phi} * f, \check{\psi} \rangle = \langle \psi * f, \phi \rangle$. It follows that $\{\check{\phi} * f \,;\, \phi \in B\}$ is an equicontinuous subset of $(\mathscr{D}_{-K})'$, which means that there exist $n \in \mathbb{N}$ and $C \in \mathbb{R}_+$ such that

$$|\langle \rho * f, \phi \rangle| \leq C \cdot \max\{\|\partial^\alpha \rho\|_\infty \,;\, \Sigma\alpha \leq n\} \qquad \text{for all } \phi \in B,\ \rho \in \mathscr{D}_K^n. \tag{4.28}$$

Let n_B denote the smallest integer such that Equation (4.28) holds for some $C \in \mathbb{R}_+$. The space $F \cap \mathscr{D}$, endowed with the subspace topology, is metrizable and thus bornological. If the least upper bound of $\{n_B \,;\, B \in \mathfrak{B}(F \cap \mathscr{D})\}$ was infinite we could derive a contraction from the fact that in a metrizable space every sequence of bounded subsets is absorbed by a single bounded subset from the space. Therefore, $n \in \mathbb{N}$ can be chosen independently of B in Equation (4.28).

Let $\rho \in \mathscr{D}_K^n$ be arbitrary. Because $F \cap \mathscr{D}$, endowed with the subspace topology induced by F, is a bornological space Equation (4.28) implies that the functional $\phi \mapsto \langle \rho * f, \phi \rangle$ is continuous on $F \cap \mathscr{D}$ and can be extended in a unique way to a continuous functional $\tilde{f}$ on F. This means $\rho * f \in \mathscr{D}'\left[F'\right]$ by the definition of $\mathscr{D}'\left[F'\right]$. The proof is completed via the parametrix method as in the proof of "$(iii) \Rightarrow (iv)$" in Theorem 3 from [12]. □

Lemma 4.14. *Let $w, v \in \mathscr{I}_{+\mathrm{lb}}$ and $K \in \mathfrak{K}$. The inequality $1_{-K} * w \leq v$ implies the inclusion*

$$\mathscr{D}_K * \mathscr{D}'\left[(\dot{\mathscr{B}}_{1/v})'\right] \subseteq \mathscr{D}'[\mathscr{M}_w]. \tag{4.29}$$

Proof. The inclusion $1_K * \left|\mathscr{D}'\left[\mathscr{M}_v\right]\right| \subseteq \mathscr{D}'\left[\mathscr{M}_w\right]$ holds. Indeed, transposing the convolution operator $(1_K * -)$ yields

$$\int (1_K * |\mu|)(x) w(x)\,\mathrm{d}x = \int (1_{-K} * w)(x)\,\mathrm{d}|\mu|(x) \leq \int v(x)\,\mathrm{d}|\mu|(x) \qquad \text{for all } \mu \in \mathscr{M} \text{ and } w \in \mathscr{I}_+. \tag{4.30}$$

Due to Corollary 4.12 and because all $\phi \in \mathscr{D}_K$ satisfy $|\phi| \leq C \cdot 1_K$ for some $C \in \mathbb{R}_+$ this inclusion proves the lemma. □

Theorem 4.15. *The convolutor space $\mathscr{O}'_C(\mathscr{D}, L^1_W)$ is the dual space of the inductive limit of the weighted spaces of smooth functions $\dot{\mathscr{B}}_{1/w}$ with $w \in W$. The topology on $\mathscr{O}'_C(\mathscr{D}, L^1_W)$ coincides with the topology of uniform convergence on those sets that are contained and bounded in $\dot{\mathscr{B}}_{1/w}$ for some $w \in W$.*

Proof. Let $f \in \mathscr{O}'_C(\mathscr{D}, L^1_W)$. Then, by Theorem 3.11, $\phi * f \in \mathscr{M}^1_w$ for all $w \in W$, $\phi \in \mathscr{D}_K$ and all $K \in \mathfrak{K}$. According to Proposition 4.13, applied to the Banach space $F = \mathscr{C}_{0,1/w}$, there exists $k \in \mathbb{N}$ such that $f \in \Delta^k(\mathscr{D}'\left[\mathscr{M}^1_w\right]) + \mathscr{D}'\left[\mathscr{M}^1_w\right]$. According to Corollary 4.12, this entails $f \in \mathscr{D}'\left[(\dot{\mathscr{B}}_{1/w})'\right]$ for all $w \in W$ and thus

$$f \in \bigcap_{w \in W} \mathscr{D}'\left[(\dot{\mathscr{B}}_{1/w})'\right] = \mathscr{D}'\left[\left(\varinjlim_{w \in W} \dot{\mathscr{B}}_{1/w}\right)'\right] = \left(\varinjlim_{w \in W} \dot{\mathscr{B}}_{1/w}\right)'.$$

In the second equality, it was used that $\mathscr{K}^+ \subseteq W$. Conversely, it is immediate from Theorem 3.11, Lemma 4.14 and the relation $L^{\infty,+}_{\mathfrak{K}} * W \subseteq W$ that every $f \in \bigcap_{w \in W}(\dot{\mathscr{B}}_{1/w})'$ is contained in $\mathscr{O}'_C(\mathscr{D}, L^1_W)$. Here, $L^{\infty,+}_{\mathfrak{K}} * W \subseteq W$ follows from the condition $\overline{\mathrm{T}}_K W \subseteq W$ for all $K \in \mathfrak{K}$. The characterization of the topology on $\mathscr{O}'_C(\mathscr{D}, L^1_W)$ is just a reformulation of its description in Theorem 4.10. □

5 | NEW IDENTITIES FOR LOCALLY CONVEX DISTRIBUTION SPACES

In this section, two classes of distribution spaces, given by their classical definitions, will be represented in the form $\mathscr{O}'_C(\mathscr{D}, E)$ with E a solid regularization-invariant space.

5.1 | Duals of solid regularization-invariant Fréchet spaces

Let E be a fixed solid regularization-invariant space that satisfies $E \subseteq L^1_{\mathrm{loc}}$, has $\mathscr{K}$ as a dense subset and is a Fréchet space. The primary purpose of this section is to prove the identity of locally convex spaces $\mathscr{D}'_{E'} = \mathscr{O}'_C(\mathscr{D}, E')$. We also prove the existence of a moderated cone ideal $W(E)$ such that $(\mathscr{D}_E)'_\sigma = \mathscr{O}'_C(\mathscr{D}, L^1_{W(E)})_\sigma$ with σ indicating weak topologies.

The space $\mathscr{D}'_{E'} = (\mathscr{D}_E)'_{\mathrm{b}}$ is the strong dual of the space $\mathscr{D}_E$, as defined in Equation (4.23), and $E' = (E)'_{\mathrm{b}}$ is the strong dual of E. Because E is a solid subspace of $\mathscr{M}$, the space $\mathscr{D}_E$ can be represented as the union

$$\mathscr{D}_E = \bigcup \left\{\mathscr{B}_{1/w}\,;\, w \in E^+\right\}. \tag{5.1a}$$

The topology of $\mathscr{D}_E$ is generated by the increasing fundamental system of seminorms $(p_n)_{n \in \mathbb{N}_0}$, given by

$$h \mapsto p_n(h) := \|\max\{|\partial^\alpha h|\,;\, \Sigma\,\alpha \le n\}\|_{E,n}, \tag{5.1b}$$

where $(\|\cdot\|_{E,n})_{n \in \mathbb{N}}$ denotes a fixed increasing fundamental system of seminorms for E and $\Sigma\,\alpha := \sum_{i=1}^d \alpha_i$. Here, continuity of binary supremum formation, see Proposition 4.5 from [23, p. 103] (see also [27, p. 234], [1, Thm. 2.17]), allows to write the maximum inside of the norm.

Let us summarize some facts about the dual space E'. The assumptions on E guarantee that E' is a (DF)-space that satisfies the continuous inclusions $L^\infty_{\mathfrak{K}} \subseteq E' \subseteq \mathscr{M}$, but $L^\infty_{\mathfrak{K}}$ is not necessarily dense in E'. According to Proposition 4.17 from [23, p. 108] (see also [27, p. 237], [1, p. 80]) E' is a locally convex lattice. Duality implies that $\mathscr{D}$ operates continuously and linearly on E. In particular, E' is a solid regularization-invariant space.

Every positive linear functional on E is continuous because E satisfies the assumptions of Proposition 2.16 from [23, p. 86] (see also [27, Thm. p. 228], [1, Thm. 5.23]). In particular, one has the equation

$$E' \cap \mathscr{I}_{+\mathrm{lb}} = (E^+)^* \tag{5.2}$$

with the notation

$$M^* := \left\{ w \in \mathscr{I}_{+\mathrm{lb}} \,;\, \forall \mu \in M : \int w(x)\,\mathrm{d}\mu(x) < \infty \right\} \qquad \text{for } M \subseteq \mathscr{M}^+. \tag{5.3}$$

The duality product $\langle -, - \rangle : E' \times E \to \mathbb{C}$ satisfies

$$\langle w, \mu \rangle = \| w \cdot \mu \|_1 = \int w(x)\,\mathrm{d}\mu(x) \qquad \text{for all } w \in E' \cap \mathscr{I}_{+\mathrm{lb}},\ \mu \in E^+. \tag{5.4}$$

According to Proposition 4.3 from [23, p. 102] (see also [27, p. 212], [1, p. 80]) the seminorms

$$g \mapsto q_B(g) = \sup\{\langle |g|, |\mu| \rangle \,;\, \mu \in B\} \qquad \text{with } B \in \mathfrak{B}(E) \tag{5.5}$$

constitute a fundamental system of seminorms for E'.

The set of weights $W(E)$ is defined as the largest moderated cone ideal contained in E^+, that is,

$$W(E) := (E^+)_{\mathrm{T}} = \left\{ w \in \mathscr{I}_{+\mathrm{lb}} \,;\, \forall K \in \mathfrak{K} : \overline{\mathrm{T}}_K w \in E^+ \right\} \tag{5.6}$$

with the notation

$$M_{\mathrm{T}} := \left\{ w \in \mathscr{I}_{+\mathrm{lb}} \,;\, \forall K \in \mathfrak{K}\ \exists \mu \in M : \overline{\mathrm{T}}_K w \le \mu \right\} \qquad \text{for } M \subseteq \mathscr{M}^+. \tag{5.7}$$

Before proving the main result, we establish some relations that involve Equations (5.3) and (5.7).

Lemma 5.1. *Let $M \subseteq \mathscr{M}^+$. The relation $(M^*)_{\mathrm{T}} = ((\mathscr{K}^+ * M)^*)_{\mathrm{T}}$ holds.*

Proof. Let $w \in \mathscr{I}_{+\mathrm{lb}}$, $\mu \in \mathscr{M}^+$ and $\phi \in \mathscr{K}^+$. Transposing the convolution operator $(\phi * -)$ yields

$$\int \overline{\mathrm{T}}_K w(x) \cdot (\phi * \mu)(x)\,\mathrm{d}x = \int \left(\check{\phi} * \overline{\mathrm{T}}_K w \right)(x)\,\mathrm{d}\mu(x). \tag{5.8}$$

The functions $\phi * \overline{\mathrm{T}}_K w$ with $w \in M$, $K \in \mathfrak{K}$, $\phi \in \mathscr{K}^+$ generate the same lower set as the functions $C \cdot \overline{\mathrm{T}}_K w$ with $w \in M$, $K \in \mathfrak{K}$, $C \in \mathbb{R}_+$ (compare Equations (3.4b) and (3.4c)). Together with Equation (5.8), this concludes the proof of the lemma. □

Lemma 5.2. *If $M \subseteq \mathscr{M}^+$, then $\mathscr{K}^+ * M \subseteq (\mathscr{K}^+ * M)_{\mathrm{T}}$.*

Proof. One has the inequality $\overline{\mathrm{T}}_K(\phi * \mu) \le (\overline{\mathrm{T}}_K \phi) * \mu$ and $\overline{\mathrm{T}}_K \phi \in \mathscr{K}^+$ for all $\phi \in \mathscr{K}^+$, $\mu \in \mathscr{M}^+$ and $K \in \mathfrak{K}$. □

Lemma 5.3. *Let $M \subseteq \mathscr{M}^+$ be such that $\mathscr{K}^+ * M \subseteq M$. Then, $(M^*)_{\mathrm{T}} = ((M_{\mathrm{T}})^*)_{\mathrm{T}}$.*

Proof. Clearly, any function from M_{T} is bounded by some measure from M and therefore $(M^*)_{\mathrm{T}} \subseteq ((M_{\mathrm{T}})^*)_{\mathrm{T}}$. Conversely,

$$(M^*)_{\mathrm{T}} = ((\mathscr{K}^+ * M)^*)_{\mathrm{T}} \supseteq (((\mathscr{K}^+ * M)_{\mathrm{T}})^*)_{\mathrm{T}} \supseteq (((M)_{\mathrm{T}})^*)_{\mathrm{T}} \tag{5.9}$$

follows from Lemma 5.1, Lemma 5.2, and the assumption on M. □

Lemma 5.4. *Any lower set $M \subseteq \mathscr{M}^+$ satisfies*

$$\{ f \in \mathscr{D}' \,;\, \forall \Phi \in \mathfrak{B}(\mathscr{D}) : |f|_\Phi \in M \} = \{ f \in \mathscr{D}' \,;\, \forall \Phi \in \mathfrak{B}(\mathscr{D}) : |f|_\Phi \in M_{\mathrm{T}} \}. \tag{5.10}$$

Proof. This follows from Equations (3.4a) and (3.5). □

Proposition 5.5. *The following sets of $\mathscr{I}_{+\mathrm{lb}}$-valued seminorms on $\mathscr{E}$ are equivalent (in the sense of Equation (3.1))*

$$\{h \mapsto \overline{\mathrm{T}}_K(\max\{|\partial^\alpha h|\,;\, \Sigma\alpha \le n\})\,;\, n \in \mathbb{N}_0,\, K \in \mathfrak{K}\}, \tag{5.11a}$$

$$\{h \mapsto 1_K * (\max\{|\partial^\alpha h|\,;\, \Sigma\alpha \le n\})\,;\, n \in \mathbb{N}_0,\, K \in \mathfrak{K}\}. \tag{5.11b}$$

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable. Integrating $f(x) = f(y) - \int_x^y f'(z)\,dz$ over $[x-1, x]$ gives

$$|f(x)| \le \int_{x-1}^{x} |f(y)| + \int_{x}^{y} |f'(z)|\,dz\,dy \le \int_{x-1}^{x} |f(y)| + \int_{x-1}^{x} |f'(z)|\,dz\,dy = \int_{x-1}^{x} |f(y)| + |f'(y)|\,dy \quad \text{for all } x \in \mathbb{R}. \tag{5.12}$$

For a function $h \in \mathscr{E}$, this estimate is now applied successively in each dimension, which gives

$$|\partial^\alpha h| \le \sum_{\beta\in\{0,1\}^d} 1_{[0,1]^d} * \left|\partial^{\alpha+\beta} h\right|, \qquad \text{for all } h \in \mathscr{E}. \tag{5.13}$$

This estimate and Equation (3.7) applied to $w = 1_K$ with $K \in \mathfrak{K}$ complete the proof. □

We are now ready to prove

Theorem 5.6. *Let E be a solid regularization-invariant space that satisfies $E \subseteq L^1_{\mathrm{loc}}$, has $\mathscr{K}$ as a dense subset and is a Fréchet space. Let $W(E)$ be the set from Equation (5.6). Then, the following identities of locally convex spaces hold:*

$$(\mathscr{D}_E)'_{\mathrm{b}} = \mathscr{O}'_C(\mathscr{D}, E'), \tag{5.14}$$

$$(\mathscr{D}_E)'_\sigma = \mathscr{O}'_C(\mathscr{D}, L^1_{W(E)})_\sigma. \tag{5.15}$$

Here, on the left-hand side of Equation (5.15), σ indicates the weak topology induced on the dual space and, on the right-hand side of Equation (5.15), σ indicates the weak topology associated with the given topology.

Proof. First, the spaces $(\mathscr{D}_E)'$, $\mathscr{O}'_C(\mathscr{D}, E')$ and $\mathscr{O}'_C(\mathscr{D}, L^1_{W(E)})$ are equal as linear spaces: The inclusion $(\mathscr{D}_E)' \supseteq \mathscr{O}'_C(\mathscr{D}, E')$ follows from Propositions 4.11 and 4.13 and $(\mathscr{D}_E)' \subseteq \mathscr{O}'_C(\mathscr{D}, E')$ follows from Proposition 4.11 and Corollary 3.21. We note that Theorem 3.11 and Lemma 5.4 imply

$$\mathscr{O}'_C(\mathscr{D}, E') = \{f \in \mathscr{D}'\,;\, \forall\Phi \in \mathfrak{B}(\mathscr{D}) : |f|_\Phi \in ((E')^+)_{\mathrm{T}}\}, \tag{5.16a}$$

$$\mathscr{O}'_C(\mathscr{D}, L^1_{W(E)}) = \{f \in \mathscr{D}'\,;\, \forall\Phi \in \mathfrak{B}(\mathscr{D}) : |f|_\Phi \in ((W(E))^*)_{\mathrm{T}}\}. \tag{5.16b}$$

Then, from (i) the inclusion $E' \cap \mathscr{I}_{+\mathrm{lb}} \subseteq (E')^+$ and the definition of M_{T}, (ii) equation (5.2), (iii) the inclusion $\mathscr{K}^+ * E^+ \subseteq E^+$ and Lemma 5.3, and, (iv) Equation (5.6), we derive

$$((E')^+)_{\mathrm{T}} \overset{\text{(i)}}{=} (E' \cap \mathscr{I}_{+\mathrm{lb}})_{\mathrm{T}} \overset{\text{(ii)}}{=} ((E^+)^*)_{\mathrm{T}} \overset{\text{(iii)}}{=} (((E^+)_{\mathrm{T}})^*)_{\mathrm{T}} \overset{\text{(iv)}}{=} ((W(E))^*)_{\mathrm{T}}. \tag{5.17}$$

Finally, Equations (5.16) and (5.17) yield $\mathscr{O}'_C(\mathscr{D}, E') = \mathscr{O}'_C(\mathscr{D}, L^1_{W(E)})$.

Second, the identity $(\mathscr{D}_E)'_\sigma = \mathscr{O}'_C(\mathscr{D}, L^1_{W(E)})_\sigma$ from Equation (5.15) holds: the dual spaces of both the spaces in this identity are given in Equations (5.1a) and (4.21) of Theorem 4.10, respectively. Proposition 5.5 in connection with solidity and regularization-invariance of E implies that the sets in these equations are equal.

Third, the subspace topologies $\mathscr{T}_1$ and $\mathscr{T}_2$ induced on $\mathscr{E}'$ by $\mathscr{D}'_{E'}$ and $\mathscr{O}'_C(\mathscr{D}, E')$, respectively, are equal: let $K \in \mathfrak{K}$ with non-empty interior. Proposition 5.5 and the regularization-invariance of E imply that $B \subseteq \mathscr{D}_E$ is bounded if and only if

$$\sup\left\{\left\|\overline{\mathrm{T}}_K(\max\{|\partial^\alpha b|\,;\, \Sigma\alpha \le n\})\right\|_{E,n}\,;\, b \in B\right\} := C_n < \infty \qquad \text{for all } n \in \mathbb{N}_0. \tag{5.18}$$

Assume now, that B is bounded. Let $\lambda_n > 0$, $n \in \mathbb{N}_0$ such that

$$\sum_{n\in\mathbb{N}_0} C_n \lambda_n < \infty \tag{5.19a}$$

and define

$$\max\{|\partial^\alpha b| ; \Sigma\alpha \le n\} := w_{b,n}, \qquad w_b := \sum_{n\in\mathbb{N}_0} \lambda_n w_{b,n} \qquad \text{for } b \in B. \tag{5.19b}$$

By construction it holds $w_{b,n} \le (1/\lambda_n) \cdot w_b$ for all $n \in \mathbb{N}_0$. This results in the inclusion

$$B \subseteq \bigcup \left\{ \mathrm{B}\big(w_b ; C'\big) ; b \in B \right\} \qquad \text{with } C'_\alpha := 1/\lambda_{\Sigma\alpha} \text{ for } \alpha \in \mathbb{N}_0^d, \tag{5.20}$$

where the notation from Equation (4.3) is used. It follows from Equations (5.18) and (5.19a) that $\tilde{B} := \{\overline{\mathrm{T}}_K w_b ; b \in B\}$ is a bounded subset of E. Using Lemma 4.8 and then Equations (5.5) and (5.4), one finds $\Phi \in \mathfrak{B}(\mathscr{D})$ such that

$$|\langle f, b\rangle| \le \int |f|_\Phi(x) \overline{\mathrm{T}}_K w_b(x)\, \mathrm{d}x = q_{\{\overline{\mathrm{T}}_K w_b\}}(|f|_\Phi) \qquad \text{for all } f \in \mathscr{E}',\ b \in B. \tag{5.21}$$

Taking the supremum over $b \in B$ proves $\mathscr{T}_1 \subseteq \mathscr{T}_2$.

The estimate $\phi * |f|_\Phi \le |f|_\Psi$ for $\Psi = \|\phi\|_1 \cdot \mathrm{T}_{\mathrm{supp}\,\phi}\Phi$, Equation (3.5), Proposition 3.6, and Theorem 3.11 applied to E' imply that the topology of $\mathscr{O}'_C(\mathscr{D}, E')$ is generated by the seminorms

$$f \mapsto q(\phi * |f|_\Phi) \qquad \text{with } \phi \in \mathscr{D}, \phi \ge 0, \Phi \in \mathfrak{B}(\mathscr{D}), q \in \operatorname{clsn} E'. \tag{5.22}$$

Therefore, for the proof of the converse inequalities, let $\phi \in \mathscr{D}$ with $\phi \ge 0$, $\Phi \in \mathfrak{B}(\mathscr{D})$ and $\tilde{B}'$ a bounded solid subset of E. Using Equations (5.4) and (5.5) and then Lemma 4.9 one finds $L \in \mathfrak{K}$ and $C'' \in \mathbb{R}_+(\mathbb{N}_0^d)$ such that

$$q_{\{|\mu|\}}(\phi * |f|_\Phi) = \||f|_\Phi(\check{\phi} * |\mu|)\|_1 \le \sup\left\{ |\langle f, b\rangle| ; b \in \mathrm{B}\Big(\overline{\mathrm{T}}_L(\check{\phi} * |\mu|) ; C''\Big)\right\} \qquad \text{for all } f \in \mathscr{E}', \mu \in \tilde{B}'. \tag{5.23}$$

The boundedness of $\tilde{B}'$ in E entails boundedness in $\mathscr{D}_E$ for the set

$$B' := \bigcup \left\{ \mathrm{B}\Big(\overline{\mathrm{T}}_L(\check{\phi} * |\mu|) ; C''\Big) ; \mu \in \tilde{B}' \right\}, \tag{5.24}$$

because $\overline{\mathrm{T}}_L(\check{\phi} * |\mu|) \le (\overline{\mathrm{T}}_L \check{\phi}) * |\mu|$ and because $L^\infty_{\mathfrak{K}}$ operates continuously and linearly on E by convolution according to Proposition 3.10. Thus, taking the supremum over $\mu \in \tilde{B}'$ in Equation (5.23) proves $\mathscr{T}_1 \supseteq \mathscr{T}_2$.

Fourth, the inequalities (5.21) and (5.23) hold also for $f \in \mathscr{O}'_C(\mathscr{D}, L^1_{W(E)}) = (\mathscr{D}_E)'$: Let (θ_n) be an approximate unit. Because $\mathscr{K}^+$ is dense in $L^1_{W(E)}$ Proposition 3.24 and the last statement of Theorem 4.10 imply that $\langle \theta_n f, h\rangle \to \langle f, h\rangle$ uniformly on $h \in \mathrm{B}(w ; C)$ with fixed $w \in W(E)$ and $C \in \mathbb{R}_+(\mathbb{N}_0^d)$. On the other hand, Proposition 3.7 and Lebesgue's theorem of dominated convergence can be applied to Equations (5.21) and (5.23). As a result, these inequalities extend to distributions $f \in \mathscr{O}'_C(\mathscr{D}, L^1_{W(E)})$. Again, taking suprema over extended estimates (5.21) and (5.23), as done in step three of the proof, it follows that the strong topology on $\mathscr{D}'_{E'}$ is equal to the topology of $\mathscr{O}'_C(\mathscr{D}, E')$. This completes the proof of the identity (5.14). □

Example 5.7. Recall, that $\mathscr{E}' = (\mathscr{E})'_{\mathrm{b}}$. The space $L^\infty_{\mathfrak{K}}$ is the strong dual of L^1_{loc}, the largest space E to which Theorem 5.6 applies. From Proposition 5.5, one derives $\mathscr{E} = \mathscr{D}_{L^1_{\mathrm{loc}}}$. Therefore, Theorem 5.6 yields the identity of locally convex spaces

$$\mathscr{E}' = \mathscr{O}'_C(\mathscr{D}, L^\infty_{\mathfrak{K}}). \tag{5.25}$$

By Proposition 3.10 and monotony of $\mathscr{O}'_C(\mathscr{D}, -)$, all solid regularization-invariant spaces E satisfy the continuous inclusion

$$\mathscr{E}' \subseteq \mathscr{O}'_C(\mathscr{D}, E). \tag{5.26}$$

Example 5.8. The space of tempered distributions $\mathscr{S}'$ is defined as the strong dual of the Schwartz-space $\mathscr{S}$, which is a Fréchet space. Let L^1_P denote the weighted L^1-space, in the sense of Section 4, with the moderated cone ideal P that is generated by the non-negative polynomials. Then, the identity $\mathscr{D}_{L^1,P} = \mathscr{S}$ is immediate from the definitions. The dual space of L^1_P is the weighted Lebesgue space $L^\infty_{P^*}$, in the sense of Section 4, where P^* consists of the rapidly decreasing functions from $\mathscr{I}_{+\mathrm{lb}}$. Theorem 5.6 implies the identity of locally convex spaces

$$\mathscr{S}' = \mathscr{O}'_C(\mathscr{D}, L^\infty_{P^*}). \tag{5.27}$$

Example 5.9. The assumptions imposed on E at the beginning of this section cover the distribution spaces $\mathscr{D}'_{E'}$ associated with solid translation-invariant Banach spaces of distributions from [9]. Special cases of these are the Lebesgue spaces L^p with $1 < p \le \infty$. Thus, the case $p = 1$ being covered by Theorem 5.1 from [11], Theorem 5.6 completes the identity of locally convex spaces

$$\mathscr{D}'_{L^p} = \mathscr{O}'_C(\mathscr{D}, L^p) \qquad \text{for all } 1 \le p \le \infty. \tag{5.28a}$$

Let L^∞_0 denote the closure of $L^\infty_{\mathfrak{K}}$ in L^∞. Then, $L^\infty_0 \cap \mathscr{C} = \mathscr{C}_0$ and Proposition 3.24 yields

$$\dot{\mathscr{B}}' = \mathscr{O}'_C(\mathscr{D}, L^\infty_0) = \mathscr{O}'_C(\mathscr{D}, \mathscr{C}_0). \tag{5.28b}$$

Example 5.10. Let w be a moderated weight and $1 \le p \le \infty$. Then, L^p_w is a solid regularization-invariant space by Proposition 4.1. As in Example 5.9, Theorem 5.6 completes the identity of locally convex spaces

$$\mathscr{D}'_{L^p,w} = \mathscr{O}'_C(\mathscr{D}, L^p_w). \tag{5.29}$$

Further, the space $\mathscr{D}'_{L^p,w}$ is isomorphic to $h \cdot \mathscr{D}'_{L^p}$ via the mapping $f \mapsto h \cdot f$ for any fixed $h \in \mathscr{B}_{1/w} \cap (1/\mathscr{B}_w)$, according to Proposition 4.6 and Example 5.9. As a consequence, $\mathscr{D}'_{L^p,w}$ inherits the ultrabornologicity of $\mathscr{D}'_{L^p}$, which is proved in [19, p. 592] for instance. This extends the statements (i) and (iii) of Theorem 5.1 from [11] to $p \ne 1$. According to Theorem 3.11, the topology on $\mathscr{D}'_{L^p,w}$ is generated by the set of seminorms $f \mapsto \| |f|_\Phi \cdot w \|_p$ with $\Phi \in \mathfrak{B}(\mathscr{D})$.

An important special case are the spaces $\mathscr{D}'_{L^p,\mu,k} := \mathscr{D}'_{L^p,w^{\mu;k}}$ with $1 \le p \le \infty$, $\mu \in \mathbb{R}$ and $k \in \mathbb{Z}$ that arise as maximal domains for distributional convolution operators with quasi-homogeneous kernels [19, 21, 34]. The corresponding weighted Lebesgue spaces are denoted by $L^p_{\mu,k} := L^p_{w^{\mu;k}}$. Here, $w^{\mu;k}$ denotes the power-logarithmic weight

$$w^{\mu;k}(x) := (1 + |x|^2)^{\mu/2} (\log(\mathrm{e} + |x|^2))^k \qquad \text{for all } x \in \mathbb{R}^d, \mu \in \mathbb{R}, k \in \mathbb{Z}, \tag{5.30}$$

and with the power weights as special case $w^\mu := w^{\mu;0}$. The weights $w^{\mu;k}$ are moderated and satisfy $w^{\mu;k} \in \mathscr{B}_{1/w^{\mu;k}} \cap (1/\mathscr{B}_{w^{\mu;k}})$. Note that these space are defined via multiplication $f \mapsto w^{\mu;k} \cdot f$ in [19, p. 582] and [21, Def. 3.1.1]. Convolution on these spaces was studied in [19, 21, 34] and will be reconsidered in this work in Examples 6.6 and 6.7 of Section 6.

5.2 | Lebesgue spaces with strict topologies

In the following, let $\mathscr{D}'_{L^p,\mathrm{c}}$ denote the space $\mathscr{D}'_{L^p}$ endowed with the topology of uniform convergence on the compact subsets of $\mathscr{D}_{L^q}$ for $p > 1$ where $1/p + 1/q = 1$ and of $\dot{\mathscr{B}}$ for $p = 1$ [3]. The locally convex space $(L^p)_{\mathrm{str}}$ is the linear space L^p endowed with the topology induced by the seminorms $g \mapsto \|w \cdot g\|_p$ with $w \in \mathscr{C}^+_0$. The locally convex space $(L^p)_{\mathrm{str}}$ is equal to the weighted space $L^p_{W_0}$, as defined in Equation (4.1), where W_0 is the moderated cone ideal that consists of the lower semicontinuous functions vanishing at infinity. According to Proposition 4.1, $(L^p)_{\mathrm{str}}$ is a solid regularization-invariant space.

Theorem 5.11. *The following identity of locally convex spaces holds:*

$$\mathscr{D}'_{L^p,\mathrm{c}} = \mathscr{O}'_C(\mathscr{D}, (L^p)_{\mathrm{str}}) \qquad \textit{for all } 1 \le p \le \infty. \tag{5.31}$$

Proof. We already know from Example 5.9 that Equation (5.31) holds as an identity of linear spaces. Set $q := 1/(1 - 1/p)$ and let $K \in \mathfrak{K}$ be a fixed neighborhood of zero. If $p = 1$ let H be a bounded subset of $\dot{\mathscr{B}}$, otherwise let H be a bounded subset of $\mathscr{D}_{L^q}$.

For $p = 1$, we will use the relation $\mathcal{O}'_C(\mathcal{D}, L^1) = \mathcal{O}'_C(\mathcal{D}, (\mathcal{C}_0)')$: recall that a subset $F \subseteq \mathcal{C}_0$ is relatively compact if and only if $\sup\{|f|\,;\, f \in F\}$ is vanishing at infinity. Because $\overline{\mathrm{T}}_K : \mathcal{I}_+ \to \mathcal{I}_+$ is increasing and $\overline{\mathrm{T}}_K |\mathcal{C}_0| \subseteq \mathcal{C}_0$ it follows that $H \subseteq \dot{\mathcal{B}}$ is relatively compact if and only if there exist $g_\alpha \in \mathcal{C}_0$ such that

$$\overline{\mathrm{T}}_K |\partial^\alpha h| \leq g_\alpha \qquad \text{for all } h \in H,\ \alpha \in \mathbb{N}_0^d. \tag{5.32}$$

In the case $p > 1$, where $q < \infty$, Propositions 3.10 and 5.5 yield that the set $\{\overline{\mathrm{T}}_K |\partial^\alpha h|\ ;\ h \in H\}$ is bounded in L^q for all $\alpha \in \mathbb{N}_0^d$. Due to the inequality

$$|\mathrm{T}_x f - f| \leq r \cdot \overline{\mathrm{T}}_{B_r} |\nabla f|_2 \qquad \text{for all } f \in \mathcal{C},\ x \in B_r,\ r > 0, \tag{5.33}$$

where $B_r := \{y \in \mathbb{R}^d\,;\, |y|_2 \leq r\}$ and $|-|_2$ is the Euclidean norm on $\mathbb{R}^d$, this implies that the conditions (i) and (iii) of the criterion for relative compactness in L^q, $q < \infty$ from [29, Thm. 6.4.12, p. 140] hold. Therefore, the remaining condition (ii) of [29, Thm. 6.4.12] implies that H is relatively compact if and only if there exist $g_\alpha \in \mathcal{C}_0$ and $C_\alpha \in \mathbb{R}_+$ such that

$$\sup_{h \in H} \left\| \left(\overline{\mathrm{T}}_K |\partial^\alpha h|\right) / g_\alpha \right\|_q \leq C_\alpha \qquad \text{for all } \alpha \in \mathbb{N}_0^d. \tag{5.34}$$

Given $g_\alpha \in \mathcal{C}_0, \alpha \in \mathbb{N}_0^d$ one finds $g \in \mathcal{C}_0$ and $C'_\alpha \in \mathbb{R}_+$ such that $g_\alpha \leq C'_\alpha \cdot g$ for all $\alpha \in \mathbb{N}_0^d$. Then, for general p, it follows from Equations (5.32) and (5.34) that H is relatively compact if and only if there exist $g \in \mathcal{C}_0$ and $C''_\alpha \in \mathbb{R}_+$ such that

$$\sup_{h \in H} \left\| \left(\overline{\mathrm{T}}_K |\partial^\alpha h|\right) / g \right\|_q \leq C''_\alpha \qquad \text{for all } \alpha \in \mathbb{N}_0^d. \tag{5.35}$$

With this result, we can proceed analogously to the proof of Theorem 5.11 and show that for all $H \subseteq \mathcal{D}_{L^q}$ ($H \subseteq \dot{\mathcal{B}}$) relatively compact, $g \in \mathcal{C}_0^+$ and $\phi \in \mathcal{D}$ there exist $g' \in \mathcal{C}_0^+$ and $H' \subseteq \mathcal{D}_{L^q}$ ($H' \subseteq \dot{\mathcal{B}}$) relatively compact such that the inequalities

$$\sup\{|\langle f, h\rangle|\,;\, h \in H\} \leq \left\| |f|_\Phi \cdot g' \right\|_p \tag{5.36a}$$

$$\|(\phi * |f|_\Phi) \cdot g\|_p \leq \sup\left\{|\langle f, h\rangle|\,;\, h \in H'\right\} \tag{5.36b}$$

hold for all $f \in \mathcal{O}'_C(\mathcal{D}, (L^p)_{\mathrm{str}})$. This concludes the proof. □

The following corollary of Theorems 5.11 and 3.11 contains the dual result to Propositions 2.5 and 3.2 from [3]:

Corollary 5.12. *The topology of $\mathcal{D}'_{L^p,c}$, $1 \leq p \leq \infty$ is induced by the seminorms*

$$f \mapsto \|g \cdot |f|_\Phi\|_p \qquad \textit{with } \Phi \in \mathfrak{B}(\mathcal{D}),\ g \in \mathcal{C}_0^+, \tag{5.37}$$

or, equivalently, by the seminorms

$$f \mapsto \|g \cdot (\phi * f)\|_p \qquad \textit{with } \phi \in \mathcal{D},\ g \in \mathcal{C}_0^+. \tag{5.38}$$

6 | CONVOLUTION ON DISTRIBUTION SPACES AND REGULARIZATION

In this section, we study convolution on distribution spaces of the form $\mathcal{O}'_C(\mathcal{D}, E)$. First, we will characterize convolvability and describe a useful mapping property of convolution via generalized absolute values in Theorems 6.2 and 6.3. Combining this with Theorem 3.11 one obtains Theorem 6.4, which allows to transport several continuity properties of bilinear convolution mappings $* : E \times F \to G$ with solid regularization-invariant spaces E, F, G to the associated convolution mappings $* : \mathcal{O}'_C(\mathcal{D}, E) \times \mathcal{O}'_C(\mathcal{D}, F) \to \mathcal{O}'_C(\mathcal{D}, G)$.

Remark 6.1. Convolution products of non-negative lower semicontinuous functions $w, v \in \mathcal{I}_+$ can be defined point-wise via upper integrals [7]. Because $\mathcal{I}_+$ is closed under supremum formation it follows $w * v \in \mathcal{I}_+$. The following inequality holds:

$$\overline{\mathrm{T}}_{K+L}(w * v) \leq \overline{\mathrm{T}}_K w * \overline{\mathrm{T}}_L v \qquad \text{for all } w, v \in \mathcal{I}_+,\ K, L \in \mathfrak{K}. \tag{6.1}$$

In the following theorem, we will consider Condition (a) as the definition of convolvability for two distributions f, g on $\mathbb{R}^d$. For definitions and fundamental properties of convolution of distributions, we refer to [21, 22, 30, 32].

Theorem 6.2. *Let $f, g \in \mathscr{D}'$. The following conditions are equivalent:*

(a) The distribution $\theta^\Delta \cdot (f \otimes g)$ belongs to the space $\mathscr{D}'_{L^1}(\mathbb{R}^{2d})$ for all $\theta \in \mathscr{D}$ where $\theta^\Delta(x, y) := \theta(x + y)$.
(b) The function $|f \otimes g|_\Phi$ is integrable over K^Δ for all $K \in \mathfrak{K}$ and $\Phi \in \mathfrak{B}(\mathscr{D}(\mathbb{R}^{2d}))$ where $K^\Delta := \{(x, y) \in \mathbb{R}^{2d}\,;\, x + y \in K\}$.
(c) The $\overline{\mathbb{R}}_+$-valued function $|f|_\Phi * |g|_\Phi$ is locally integrable for all $\Phi \in \mathfrak{B}(\mathscr{D})$.
(d) The $\overline{\mathbb{R}}_+$-valued function $|f|_\Phi * |g|_\Phi$ is locally bounded for all $\Phi \in \mathfrak{B}(\mathscr{D})$.
(e) The $\overline{\mathbb{R}}_+$-valued function $|f|_\phi * |g|_\psi$ is finite-valued for all $\phi, \psi \in \mathscr{D}$.
(f) The tuple $(\phi * f, \psi * g)$ is convolvable for all $\phi, \psi \in \mathscr{D}$.

Proof. We prove "(a) ⇒ (b) ⇒ (c) ⇒ (d)". The remaining implications are either trivial or proved in [30]. According to Example 5.9, Condition (a) is equivalent to

$$\forall \theta \in \mathscr{D}, \Phi \in \mathfrak{B}(\mathscr{D}(\mathbb{R}^{2d})) \quad : \quad \left|\theta^\Delta \cdot (f \otimes g)\right|_\Phi \in L^1(\mathbb{R}^{2d}). \tag{6.2}$$

Using Equation (3.6), one concludes that Equation (6.2) is equivalent to

$$\forall \theta \in \mathscr{D}, \Phi \in \mathfrak{B}(\mathscr{D}(\mathbb{R}^{2d})) \quad : \quad \left|\theta^\Delta\right| \cdot |f \otimes g|_\Phi \in L^1(\mathbb{R}^{2d}). \tag{6.3}$$

Clearly, Equation (6.3) is equivalent to Condition (b). According to [31, Theorem 51.7], Equation (6.3) is equivalent to

$$\forall \theta \in \mathscr{D}, \Psi \in \mathfrak{B}(\mathscr{D}) \quad : \quad \left|\theta^\Delta\right| \cdot |f \otimes g|_{\Psi \otimes \Psi} \in L^1(\mathbb{R}^{2d}). \tag{6.4}$$

Using $|f \otimes g|_{\Psi\otimes\Psi} = |f|_\Psi \otimes |g|_\Psi$, it is seen that Equation (6.4) is equivalent to

$$\forall \theta \in \mathscr{D}, \Psi \in \mathfrak{B}(\mathscr{D}) \quad : \quad \int |\theta(x + y)||f|_\Psi(x)|g|_\Psi(y)\, d(x, y) < \infty. \tag{6.5}$$

The integral in Equation (6.5) is rewritten as

$$\int |\theta(x + y)||f|_\Psi(x)|g|_\Psi(y)\, d(x, y) = \int |\theta(x)|(|f|_\Psi * |g|_\Psi)(x)\, dx \tag{6.6}$$

and therefore Equation (6.5) is equivalent to $|f|_\Psi * |g|_\Psi \in L^1_{\mathrm{loc}}$ for all $\Psi \in \mathfrak{B}(\mathscr{D})$, which is Condition (c). In connection with Equations (3.4a) and (6.1), Condition (c) implies $|f|_\Psi * |g|_\Psi \in L^\infty_{\mathrm{loc}}$ for all $\Psi \in \mathfrak{B}(\mathscr{D})$, which is Condition (d). □

Theorem 6.3. *For all $\Phi \in \mathfrak{B}(\mathscr{D})$, there exists $\Psi \in \mathfrak{B}(\mathscr{D})$ such that*

$$|f * g|_\Phi \le |f|_\Psi * |g|_\Psi \qquad \text{for all convolvable } f, g \in \mathscr{D}'. \tag{6.7}$$

Proof. Let $\Phi \in \mathfrak{B}(\mathscr{D})$ and choose $\Psi \in \mathfrak{B}(\mathscr{D})$ such that $\Phi \subseteq \operatorname{acx}(\Psi * \Psi)$, which is possible by Corollary 3.3. Using the associative law [24, Prop. 1], one obtains

$$|f * g|_\Phi \le |f * g|_{\Psi*\Psi} = \sup_{\psi_1,\psi_2\in\Psi} |(\psi_1 * f) * (\psi_2 * g)| \le |f|_\Psi * |g|_\Psi$$

for all convolvable pairs of distributions f and g. □

In the following theorem, a bilinear mapping is called compactly respectively boundedly hypocontinuous if it is $(\mathfrak{E}, \mathfrak{F})$-hypocontinuous, in the sense of [16, p. 358], for $(\mathfrak{E}, \mathfrak{F}) = (\mathfrak{K}(E), \mathfrak{K}(F))$ respectively $(\mathfrak{E}, \mathfrak{F}) = (\mathfrak{B}(E), \mathfrak{B}(F))$.

Theorem 6.4. *Let E, F, G be solid regularization-invariant spaces. If convolution of measures $* : E \times F \to G$ has the property (P), where (P) stands for either "well-defined", "separately continuous", "compactly hypocontinuous", "boundedly hypocontinuous" or "continuous". Then, convolution of distributions $* : \mathscr{O}'_C(\mathscr{D}, E) \times \mathscr{O}'_C(\mathscr{D}, F) \to \mathscr{O}'_C(\mathscr{D}, G)$ has the property (P) as well.*

Proof. Let $(f,g) \in \mathscr{O}'_C(\mathscr{D},E) \times \mathscr{O}'_C(\mathscr{D},F)$. If $E * F$ is well-defined and $E * F \subseteq G$, then $|f|_\Phi * |g|_\Phi \in G$ for all $\Phi \in \mathfrak{B}(\mathscr{D})$. In particular, $|f|_\Phi * |g|_\Phi \in L^1_{\mathrm{loc}}$. Theorem 6.2 implies that f and g are convolvable and Theorem 6.3 implies $|f * g|_\Phi \in G$ for all $\Phi \in \mathfrak{B}(\mathscr{D})$. Further, it follows that for all $\Phi \in \mathfrak{B}(\mathscr{D})$ there exists $\Psi \in \mathfrak{B}(\mathscr{D})$ such that

$$p(|f * g|_\Phi) \le p(|f|_\Psi * |g|_\Psi) \qquad \text{for all } f \in E,\ g \in F,\ p \in \operatorname{clsn} G. \tag{6.8}$$

The theorem is now immediate from this inequality, Theorem 3.11, and Corollary 3.18. □

Corollary 6.5. *Convolution of distributions defines a separately continuous bilinear mapping* $* : \mathscr{E}' \times \mathscr{O}'_C(\mathscr{D},E) \to \mathscr{O}'_C(\mathscr{D},E)$ *for any solid regularization-invariant space* E.

Proof. This is immediate from Theorem 6.4, Example 5.7, and Proposition 3.10. □

Example 6.6. Continuity of convolution of distributions between the power weighted spaces $\mathscr{D}'_{L^p,\mu}$ from Example 5.10 was studied in [19, 21, 34]. Proposition 2.5 from [34] states that $* : \mathscr{D}'_{L^p,\mu} \times \mathscr{D}'_{L^q,\nu} \to \mathscr{D}'_{L^p,\mu}$ is continuous if

$$\nu > \max\left\{ \frac{|\mu|}{q} + \frac{d}{q'},\ \mu + \frac{d}{pq'},\ -\mu + \frac{d}{p'q'} \right\}, \tag{6.9}$$

where $\mu,\nu \in \mathbb{R}$, $1 \le p, p', q, p' \le \infty$ with $1/p + 1/p' = 1 = 1/q + 1/q'$ and d denotes the dimension.

Examining the proof of Proposition 2.5 from [34], which uses the proof of Proposition 2.1 in [34] and the proof of Proposition 9 in [19], we notice that the following two claims are established therein along the way:

(1) The convolution $f * g$ exists for all $(f,g) \in \mathscr{D}'_{L^p,\mu} \times \mathscr{D}'_{L^q,\nu}$.
(2) The mapping $* : L^p_\mu \times L^q_\nu \to L^p_\mu$ is well-defined, bilinear and continuous.

In [34], well-definedness and continuity of $* : \mathscr{D}'_{L^p,\mu} \times \mathscr{D}'_{L^q,\nu} \to \mathscr{D}'_{L^p,\mu}$ are then derived from these two claims.

The proof of claim (1), given in Part (a) of the proof of Proposition 2.1, is based on the convolvability condition (φ-CPS) [20, p. 315] and the multiplication relation $\mathscr{D}_{L^p,\mu} \cdot \mathscr{D}'_{L^q,\nu} \subseteq \mathscr{D}'_{L^1,\mu+\nu} \subseteq \mathscr{D}'_{L^1}$ [19, Prop. 9(i)]. The proof of claim (2) is contained in Parts (a) and (b) of the proof of Proposition 2.5 in [34], while referring to Parts (b)–(e) of the proof of Proposition 2.1 in [34]. The proof is based on interpolation theorems from [4]. Only the proof for continuity in one variable of this mapping is explicitly described in [34]. However, continuity follows when taking into account the estimates for the norms of the functions F_p [34, Equation (2.3)] and F_q [34, p. 476], that are provided by Theorem 1.1.1 and Corollary 5.5.4 from [4].

After establishing claims (1) and (2), it is proceeded as follows: from a representation formula of the form (4.25) and claims (1) and (2) it is concluded that $* : \mathscr{D}'_{L^p,\mu} \times \mathscr{D}'_{L^q,\nu} \to \mathscr{D}'_{L^p,\mu}$ is a well-defined bilinear mapping. The separate continuity of this mapping is then established using the closed graph theorem and the reasoning from [19, p. 592]. This reasoning is based on the hypocontinuity of multiplication $\cdot : \mathscr{D}_{L^p,\mu} \times \mathscr{D}'_{L^q,\nu} \to \mathscr{D}'_{L^r,\mu+\nu}$ for $1/p + 1/q \ge 1/r$ [19, Prop. 9(i)] and on the fact that $\mathscr{D}'_{L^p,\mu}$ is ultrabornological. Finally, Proposition 1.4.3 from [21] is applied, according to which separately continuous bilinear mappings defined on topological products of two barreled (DF)-spaces are continuous.

Using Theorem 6.4 and the description of the spaces $\mathscr{D}'_{L^p,\mu}$ from Example 5.10, the proof from [34] can be simplified drastically. The continuity of $* : \mathscr{D}'_{L^p,\mu} \times \mathscr{D}'_{L^q,\nu} \to \mathscr{D}'_{L^p,\mu}$ is just a consequence of Theorem 6.4, the identity (5.29) from Example 5.10 and the continuity of $* : L^p_\mu \times L^q_\nu \to L^p_\mu$, given in claim (2) above. In this way, it becomes superfluous to prove claim (1) above separately, because this is a consequence of Theorem 6.4. Also, it is not necessary to apply the closed graph theorem and [21, Prop. 1.4.3], which requires to know that $\mathscr{D}'_{L^p,\mu}$ is an ultrabornological (DF)-space. Also, the representation formula (4.25) is not needed.

Example 6.7. Let $1 \le p,q,r,t \le \infty$ such that $1/p + 1/q = 1/r + 1/t$ and $1 \le t \le \min\{p,q,r\} \le \infty$. Let p', q', r', t' denote the corresponding conjugate Hölder exponents. Let $w, v, u \in \mathscr{C}^+$ be moderated weights and $C \in \mathbb{R}_+$ a constant such that

$$w^{-t'} * v^{-t'} \le C u^{-t'} \quad \text{if } t' < \infty, \qquad u(x) \le C w(y) v(x-y) \text{ for all } x, y \in \mathbb{R}^d \quad \text{if } t' = \infty. \tag{6.10a}$$

Using a modified convolution $*_{t'}$ the condition (6.10a) can be written in a unified form (by reformulating Equation (4) in [6])

$$\frac{1}{w} *_{t'} \frac{1}{v} \leq C\frac{1}{u} \qquad \text{with } (f *_{t'} g)(x) := \|f \cdot \mathrm{T}_x \check{g}\|_{t'} \text{ for } f, g \in \mathscr{I}_+, x \in \mathbb{R}^d. \tag{6.10b}$$

According to Proposition 2.2 from [6] Equation (6.10b) is a sufficient condition for convolution of measures being a well-defined continuous bilinear mapping $* : L^p_w \times L^q_v \to L^r_u$. Theorem 6.4 and Example 5.10 entail then that convolution of distributions defines a continuous bilinear mapping $* : \mathscr{D}'_{L^p,w} \times \mathscr{D}'_{L^q,v} \to \mathscr{D}'_{L^r,u}$.

In the case of power weights, as in Example 6.6, the condition (6.10b) yields Proposition 3.15 from [6]. From this proposition, applied with $\phi = 1$ and $t > 1$, one obtains the three sufficient conditions

$$\max\{\mu, \nu\} < d/t', \qquad \rho \leq \mu + \nu - d/t', \quad \mu + \nu > d/t \tag{6.11a}$$

$$\max\{\mu, \nu\} = d/t', \qquad \rho < \min\{\mu, \nu\}, \quad \mu + \nu > d/t \tag{6.11b}$$

$$\max\{\mu, \nu\} > d/t', \qquad \rho \leq \min\{\mu, \nu\}, \quad \mu + \nu > d/t \tag{6.11c}$$

for the continuity of $* : L^p_\mu \times L^q_\nu \to L^r_\rho$ with the power weighted Lebesgue spaces L^p_μ from Examples 5.10 and 6.6.

In order to compare these conditions to Example 6.6, assume $\rho = \mu$ and $p = r$. This results in the sufficient conditions

$$1 < q \leq p, \quad \nu > d/q', \quad \nu \geq \mu \tag{6.12}$$

for the continuity of $* : L^p_\mu \times L^q_\nu \to L^p_\mu$. Comparing to Example 6.6, note, that $\max\{d/q', \mu\}$ is always strictly less than the right-hand side of Equation (6.9). On the other hand, the criterion (6.9) can always be satisfied by choosing ν large enough, for any given $\mu \in \mathbb{R}$ and $1 \leq p, q \leq \infty$.

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REFERENCES

[1] C. Aliprantis and O. Burkinshaw, *Locally solid Riesz spaces with applications to economics*, American Mathematical Society, Providence, RI, 2003.
[2] C. Bargetz, E. Nigsch, and N. Ortner, *Convolvability and regularization of distributions*, Ann. Mat. Pura Appl. **196** (2017), 2239–2251.
[3] C. Bargetz, E. Nigsch, and N. Ortner, *A simpler description of the κ-topologies on the spaces* $\mathscr{D}_{L^p}$, L^p, $\mathscr{M}^1$, Math. Nachr. **293** (2020), 1691–1706.
[4] J. Bergh and J. Löfström, *Interpolation spaces*, Springer, Berlin, 1976.
[5] G. Birkhoff, *Lattice theory*, 3rd ed., American Mathematical Society, Providence, RI, 1973.
[6] A. Biswas and D. Swanson, *Navier–Stokes equations and weighted convolution inequalities in groups*, Commun. Part. Diff. Equ. **35** (2010), 559–589.
[7] N. Bourbaki, *Elements of mathematics: integration I*, Springer, Berlin, 2004.
[8] B. Davey and H. Priestley, *Introduction to lattices and order*, 2nd ed., Cambridge University Press, Cambridge, 2002.
[9] A. Debrouwere, *Quasinormable* C_0*-groups and translation-invariant Fréchet spaces of type* $\mathcal{D}_E$, Results Math. **74** (2019), 1–17.
[10] A. Debrouwere, B. Prangoski, and J. Vindas, *Factorization in Denjoy–Carleman classes associated to representations of* $(\mathbb{R}^d, +)$, J. Funct. Anal. **280** (2021), 108831.
[11] A. Debrouwere and J. Vindas, *Topological properties of convolutor spaces via the short-time Fourier transform*, Trans. AMS **374** (2021), 829–861.
[12] P. Dimovski, S. Pilipovic, and J. Vindas, *New distribution spaces associated to translation-invariant Banach spaces*, Monatsh. Math. **177** (2015), 495–515.
[13] J. Dixmier and P. Malliavin, *Factorisation de Fonctions et de Vecteurs Indefiniment Differentiables*, Bull. Sci. Math. **102** (1978), 305–330.
[14] A. Grothendieck, *Produits tensoriels topologiques et espaces nucleaires*, American Mathematical Society, Providence, RI, 1966.
[15] A. Grothendieck, *Topological Vector Spaces*, Gordon and Breach, New York, 1973.
[16] J. Horvath, *Topological vector spaces and distributions*, Addison-Wesley, Reading, MA, 1966.
[17] T. Kleiner and R. Hilfer, *Convolution operators on weighted spaces of continuous functions and supremal convolution*, Ann. Mat. Pura Appl. **199** (2020), 1547–1569. https://link.springer.com/article/10.1007/s10231-019-00931-z.
[18] S. Kostadinova, S. Pilipović, K. Saneva, and J. Vindas, *The short-time Fourier transform of distributions of exponential type and Tauberian theorems for shift-asymptotics*, Filomat **30** (2016), 3047–3061.
[19] N. Ortner and P. Wagner, *Applications of* $\mathcal{D}'_{L^p}$*-spaces to the convolution of distributions*, Bull. Pol. Acad. Sci. Math. **37** (1989), 579–595.
[20] N. Ortner, *On convolvability conditions for distributions*, Monatsh. Math. **160** (2010), 313–335.

[21] N. Ortner and P. Wagner, *Distribution-valued analytic functions—theory and applications*, tredition GmbH, Hamburg, 2013.
[22] N. Ortner and P. Wagner, *Fundamental solutions of linear partial differential operators*, Springer, Cham, 2015.
[23] A. Peressini, *Ordered topological vector spaces*, Harper & Row, New York, 1967.
[24] B. Roider, *Sur la Convolution des Distributions*, Bull. Sci. Math. **100** (1976), 193–199.
[25] L. Rubel, W. Squires, and B. Taylor, *Irreducibility of certain entire functions with applications to harmonic analysis*, Ann. Math. **108** (1978), 553–567.
[26] W. Rudin, *Functional analysis*, McGraw-Hill, New York, 1973.
[27] H. Schaefer and M. Wolf, *Topological vector spaces*, Springer, Berlin, 1999.
[28] L. Schwartz, *Theorie des distributions*, Hermann, Paris, 1966.
[29] L. Schwartz, *Analyse IV: applications à la théorie de la mesure*, Hermann, Paris, 1993.
[30] R. Shiraishi, *On the definition of convolutions for distributions*, J. Sci. Hiroshima Univ. A **23** (1959), 19–32.
[31] F. Treves, *Topological vector spaces, distributions and kernels*, Academic Press, New York, 1967.
[32] V. Vladimirov, *Methods of the theory of generalized functions*, Taylor and Francis, London, 2002.
[33] J. Voigt, *Factorization in some Fréchet algebras of differentiable functions*, Studia Math. **LXVII** (1984), 333–348.
[34] P. Wagner, *On convolution in weighted* $\mathcal{D}'_{L^p}$*-spaces*, Math. Nachr. **287** (2014), 472–477.
[35] A. Wilansky, *Modern methods in topological vector spaces*, McGraw Hill, New York, 1978.

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