

Dimensional analysis and upscaling of two-phase flow in porous media with piecewise constant heterogeneities

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Abstract

Dimensional analysis of the traditional equations of motion for two-phase flow in porous media allows to quantify the influence of heterogeneities. The heterogeneities are represented by position dependent capillary entry pressures and position dependent permeabilities. Dimensionless groups quantifying the influence of random heterogeneities are identified. For the case of heterogeneities with piecewise constant constitutive parameters (e.g., permeabilities, capillary pressures) we find that the upscaling ratio defined as the ratio of system size and the scale at which the constitutive parameters are known has to be smaller than the fluctuation strength of the heterogeneities defined, e.g., as the ratio of the standard deviation to the mean value of a fluctuating quantity.

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1. Introduction and motivation

An important problem for two phase flow in porous media is the influence of heterogeneities and fluctuations of permeability, porosity and capillary pressure curves on fluid flow at different length scales [1,2]. Displacing a fluid with another immiscible fluid in a porous medium produces bypass and trapping effects that are sensitive to spatial heterogeneities of the medium on all scales.

Most workers in the field are familiar with the importance of heterogeneities for immiscible fluid flow in porous media but surprisingly few results seem to exist that allow to quantify the scale dependence of heterogeneities for the purposes of upscaling. A possible reason

might be that immiscible fluid flow and upscaling are only poorly understood even for homogeneous media [2,3]. In this paper we introduce heterogeneities into the dimensional analysis for the macroscopic two phase flow equations. Our main motivation is to obtain simple dimensionless numbers quantifying the importance of heterogeneities. Random heterogeneities may be characterized by a correlation length, the typical or most probable value and the magnitude of the fluctuations. Extending the dimensional analysis to the case of heterogeneities requires not only to characterize the heterogeneities. More importantly it becomes necessary to introduce the constitutive scale ℓ as the length scale at which the constitutive parameters are known or measured.

Dimensionless groups can be used for upscaling considerations. Estimating the constitutive parameters or solution behaviour when increasing the constitutive scale is what we shall call “upscaling on the basis of

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Nomenclature

List of symbols

\mathbf{x}	position vector in \mathbb{R}^3	$\tilde{X}(\ell)$	deviation of locally fluctuating quantity $X(\mathbf{x}; \ell)$ at scale ℓ from its typical value $\bar{X}(\ell)$ where $X = \phi, k, P_e$
\mathbf{g}	acceleration of gravity	$\sigma_X(\ell)$	standard deviation of locally fluctuating quantity $X(\mathbf{x}; \ell)$ at scale ℓ from its typical value $\bar{X}(\ell)$ where $X = \phi, k, P_e$
\mathbf{v}	velocity	R	upscaling ratio
\mathbb{P}	pore space	u	relative uncertainty
\mathbb{M}	matrix space	S	saturation
\mathbb{S}	porous sample	S_{Wi}	irreducible wetting fluid (water) saturation
\mathbb{W}	water, wetting fluid	S_{Or}	residual nonwetting (oil) saturation
\mathbb{O}	oil, nonwetting fluid	P	pressure
$\mathbb{K}(\mathbf{x}, \ell)$	measurement cell of size ℓ centered at \mathbf{x}	\hat{P}	dimensionless pressure
$\mathbb{B}(\mathbf{x}, d)$	ball of diameter d centered at \mathbf{x}	P_c	capillary pressure
ℓ	constitutive scale, i.e. scale at which the constitutive parameters are given or known from experiment and Eqs. (10)–(12) hold	P_e	entry pressure
ℓ_p	average pore size	\hat{P}_e	dimensionless entry pressure
L	system size	A	dimensionless form of quantity A
ξ	correlation length	μ	viscosity of fluid
$ \mathbb{S} $	(threedimensional) volume of the set \mathbb{S}	ρ	density of fluid
ϕ	volume fraction, porosity	J	Leverett J -function
$\phi(\mathbf{x}, \ell)$	local porosity in $\mathbb{K}(\mathbf{x}, \ell)$	Ca	capillary number
k	permeability	Gl	gravillary number
$k(\mathbf{x}, \phi(\mathbf{x}); \ell)$	local permeability in $\mathbb{K}(\mathbf{x}, \ell)$	He	heterogeneity number
k^r	relative permeability	Δ	Laplace operator
P_e	capillary entry pressure	$\mu(X; \ell)$	probability density (empirical histogram) of observable X inside a measurement cell of length ℓ
$P_e(\mathbf{x}; \ell)$	local capillary entry pressure in $\mathbb{K}(\mathbf{x}, \ell)$		
$\bar{X}(\ell)$	typical value (e.g., average) of locally fluctuating quantity $X(\mathbf{x}; \ell)$ at scale ℓ where $X = \phi, k, P_e$		

dimensional arguments". In this paper we show quantitatively how the relative uncertainty determines the upscaling ratio defined as the ratio of the system size to the constitutive scale.

Given our main objective of quantifying the influence of heterogeneities through simple dimensionless groups derived from the traditional equations we have structured the paper as follows. Length scales and geometric descriptors for heterogeneous media are introduced and discussed in the next section. Once the heterogeneities have been characterized geometrically the traditional equations of motion for immiscible two phase flow in heterogeneous media are discussed in Section 3. Rewriting these equations in dimensionless form is carried out in Section 4. In Section 5 we discuss upscaling in the sense of changing the constitutive scale. After that special solutions are discussed in Section 6 and examples in Section 7. Much information can be extracted from our simple dimensional arguments and this suggests that the dimensionless heterogeneity groups can be useful in practical situations where detailed solution of the underlying equations is not feasible.

2. Descriptors of heterogeneous porous media

Consider a porous sample $\mathbb{S} = (\mathbb{P} \cup \mathbb{M}) \subset \mathbb{R}^3$ whose pore space $\mathbb{P} = \mathbb{W} \cup \mathbb{O}$ is filled with a wetting fluid \mathbb{W} called water and a nonwetting fluid \mathbb{O} called oil. The sample \mathbb{S} is assumed to be a compact and convex subset of \mathbb{R}^3 . Its volume is $|\mathbb{S}| = \int_{\mathbb{S}} d^3\mathbf{x}$ and its characteristic size is called $L = |\mathbb{S}|^{1/3}$. The solid matrix \mathbb{M} of the porous medium is assumed to be rigid. The sets are illustrated schematically in Fig. 1.

The pore space \mathbb{P} is assumed to have an average pore size ℓ_p defined as $\ell_p = \int_{\mathbb{P}} l(\mathbf{x}) d^3\mathbf{x} / |\mathbb{P}|$ where $|\mathbb{P}| = \int_{\mathbb{P}} d^3\mathbf{x}$ is the volume of the pore space and $l(\mathbf{x}) = \sup\{d : \mathbb{B}(\mathbf{x}, d) \subset \mathbb{P}\}$ is the diameter of the largest ball $\mathbb{B}(\mathbf{x}, d)$ of diameter d centered at \mathbf{x} that is fully contained in \mathbb{P} .

For heterogeneous porous media the geometric and material parameters depend on length scale. Consider for example the porosity $\phi(\mathbb{S}) = \phi = |\mathbb{P}| / |\mathbb{S}|$ defined as the volume fraction of pore space inside the sample. The local porosity at scale ℓ introduced in [4] is defined as

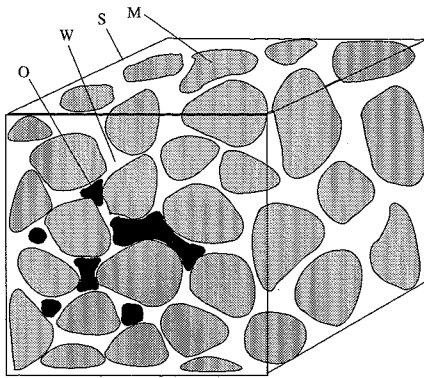


Fig. 1. Schematic illustration of the sets S, P, M, W and O.

$$\phi(\mathbf{x}; \ell) = \frac{|\mathbb{P} \cap \mathbb{K}(\mathbf{x}, \ell)|}{|\mathbb{K}(\mathbf{x}, \ell)|} \quad (1)$$

where $\mathbb{K}(\mathbf{x}, \ell)$ denotes a measurement cell (e.g., a ball) of diameter ℓ centered at \mathbf{x} . For more information on local porosities see the review [5].

To characterize the heterogeneity of the medium we proceed as follows. Let X be a physical or geometrical parameter such as porosity ϕ or permeability k . For simplicity we assume here that k is scalar, although it is in general a tensor. Then we denote by $X(\mathbf{x}; \ell)$ the local value of this observable inside a measurement cell $\mathbb{K}(\mathbf{x}; \ell)$ of size ℓ centered at \mathbf{x} . In theory it is then possible to define the distribution of $X(\mathbf{x}; \ell)$ by an ensemble average of a Dirac δ -function (see Eq. (34) in [5]). In practice one calculates the local X -density as

$$\mu(X; \ell) = \frac{1}{m} \sum_{i=1}^m \delta(X - X(\mathbf{x}_i; \ell)) \quad (2)$$

where m is the number of placements of the measurement cell and $\delta(x)$ denotes the Dirac δ -function. Then, assuming the integrals exist,

$$\bar{X}(\ell) = \int X \mu(X; \ell) dX \quad (3)$$

is the average value of X and

$$\sigma_X(\ell) = \left(\int (X - \bar{X}(\ell))^2 \mu(X; \ell) dX \right)^{1/2} \quad (4)$$

its standard deviation. The strength of heterogeneities can be characterized by decomposing the observable X as

$$X(\mathbf{x}; \ell) = \bar{X}(\ell) + [X(\mathbf{x}; \ell) - \bar{X}(\ell)] = \bar{X}(\ell) + \tilde{X}(\mathbf{x}; \ell). \quad (5)$$

This can be rewritten as

$$X(\mathbf{x}; \ell) = \bar{X}(\ell) \left(1 + u_X(\ell) \frac{\tilde{X}(\mathbf{x}; \ell)}{\sigma_X(\ell)} \right). \quad (6)$$

where

$$u_X(\ell) = \frac{\sigma_X(\ell)}{\bar{X}(\ell)} \quad (7)$$

are the dimensionless *relative uncertainties* or *relative errors* of X . In the following the quantities u_X will be used to quantify the heterogeneity or disorder in X . Two important cases have to be distinguished. For $u_X < 1$ we speak of weak disorder while for $u_X > 1$ we speak of strong disorder. The observable X will be chosen below as porosity $X = \phi$, permeability $X = k$ or capillary entry pressure $X = P_c$. The porosity disorder will normally be weak because the porosity cannot exceed unity. Hence strong porosity disorder is only possible at very small porosities. On the other hand the permeability disorder is often very strong, i.e. $u_k \gg 1$. An example would be fractured media, where the average and variance of the permeability may be infinite, and the most probable permeability is often much smaller than the half width of the local permeability distribution.

If the disorder is so strong that the integrals of the local X -density in (3) and (4) diverge, then we define $\bar{X}(\ell)$ to be the most probable value of X and $\sigma_X(\ell)$ to be the halfwidth of the distribution.

For the case of porosity $X = \phi$ measurements of $\bar{\phi}(\ell)$ and $\tilde{\phi}(\ell)$ for sedimentary rocks can, e.g., be found in [6,7,5]. In homogeneous (stationary) media there exists a length scale $\ell_{\text{rev}}(\phi; \delta) \ll L$ such that for every fixed $\delta > 0$ one has $\sigma_\phi(\ell) < \delta$ for all $\ell > \ell_{\text{rev}}(\phi; \delta)$. The length scale $\ell_{\text{rev}}(\phi; \delta)$ determines the size of the representative elementary volume (Fig. 1.3.2 in [8], p. 20) beyond which the porosity is constant with accuracy δ . For heterogeneous media on the other hand the length scale $\ell_{\text{rev}}(\phi; \delta) \approx L$ is of order of the system size. Similar considerations apply to permeability, i.e. $X = k$.

The fluctuations of porosity and permeability are not independent. In principle it is possible that these observables have different correlation lengths. In this paper it is assumed that porosity and permeability have the same correlation length ξ . The same correlation length is also assumed for other parameter heterogeneities (such as the local entry pressure fluctuations below). The correlation length reflects the typical size of the heterogeneities. We assume here that $\ell_P \ll \xi \ll L$ holds [9].

3. Macroscopic equations

The pore space $\mathbb{P} = \mathbb{W}(t) \cup \mathbb{O}(t)$ is filled with a wetting fluid called water and a nonwetting fluid called oil. The wetting fluid occupies a temporally changing subset $\mathbb{W}(t)$, the nonwetting fluid occupies the set $\mathbb{O}(t)$. The fluids have densities ρ_W, ρ_O and viscosities μ_W, μ_O , respectively. On length scales ℓ local volume fractions

$$\phi_w(\mathbf{x}, t; \ell) = \frac{|\mathbb{W}(t) \cap \mathbb{K}(\mathbf{x}, \ell)|}{|\mathbb{K}(\mathbf{x}, \ell)|} \quad (8a)$$

$$\phi_o(\mathbf{x}, t; \ell) = \frac{|\mathbb{O}(t) \cap \mathbb{K}(\mathbf{x}, \ell)|}{|\mathbb{K}(\mathbf{x}, \ell)|} \quad (8b)$$

of water and oil are defined as functions of position and time, where $\mathbb{K}(\mathbf{x}, \ell)$ denotes again a measurement cell of diameter ℓ centered at \mathbf{x} . The dependence on length scale is usually suppressed to simplify the notation. The volume fractions may be expressed as $\phi_w(\mathbf{x}, t) = \phi(\mathbf{x}, t)S_w(\mathbf{x}, t)$ and $\phi_o(\mathbf{x}, t) = \phi(\mathbf{x}, t)S_o(\mathbf{x}, t)$ in terms of the (possibly space and time dependent) porosity $\phi(\mathbf{x}, t)$ and the fluid saturation fields S_w, S_o . The saturations obey

$$S_w(\mathbf{x}, t) + S_o(\mathbf{x}, t) = 1 \quad (9)$$

for all \mathbf{x} and t . The fluid pressures are denoted as $P_w(\mathbf{x}, t), P_o(\mathbf{x}, t)$ and the fluid velocities are written as $\mathbf{v}_w(\mathbf{x}, t), \mathbf{v}_o(\mathbf{x}, t)$. The pressures and velocities are defined as spatial averages of the microscopic (i.e. pore scale) pressures and velocities over the cell $\mathbb{K}(\mathbf{x}, \ell)$. Again their dependence on ℓ is suppressed.

Mass conservation of the fluids is assumed and requires

$$\frac{\partial}{\partial t}(\phi_w \varrho_w) + \nabla \cdot (\phi_w \varrho_w \mathbf{v}_w) = 0 \quad (10a)$$

$$\frac{\partial}{\partial t}(\phi_o \varrho_o) + \nabla \cdot (\phi_o \varrho_o \mathbf{v}_o) = 0 \quad (10b)$$

We assume that fluids are incompressible $\varrho_w(\mathbf{x}, t) = \varrho_w = \text{const}$ and $\varrho_o(\mathbf{x}, t) = \varrho_o = \text{const}$. As usual a generalized Darcy law without offdiagonal terms

$$\phi_w(\mathbf{x}, t) \mathbf{v}_w(\mathbf{x}, t) = -\frac{1}{\mu_w} k(\mathbf{x}, \phi(\mathbf{x})) k_w^r(\mathbf{x}, S_w) \times (\nabla P_w(\mathbf{x}, t) - \varrho_w \mathbf{g}) \quad (11a)$$

$$\phi_o(\mathbf{x}, t) \mathbf{v}_o(\mathbf{x}, t) = -\frac{1}{\mu_o} k(\mathbf{x}, \phi(\mathbf{x})) k_o^r(\mathbf{x}, S_o) \times (\nabla P_o(\mathbf{x}, t) - \varrho_o \mathbf{g}) \quad (11b)$$

is assumed to govern the viscous momentum exchange between the fluids and the rigid matrix. Note that the permeability is in general a tensor. The constitutive theory is completed by assuming

$$P_o(\mathbf{x}, t) - P_w(\mathbf{x}, t) = P_c(\mathbf{x}, S_w(\mathbf{x}, t)) \quad (12)$$

with a spatially varying capillary pressure function.

It is assumed that the constitutive functions such as $k(\mathbf{x}, \phi; \ell)$ and $P_c(\mathbf{x}, S_w; \ell)$ change with length scale ℓ while the form of Eqs. (10)–(12) remains unchanged for a wide range of length scales $\ell_p \leq \ell \leq L$. Of course the choice of ℓ_p as the lower limit of validity is somewhat too low, and in practice one might use a larger value for ℓ_p . Such a change would not affect the main

results of our present analysis. In the following the length scale ℓ at which the constitutive functions are known will be called the *constitutive scale*.

Inserting the constitutive assumptions into Eqs. (10) leads to

$$\phi(\mathbf{x}) \frac{\partial S_w}{\partial t} = \frac{1}{\mu_w} \nabla \cdot \left[k(\mathbf{x}, \phi(\mathbf{x})) k_w^r(\mathbf{x}, S_w(\mathbf{x}, t)) \times \left(\nabla P(\mathbf{x}, t) - \frac{1}{2} \nabla P_c(\mathbf{x}, S_w(\mathbf{x}, t)) - \varrho_w \mathbf{g} \right) \right] \quad (13a)$$

$$\phi(\mathbf{x}) \frac{\partial S_o}{\partial t} = \frac{1}{\mu_o} \nabla \cdot \left[k(\mathbf{x}, \phi(\mathbf{x})) k_o^r(\mathbf{x}, S_o(\mathbf{x}, t)) \times \left(\nabla P(\mathbf{x}, t) + \frac{1}{2} \nabla P_c(\mathbf{x}, S_o(\mathbf{x}, t)) - \varrho_o \mathbf{g} \right) \right] \quad (13b)$$

where

$$P(\mathbf{x}, t) = \frac{1}{2} (P_w(\mathbf{x}, t) + P_o(\mathbf{x}, t)) \quad (14)$$

Together with Eq. (9) these are two equations for the two unknown fields $P(\mathbf{x}, t)$ and $S_w(\mathbf{x}, t)$.

Next we carry out the differentiation outside the square bracket in (13) to obtain for the wetting fluid

$$\begin{aligned} \frac{\partial S_w}{\partial t} = \frac{k k_w^r}{\mu_w \phi} \left\{ \Delta P - \frac{1}{2} \frac{\partial P_c}{\partial S_w} \Delta S_w \right. \\ - \frac{1}{2} \left(\frac{\partial^2 P_c}{\partial S_w^2} + \frac{\partial P_c}{\partial S_w} \frac{\partial \log k_w^r}{\partial S_w} \right) (\nabla S_w)^2 \\ + \frac{\partial \log k_w^r}{\partial S_w} (\nabla P - \varrho_w \mathbf{g}) \cdot \nabla S_w \\ - \frac{1}{2} \left[\frac{\partial \log k_w^r}{\partial S_w} \mathbf{grad} P_c + \frac{\partial \mathbf{grad} P_c}{\partial S_w} + \mathbf{grad} \frac{\partial P_c}{\partial S_w} \right. \\ \left. + \frac{\partial P_c}{\partial S_w} \left(\mathbf{grad} \log k + \frac{\partial \log k}{\partial \phi} \nabla \phi + \mathbf{grad} \log k_w^r \right) \right] \\ \cdot \nabla S_w + \left(\mathbf{grad} \log k + \frac{\partial \log k}{\partial \phi} \nabla \phi + \mathbf{grad} \log k_w^r \right) \\ \cdot (\nabla P - \varrho_w \mathbf{g}) - \frac{1}{2} \left(\mathbf{grad} \log k + \frac{\partial \log k}{\partial \phi} \nabla \phi \right. \\ \left. + \mathbf{grad} \log k_w^r \right) \cdot \mathbf{grad} P_c - \frac{1}{2} (\mathbf{grad} \cdot \mathbf{grad}) P_c \left. \right\} \quad (15) \end{aligned}$$

where the notation $\mathbf{grad} k(\mathbf{x}, \phi(\mathbf{x}))$ means the gradient with respect to the first vector variable of a parameter function as indicated by the arrow. The gradient of the pressure or saturation fields is written as before, e.g., $\nabla S_w(\mathbf{x}, t)$. A similar equation is obtained from (13b) for the nonwetting fluid.

To simplify the subsequent discussion we assume from now on that the capillary pressure factorizes

$$P_c(\mathbf{x}, S_w) = P_c(\mathbf{x}) J(S_w) \quad (16)$$

into a position dependent entry pressure $P_e(\mathbf{x})$ and a dimensionless function of saturation $J(S_W)$ reminiscent of the Leverett- j -function. Then Burdines equations [10] suggest that the assumption

$$k_{\mathbb{W}}^r(\mathbf{x}, S_W) = k_{\mathbb{W}}^r(S_W) \quad (17)$$

holds, i.e. that the relative permeabilities are spatially homogeneous. With these assumptions Eq. (15) simplifies to become

$$\begin{aligned} \frac{\partial S_{\mathbb{W}}}{\partial t} = & \frac{k k_{\mathbb{W}}^r}{\mu_{\mathbb{W}} \phi} \left\{ \Delta P - \frac{P_e}{2} \frac{dJ}{dS_{\mathbb{W}}} \Delta S_{\mathbb{W}} \right. \\ & - \frac{P_e}{2} \left(\frac{d^2 J}{dS_{\mathbb{W}}^2} + \frac{dJ}{dS_{\mathbb{W}}} \frac{d \log k_{\mathbb{W}}^r}{dS_{\mathbb{W}}} \right) (\nabla S_{\mathbb{W}})^2 \\ & + \frac{d \log k_{\mathbb{W}}^r}{dS_{\mathbb{W}}} (\nabla P - \rho_{\mathbb{W}} \mathbf{g}) \cdot \nabla S_{\mathbb{W}} \\ & - \frac{1}{2} \left(J \frac{d \log k_{\mathbb{W}}^r}{dS_{\mathbb{W}}} \nabla P_e + 2 \frac{dJ}{dS_{\mathbb{W}}} \nabla P_e \right. \\ & + P_e \frac{dJ}{dS_{\mathbb{W}}} \nabla \log k \left. \right) \cdot \nabla S_{\mathbb{W}} + \nabla \log k \\ & \left. \cdot (\nabla P - \rho_{\mathbb{W}} \mathbf{g}) - \frac{J}{2} (\nabla \log k \cdot \nabla P_e + \Delta P_e) \right\} \quad (18) \end{aligned}$$

4. Dimensional analysis

We now use Eq. (5) for porosity $X = \phi$, permeability $X = k$ and entry pressure $X = P_e$ and insert these three equations into Eq. (18) to find

$$\begin{aligned} \frac{\partial S}{\partial t} = & \frac{k(\mathbf{x}, \phi) k^r(S)}{\mu \phi(\mathbf{x})} \left\{ \Delta P - \frac{P_e(\mathbf{x})}{2} \frac{dJ(S)}{dS} \Delta S \right. \\ & - \frac{P_e(\mathbf{x})}{2} \left(\frac{d^2 J(S)}{dS^2} + \frac{dJ(S)}{dS} \frac{d \log k^r}{dS} \right) (\nabla S)^2 \\ & + \frac{d \log k^r}{dS} (\nabla P - \rho \mathbf{g}) \cdot \nabla S - \frac{1}{2} \left[J(S) \frac{d \log k^r}{dS} \nabla \tilde{P}_e \right. \\ & + 2 \frac{dJ(S)}{dS} \nabla \tilde{P}_e + \frac{P_e(\mathbf{x})}{k(\mathbf{x}, \phi)} \frac{dJ(S)}{dS} \left(\mathbf{grad} \tilde{k}(\mathbf{x}, \phi) \right. \\ & \left. + \frac{\partial \tilde{k}}{\partial \phi} \nabla \tilde{\phi} \right) \left. \right] \cdot \nabla S + \frac{1}{k(\mathbf{x}, \phi)} \left(\mathbf{grad} \tilde{k}(\mathbf{x}, \phi) \right. \\ & \left. + \frac{\partial \tilde{k}}{\partial \phi} \nabla \tilde{\phi} \right) \cdot (\nabla P - \rho \mathbf{g}) - \frac{J(S)}{2} \left[\frac{1}{k(\mathbf{x}, \phi)} \right. \\ & \left. \times \left(\mathbf{grad} \tilde{k}(\mathbf{x}, \phi) + \frac{\partial \tilde{k}}{\partial \phi} \nabla \tilde{\phi} \right) \cdot \nabla \tilde{P}_e + \Delta \tilde{P}_e \right] \left. \right\} \quad (19) \end{aligned}$$

where the index \mathbb{W} was suppressed to simplify the notation.

We introduce dimensionless variables

$$\mathbf{x} = L \hat{\mathbf{x}} \quad (20)$$

$$\nabla = \frac{1}{L} \hat{\nabla} \quad (21)$$

$$t = \tau \hat{t} \quad (22)$$

dimensionless pressure

$$P(\mathbf{x}, t) = \overline{P}_e \hat{P}(\hat{\mathbf{x}}, \hat{t}) \quad (23)$$

and dimensionless parameter functions

$$P_e(\mathbf{x}) = \overline{P}_e \hat{P}_e(\hat{\mathbf{x}}) \quad (24)$$

$$\tilde{P}_e(\mathbf{x}) = \sigma_{P_e} \hat{\tilde{P}}_e(\hat{\mathbf{x}}) \quad (25)$$

$$k(\mathbf{x}, \phi) = \bar{k} \hat{k}(\hat{\mathbf{x}}, \phi) \quad (26)$$

$$\tilde{k}(\mathbf{x}, \phi) = \sigma_k \hat{\tilde{k}}(\hat{\mathbf{x}}, \phi) \quad (27)$$

We also introduce the abbreviations

$$\mathbf{P} = \hat{\nabla} \hat{\tilde{P}}_e \quad (28a)$$

$$\mathbf{K} = \frac{1}{\bar{k}} \left(\widehat{\mathbf{grad}} \hat{k} + \frac{\partial \hat{k}}{\partial \hat{\phi}} \hat{\nabla} \hat{\phi} \right) \quad (28b)$$

$$D = \hat{\Delta} \hat{\tilde{P}}_e \quad (28c)$$

and $\hat{\phi}(\hat{\mathbf{x}}) = \phi(L \hat{\mathbf{x}})$.

Inserting these we obtain

$$\begin{aligned} \frac{\partial S}{\partial \hat{t}} = & \frac{\tau \bar{k} \overline{P}_e}{L^2 \mu} \frac{\hat{k}(\hat{\mathbf{x}}, \phi) k^r(S)}{\hat{\phi}(\hat{\mathbf{x}})} \left\{ \hat{\Delta} \hat{P} - \frac{\hat{P}_e(\hat{\mathbf{x}})}{2} \frac{dJ(S)}{dS} \hat{\Delta} S \right. \\ & - \frac{\hat{P}_e(\hat{\mathbf{x}})}{2} \left(\frac{d^2 J(S)}{dS^2} + \frac{dJ(S)}{dS} \frac{d \log k^r}{dS} \right) (\hat{\nabla} S)^2 \\ & + \frac{d \log k^r}{dS} \left(\hat{\nabla} \hat{P} - \frac{\rho \mathbf{g} L}{\overline{P}_e} \right) \cdot \hat{\nabla} S \\ & - \frac{1}{2} \left[\frac{\sigma_{P_e}}{\overline{P}_e} \left(J(S) \frac{d \log k^r}{dS} + 2 \frac{dJ(S)}{dS} \right) \mathbf{P}(\hat{\mathbf{x}}) \right. \\ & + \frac{\sigma_k}{\bar{k}} \frac{dJ(S)}{dS} \hat{P}_e(\hat{\mathbf{x}}) \mathbf{K}(\hat{\mathbf{x}}) \left. \right] \cdot \hat{\nabla} S \\ & + \frac{\sigma_k}{\bar{k}} \mathbf{K}(\hat{\mathbf{x}}) \cdot \left(\hat{\nabla} \hat{P} - \frac{\rho \mathbf{g} L}{\overline{P}_e} \right) \\ & \left. - \frac{J(S)}{2} \left[\frac{\sigma_k}{\bar{k}} \frac{\sigma_{P_e}}{\overline{P}_e} \mathbf{K}(\hat{\mathbf{x}}) \cdot \mathbf{P}(\hat{\mathbf{x}}) + \frac{\sigma_{P_e}}{\overline{P}_e} D(\hat{\mathbf{x}}) \right] \right\} \quad (29) \end{aligned}$$

as the dimensionless form of the equation. Eq. (29) allows one to identify the macroscopic capillary number [11,12]

$$Ca = \frac{L^2 \mu}{\tau \bar{k}(\ell) \overline{P}_e(\ell)} \quad (30)$$

and the gravillary number, defined as the ratio of gravitational to capillary forces [11,12]

$$Gl = \frac{\rho |g| L}{P_c(\ell)} \tag{31}$$

as usual. Note that the gravillary number differs from the commonly used bond number in that the latter involves the density difference. The constitutive scale ℓ has been indicated explicitly because the constitutive parameters are often not known at the scale of the system size L . These dimensionless numbers allow to quantify the physical force balance between capillarity, viscosity and gravity. Eq. (29) shows that the slope and curvature of the Leverett function as well as the slope of the logarithm of the relative permeabilities have a strong influence on two phase flow.

In Eq. (29) we have separated the dimensionless terms arising from the heterogeneities into the last four lines. The last line shows that heterogeneities act as a kind of external pseudoforce. It also shows that a constant saturation field cannot be a solution in the presence of heterogeneities. The heterogeneity terms appear in the combinations $D\sigma_{P_c}/\overline{P_c}$, $P\sigma_{P_c}/\overline{P_c}$ and $K\sigma_k/\overline{k}$. In order to estimate their importance it is necessary to estimate the magnitude of these terms.

Note that the dimensionless ratio $\sigma_{P_c}/\overline{P_c}$ measures the variability or contrast of the heterogeneities while the dimensionless gradient \mathbf{P} characterizes spatial rapidity of fluctuations.

5. Upscaling

Upscaling means in this paper to increase the constitutive scale ℓ within the allowed range $\ell_p \lesssim \ell \leq L$. The ratio

$$R = \frac{L}{\ell} \geq 1 \tag{32}$$

will be called the upscaling ratio. We assume throughout that the constitutive scale is less than or equal to the system size, so that the upscaling ratio is always larger than unity.

This section discusses the upscaling behaviour of the heterogeneity terms. Du_{P_c} , Pu_{P_c} and Ku_k . Estimating them requires knowledge of the position dependence of the permeability and entry pressure at the constitutive scale ℓ . In this paper we have assumed that the averaged equations of motion remain valid down to length scales of order $\ell \approx \ell_p$ although in applications one might also take some multiple of ℓ_p . To continue the analysis we assume here that the heterogeneities are block discontinuities. By this we mean that permeability and entry pressure are piecewise constant functions of position. This assumption is not essential for the dimensional analysis method but is made here because it represents an idealized special case of considerable practical importance. For block discontinuities it is clear that at scale ℓ

the dimensionless gradients \mathbf{P} , \mathbf{K} change by unity over a distance of order ℓ in the limit $\ell \rightarrow \ell_p$. Similarly the dimensionless curvature D will be of order L^2/ℓ^2 . This suggests to extract the dependence on L/ℓ and to write

$$\mathbf{P} = R\widehat{\mathbf{P}} = \frac{L}{\ell}\widehat{\mathbf{P}} \tag{33a}$$

$$\mathbf{K} = R\widehat{\mathbf{K}} = \frac{L}{\ell}\widehat{\mathbf{K}} \tag{33b}$$

$$D = R^2\widehat{D} = \frac{L^2}{\ell^2}\widehat{D} \tag{33c}$$

thereby defining nondimensional functions $\widehat{\mathbf{P}}$, $\widehat{\mathbf{K}}$, \widehat{D} that are expected to vary only little with length scale.

The influence of heterogeneities can be quantified by defining the heterogeneity numbers at scale ℓ as

$$He_k = Ru_k = \frac{L\sigma_k(\ell)}{\ell\overline{k}(\ell)} \tag{34}$$

$$He_{P_c} = Ru_{P_c} = \frac{L\sigma_{P_c}(\ell)}{\ell\overline{P_c}(\ell)} \tag{35}$$

for permeability and capillarity. Heterogeneity effects are strong for $He > 1$ and weak for $He < 1$.

With the help of the dimensionless groups (30), (31), (34) and (35) Eq. (29) becomes

$$\begin{aligned} \frac{\partial S}{\partial t} = & \frac{1}{Ca} \frac{\hat{k}(\hat{\mathbf{x}}, \phi)k^r(S)}{\phi(\hat{\mathbf{x}})} \left\{ \widehat{\Delta}\widehat{P} - \frac{\widehat{P}_c(\hat{\mathbf{x}})}{2} \frac{dJ(S)}{dS} \widehat{\Delta}S \right. \\ & - \frac{\widehat{P}_c(\hat{\mathbf{x}})}{2} \left(\frac{d^2J(S)}{dS^2} + \frac{dJ(S)}{dS} \frac{d \log k^r}{dS} \right) (\widehat{\nabla}S)^2 \\ & + \frac{d \log k^r}{dS} \left(\widehat{\nabla}\widehat{P} - Gl \frac{\mathbf{g}}{|\mathbf{g}|} \right) \cdot \widehat{\nabla}S \\ & - \frac{1}{2} \left[He_{P_c} \left(J(S) \frac{d \log k^r}{dS} + 2 \frac{dJ(S)}{dS} \right) \widehat{\mathbf{P}}(\hat{\mathbf{x}}) \right. \\ & + He_k \frac{dJ(S)}{dS} \widehat{P}_c(\hat{\mathbf{x}})\widehat{\mathbf{K}}(\hat{\mathbf{x}}) \cdot \widehat{\nabla}S \\ & + He_k \widehat{\mathbf{K}}(\hat{\mathbf{x}}) \cdot \left(\widehat{\nabla}\widehat{P} - Gl \frac{\mathbf{g}}{|\mathbf{g}|} \right) \\ & \left. - \frac{J(S)}{2} He_{P_c} \left[He_k \widehat{\mathbf{K}}(\hat{\mathbf{x}}) \cdot \widehat{\mathbf{P}}(\hat{\mathbf{x}}) + \frac{L}{\ell} \widehat{D}(\hat{\mathbf{x}}) \right] \right\} \tag{36} \end{aligned}$$

where $\mathbf{g}/|\mathbf{g}|$ denotes the unit vector in the direction of gravity and the order of terms is the same as in Eq. (36). Note that the dimensionless groups Ca , Gl , He_{P_c} , He_k all depend on the system size L .

To estimate the magnitude of the heterogeneity numbers it is necessary to estimate how \overline{k} , $\overline{P_c}$ and σ_k , σ_{P_c} depend on length scale ℓ . We assume here

$$\overline{k}(\ell) = \overline{k} \tag{37}$$

$$\overline{P_c}(\ell) = \overline{P_c} \tag{38}$$

that the typical values \bar{k}, \bar{P}_e are independent of ℓ . For the magnitude of the fluctuations σ_k, σ_{P_e} such an assumption is not possible because they must vanish in the limit $\ell \rightarrow L$. For $\ell < \xi$ we expect that the distribution of local permeabilities and entry pressures does not change and hence its half width or standard deviation would also be constant. Thus we assume that $\sigma_k(\ell) = \sigma_k$ for $\ell_p \leq \ell \leq \xi$ and that $\sigma_k(L) = 0$ for $\ell = L$. A simple linear interpolation would suggest the form

$$\sigma_k(\ell) = \begin{cases} \sigma_k & \text{for } \ell_p \leq \ell \leq \xi \\ \sigma_k \frac{\xi - \ell}{\xi - L} & \text{for } \xi \leq \ell \leq L \end{cases} \quad (39)$$

and similarly for σ_{P_e} . If for $\ell > \xi$ one assumes that the fluctuations are governed by the central limit theorem then one expects, e.g., $\sigma_k \sim (L - \ell)^{-3/2}$ at least for some range when ℓ and L are large.

The heterogeneity numbers can be used for upscaling considerations. Eqs. (34) and (35) imply that heterogeneity effects can be neglected ($He < 1$) if

$$u_X < \frac{1}{R} \quad \text{or} \quad \frac{\sigma_X}{\bar{X}} < \frac{\ell}{L} \quad (40)$$

holds (where $X = k, P_e$). In other words heterogeneity effects are negligible when the relative uncertainty is smaller than the inverse upscaling ratio.

Suppose that laboratory measurements of the permeability and capillary pressure have been performed on samples of size ℓ taken from different locations of a larger medium. If the measurements give an average permeability \bar{k} with standard deviation σ_k then the heterogeneity numbers allow to estimate the maximal system size for a homogeneous model having average permeability \bar{k} from the condition $He_k < 1$. One finds

$$L < \frac{\ell \bar{k}}{\sigma_k} \quad (41)$$

for the maximal system size. Such information could also be useful for upscaling in numerical simulations in order to estimate the maximal size of blocks in a large scale simulation. Eq. (41) shows that for systems with strong permeability disorder (i.e. for $\sigma_k > \bar{k}$) upscaling is not possible (because L would have to fall below ℓ).

6. Solutions

For large L heterogeneity dominates and Eq. (36) simplifies into an equation for saturation alone

$$Ca \frac{\partial S}{\partial t} = - \frac{\hat{k}(\hat{\mathbf{x}}, \phi) k^f(S)}{\phi(\hat{\mathbf{x}})} \left\{ Gl He_k \mathbf{K}(\hat{\mathbf{x}}) \cdot \mathbf{e}_g + \frac{J(S)}{2} He_{P_e} \left[He_k \mathbf{K}(\hat{\mathbf{x}}) \cdot \mathbf{P}(\hat{\mathbf{x}}) + \frac{L}{\ell} D(\hat{\mathbf{x}}) \right] \right\} \quad (42)$$

This equation can be solved in the case of stationarity, i.e. for $\partial S / \partial t = 0$. The solution becomes

$$S(\hat{\mathbf{x}}) = J^{-1} \left(- \frac{2 \frac{\ell_0}{\sigma_{P_e}} \mathbf{g} \cdot \mathbf{K}(\hat{\mathbf{x}})}{\mathbf{K}(\hat{\mathbf{x}}) \cdot \mathbf{P}(\hat{\mathbf{x}}) + \frac{k}{\sigma_k} D(\hat{\mathbf{x}})} \right) \quad (43)$$

The vectors \mathbf{K} and \mathbf{P} are oriented parallel and opposite to each other, hence their scalar product is negative. Of course this solution is only approximate. It contains only heterogeneity and gravity forces while viscous forces are absent.

Note that a stationary solution can be obtained without any approximation directly from Darcy's law. It reads

$$S(\mathbf{x}) = J^{-1} \left(\frac{P_0 + (\rho_0 - \rho_w) \mathbf{g} \cdot \mathbf{x}}{P_c(\mathbf{x})} \right) \quad (44)$$

where P_0 is an integration constant.

7. Examples

Consider the following example from Ref. [13, p. 115, 126] where $\ell = \xi = 1, L = 64$, and therefore

$$u_k = \frac{\sigma_k(\ell)}{\bar{k}(\ell)} = \frac{4.375 \times 10^{-5}}{2.3 \times 10^{-4}} = 0.19 \quad (45)$$

Thus the relative error in the permeability is roughly 20%. This gives

$$He_k = \frac{L \sigma_k}{\bar{k}} = 12 \gg 1 \quad (46)$$

showing that heterogeneity in the permeabilities cannot be neglected at the chosen system size. If one demands that $He_k < 1$ then permeability fluctuations can only be neglected up to a system size $L < 5$.

Assuming a Leverett correlation between permeability and entry pressure Braun uses the relation [13, p. 115]

$$P_c = \bar{P}_c \sqrt{\frac{k}{\bar{k}}} \quad (47)$$

and hence one expects entry pressure variations of order

$$|\Delta P_c| \approx \frac{\bar{P}_c}{2\bar{k}} |\Delta k| \quad (48)$$

from permeability variations of order $|\Delta k|$. Taking $\sigma_{P_e} \approx |\Delta P_c|$ and $\sigma_k \approx |\Delta k|$ this amounts to assuming

$$\frac{\sigma_{P_e}}{P_c} \approx \frac{\sigma_k}{2\bar{k}} \approx 0.095, \quad (49)$$

and gives a heterogeneity number

$$He_{P_e} = \frac{L \sigma_{P_e}}{\bar{P}_c} = 6 > 1 \quad (50)$$

larger than unity. Again this implies that also the fluctuations in capillary entry pressures become important when the system size reaches $L \approx 10$.

These estimates show that even with a small relative error of only 20% in the knowledge of the permeability

at the microscopic (cell) scale one cannot expect reliable results based on homogeneous regions if the regions that are assumed to be homogeneous are larger than 5 cells! For any simulation region larger than five (or at most) ten microscopic cells the heterogeneities become dominant, and need to be taken into account.

Finally we discuss the experimental example of coarse and fine sand embedded in a medium grained sand discussed in [13, p. 103]. In this example the constitutive parameters of the sand are known on scales $\ell \approx 1$ cm. The scale of the heterogeneities is $\xi \approx 10$ cm in the x -direction and $\xi \approx 2$ cm in the z -direction. The system size is $L \approx 1$ m. The local permeability and local entry pressure distributions at scale ℓ are approximately sums of three δ -distributions. The three types of sands have volume fraction ratios of fine:medium:coarse of 0.1:0.8:0.1. The most probable values are $\bar{k} \approx 1.2 \times 10^{-10} \text{ m}^2$ and $\bar{P}_e \approx 540 \text{ Pa}$. The half widths are roughly $\sigma_k \approx 0.6 \times 10^{-10} \text{ m}^2$ and $\sigma_{P_e} \approx 200 \text{ Pa}$. Therefore we find $\text{He}_k \approx 50$ and $\text{He}_{P_e} \approx 40$. This shows that upscaling is impossible for this example. Even if the constitutive functions were known at the horizontal correlation length scale of $\xi \approx 10$ cm upscaling would still not be possible. Of course upscaling might become possible if the constitutive parameters are measured or calculated repetitively on scales significantly larger than ξ .

8. Conclusion

Our main conclusion is that the typical permeability and capillary properties need to be known with very high precision if one wants to upscale a homogeneous model. Upscaling is not possible for systems with strong disorder (i.e. $u > 1$). For weak disorder governed by Gaussian statistics high precision of the average value is equivalent to a small standard deviation. This means that homogeneous modeling requires large averaging volumes (REV's) over which permeability and capillary properties need to be averaged. Here we have given formulae to quantify these considerations for weak and strong disorder.

We hope that our heterogeneity groups will be helpful to obtain rough quantitative estimates for the impor-

tance of heterogeneity effects in practical situations where an expensive and time consuming solution of the underlying equations is not possible.

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