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Erosion–dilation analysis for experimental and synthetic microstructures of sedimentary rock

A. Tscheschel[a], D. Stoyan[a,*], R. Hilfer[b,c]

[a] *Institut für Stochastik, TU Bergakademie Freiberg, 09596 Freiberg, Germany*
[b] *Institut für Computerphysik, Universität Stuttgart, Pfaffenwaldring 27, 70569 Stuttgart, Germany*
[c] *Institut für Physik, Universität Mainz, 55099 Mainz, Germany*

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Abstract

Microstructures such as rock samples or simulated structures can be described and characterized by means of ideas of spatial statistics and mathematical morphology. A powerful approach is to transform a given 3D structure by operations of mathematical morphology such as dilation and erosion. This leads to families of structures, for which various characteristics can be determined, for example, porosity, specific connectivity number or correlation and connectivity functions. An application of this idea leads to a clear discrimination between a sample of Fontainebleau sandstone and two simulated samples. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper continues the study of statistical methods for the characterization of porous media as in Refs. [1,2]. The aim of these investigations is twofold. First, the prediction of physical properties of porous media based on geometrical information has to be improved [1–4]. Second, efficient tools are necessary for testing the goodness-of-fit of possible models for such structures. For both purposes the traditional methods consisting in determination of porosity, specific surface content, correlation functions [2,3], and contact distribution functions [5–7] are not sufficient. More sophisticated approaches are necessary. In the direction of the first aim above, local porosity theory (LPT) was used to include connectivity and percolation probability to predict conductivity [4,8].

* Corresponding author.
E-mail address: tschesch@math.tu-freiberg.de (A. Tscheschel).

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As usual, in Refs. [1,2] the characteristics were determined for the pore space $\mathbb{P}$. The present paper suggests to extend $\mathbb{P}$ to a family $\{\mathbb{P}(d)\}$ of enlarged or diminished sets defined in Section 2.3, where $\mathbb{P} = \mathbb{P}(0)$, and to carry out the statistical analysis in parallel for all suitable d. For positive d, $\mathbb{P}(d)$ is the set $\mathbb{P}$ enlarged by dilation by a sphere of radius d, and for negative d, $\mathbb{P}(d)$ is the set $\mathbb{P}$ diminished by erosion by a sphere of radius d. (The concepts of dilation and erosion belong to 'mathematical morphology' and are explained, for example, in Refs. [6,7,9]; see also Section 2.3.)

Since statistical characteristics estimated for the $\mathbb{P}(d)$ depend on d, this approach produces curves (depending on the variable d) that describe aspects of the spatial distribution of $\mathbb{P}$ which the classical characteristics do not divulge. This is a combination and extension of approaches developed by Mecke [10] and Sivakumar and Goutsias [11]. In the latter paper, Sivakumar and Goutsias have considered (translated in the terms of this paper) the porosity $\langle\phi(d)\rangle$ of $\mathbb{P}(d)$, have studied estimation techniques for $\langle\phi(d)\rangle$ and have shown that $\langle\phi(d)\rangle$ is a powerful statistical tool for the geometrical characterization of porous media. (Sivakumar and Goutsias also used opening and closing instead of dilation and erosion, see also Section 4.1.) Mecke studied only the case of positive d but, additionally to $\langle\phi(d)\rangle$, also the specific surface content and the specific connectivity number, and further characteristics related to the so-called Minkowski measures.

The present paper considers positive and negative d as well as further characteristics such as correlation and contact distribution functions, which are not directly related to Minkowski measures. For the case of negative d (i.e., diminishing of $\mathbb{P}$) the characteristics are not only of statistical interest. For example, in the study of percolation, usually only the existence of paths inside the pore space is considered. However, one could also ask for the existence of paths which a ball of given radius d could go. This should be important for the study of fluid flow and filtration in porous media.

In this paper we apply dilation and erosion to structures analysed already in Ref. [1]. One of them is an experimental sample of Fontainebleau sandstone, while the two other microstructures are synthetic samples obtained by computer simulation.

In Section 2, we introduce and define the various geometrical characteristics that will be used for describing the microstructures and their dilated and eroded counterparts. In this context we also discuss some stochastic models. Then in Section 3 we discuss computational aspects with particular emphasis to problems with lattice geometry. Finally, in Section 4 we present the results for the three microstructures and discuss the ability of the characteristics to discriminate between the microstructures.

2. Measured quantities

2.1. Some characteristics for the original system of pores

Consider a two-component porous medium in $\mathbb{R}^3$, which is statistically homogeneous (stationary), isotropic and ergodic. The two components are the pore space $\mathbb{P}$ and its complement, the solid matrix $\mathbb{M}$. In practical statistics, the data are given as lattice structures. In Section 3 below some aspects in the context of computations in the 3D lattice will be discussed.

The average porosity is given as

$$\langle \phi \rangle = \text{Prob}\{\mathbf{o} \in \mathbb{P}\} = \langle \chi_{\mathbb{P}}(o) \rangle , \tag{2.1}$$

where $\mathbf{o}$ is the origin of the coordinate system; instead of $\mathbf{o}$ any other point could be used because of the homogeneity assumption. $\chi_{\mathbb{P}}$ is the characteristic function of $\mathbb{P}$.

The (two-point porosity–porosity) correlation function of $\mathbb{P}$ is given as

$$G(r) = \frac{\text{Prob}\{\mathbf{o} \in \mathbb{P}, \mathbf{r} \in \mathbb{P}\} - \langle \phi \rangle^2}{\langle \phi \rangle (1 - \langle \phi \rangle)} , \tag{2.2}$$

where $\mathbf{r}$ is any position vector of length r. It is $\text{Prob}\{\mathbf{o} \in \mathbb{P}, \mathbf{r} \in \mathbb{P}\} = \langle \chi_{\mathbb{P}}(\mathbf{o}) \chi_{\mathbb{P}}(\mathbf{r}) \rangle$.

Note that for a homogeneous but non-isotropic structure the correlation function can be used to detect and characterize anisotropy. Then the orientation of $\mathbf{r}$ is relevant; for different orientations of $\mathbf{r}$ the correlation functions can be different thus showing the degree of anisotropy (see Ref. [6, p. 210]).

The correlation function can be refined by considering connectivity, what leads to the pair connectness function [12]

$$G_p(r) = \frac{\text{Prob}\{\text{there is any path in } \mathbb{P} \text{ from } \mathbf{o} \text{ to } \mathbf{r}\} - \langle \phi \rangle^2}{\langle \phi \rangle (1 - \langle \phi \rangle)} . \tag{2.3}$$

For typical porous media with totally connected pores (i.e., for any two pore points there exists a connecting path in $\mathbb{P}$) it is clearly

$$G(r) = G_p(r) \quad \text{for all } r . \tag{2.4}$$

Instead of any path from $\mathbf{o}$ to $\mathbf{r}$ one can ask for lineal paths. This leads to the lineal path function $L^{(i)}(z)$ [13] or to

$$G_l(r) = \frac{\text{Prob}\{s(r) \text{ is completely in } \mathbb{P}\} - \langle \phi \rangle^2}{\langle \phi \rangle (1 - \langle \phi \rangle)} . \tag{2.5}$$

Here $s(r)$ is the line segment starting in $\mathbf{o}$ and ending in $\mathbf{r}$. It is $G(0)=G_p(0)=G_l(0)=1$ and $G(\infty) = 0$. In contrast, $G_p(\infty)$ and $G_l(\infty)$ may be negative. The lineal-path function is closely related to the classical concept of Delfiner [5] of size measurement of random sets and to the related concept of linear contact distribution functions [5–7]. The linear contact distribution function $H_l(r)$ is defined as

$$H_l(r) = 1 - \frac{\text{Prob}\{s(r) \text{ is completely in } \mathbb{P}\}}{\langle \phi \rangle} . \tag{2.6}$$

In the case of isotropy, the direction of the segment does not matter. But if the structure is anisotropic, various linear contact distribution functions corresponding to different directions can be defined (see Ref. [6, p. 211]). Note that there is a close relationship to the chord length distribution function (see Ref. [6, p. 208]).

The spherical contact distribution function $H_s(r)$ is defined as

$$H_s(r) = 1 - \frac{\text{Prob}\{B(\mathbf{o}, r) \text{ is completely in } \mathbb{P}\}}{\langle \phi \rangle} . \tag{2.7}$$

Also this function was introduced by Delfiner [5]; in some papers it is called pore-size distribution [14]. Here $B(\mathbf{o}, r)$ is the sphere of radius r centered at $\mathbf{o}$. The spherical contact distribution can be interpreted as the distribution function of the random distance from a randomly chosen point in $\mathbb{P}$ to its nearest neighbour in $\mathbb{M}$. It is well known in spatial statistics that $H_s(r)$ contains usually more interesting information than $H_l(r)$ and $G_l(r)$ and leads to more powerful goodness-of-fit tests; see also the discussion below for the Boolean model. By the way, in the terminology of Ref. [6], the contact distribution functions introduced here are the contact distribution functions with respect to $\mathbb{M}$, the complement of $\mathbb{P}$.

A quantity describing the connectivity or topological structure of $\mathbb{P}$ is the connectivity (or Euler–Poincaré) number per unit volume N_V [6,7,10,15–18]. It can be interpreted as the limit

$$N_V = \lim_{\mathbb{S} \to \mathbb{R}^d} \frac{E(\mathbb{S} \cap \mathbb{P})}{V(\mathbb{S})} , \tag{2.8}$$

where $\mathbb{S}$ is a sample set of volume $V(\mathbb{S})$ and $E(\mathbb{S} \cap \mathbb{P})$ the Euler–Poincaré characteristic of $\mathbb{S} \cap \mathbb{P}$.

Analogously, also other characteristics related to Minkowski measures can be used [10], namely mean surface content per volume unit for the boundary of $\mathbb{P}$ and a characteristic related to mean curvature.

2.2. Three simple stochastic models for porous media

In order to familiarize the reader with the complicated characteristic N_V, we give its values for some stochastic models.

Note first that $\mathbb{P}$ and $\mathbb{M}$ have the same N_V for a statistically homogeneous three-dimensional structure in $\mathbb{R}^3$. Furthermore, the Euler–Poincaré characteristic is a topological invariant. This implies that in the examples below N_V remains invariant under suitable continuous transformations.

2.2.1. Systems of spheres

Let the pore space $\mathbb{P}$ be a system of 'hard' (nonoverlapping) spheres (as in some Swiss cheese), which are completely isolated. Then N_V is positive and equals the mean number n of spheres per volume unit.

If $\mathbb{P}$ is a 'dense' packing of spheres where spheres close together are in contact, then N_V may be negative. It holds

$$N_V = n\left(1 - \frac{\bar{c}}{2}\right) ,$$

where n is again the mean number of spheres per volume unit and $\bar{c}$ is the mean number of spheres in contact with a typical sphere. The same value is also obtained if the spheres are deformed in such a way that no contacts are severed and not new contacts are generated.

In the case of a dense packing also the complementary set can serve as a model for a pore space $\mathbb{P}$. In this case N_V is the same as for the sphere system. In contrast, differences in the connectivity behaviour are possible; while $\mathbb{P}$ is totally connected, this is not necessarily the case for the system of spheres.

2.2.2. Edges of a spatial mosaic

Consider a statistically homogeneous spatial mosaic or tessellation [6]. An example is the Voronoi tessellation with respect to a Poisson point process. Let $\mathbb{P}$ be the system of all edges. (A more realistic model could be obtained by dilation, where the edges are enlarged to tubes.) In this case

$$N_V = n_v - n_e ,$$

n_v = number of vertices per volume unit and n_e = number of edges per volume unit.

Also here N_V is negative. Probably, Mecke [19] had similar models in mind when he associated negative values of N_V with 'netlike' structures.

2.2.3. Boolean model

The well-known Boolean model is probably not a good model for porous media. But it is of great value as a benchmark model. According to this model, $\mathbb{P}$ consists of randomly scattered 'grains' which can overlap. A particular case of special interest in the given context is the Boolean model with Poisson polyhedra as grains [6]. Indeed, Fig. 1 in [1] supports, the idea that the porous space of the Fontainebleau sandstone sample considered there can be approximated by a union of randomly scattered polyhedra. This model depends on two scalar parameters λ and ϱ only. Here λ is the mean number of grains per volume unit and ϱ characterizes the size of the grains; their mean volume is $6/(\pi\varrho^3)$ (see Ref. [6, p. 84]). The average porosity (i.e., the volume fraction of space occupied by the Boolean model) is $1 - \exp(-6\lambda/\pi\varrho^3)$. The planar section of such a model is again a Boolean model, namely a planar Boolean model with Poisson polygons as grains; the parameters are then λ_A and ϱ_A with $\lambda_A = 3\lambda/(2\varrho)$ and $\varrho_A = \varrho$.

There are formulae for $H_l(r)$, $H_s(r)$ and $C(r)\,(=\mathrm{Prob}\,(o \in \mathbb{P}$ and $\mathbf{r} \in \mathbb{P}))$, namely

$$H_l(r) = 1 - \exp(-6\lambda r/(\pi\varrho^2)) ,$$

$$H_s(r) = 1 - \exp\left(-\lambda r\left(\frac{24}{\pi\varrho^2} + \frac{3\pi}{\varrho} r + \frac{4}{3}\pi r^2\right)\right) ,$$

$$C(r) = 2\langle\phi\rangle - 1 + (1 - \langle\phi\rangle)^2 \exp(6\lambda \exp(-\varrho r)/(\pi\varrho^3)) .$$

For the planar section analogous characteristics can be defined which satisfy

$$C_A(r) = C(r), \quad H_{l,A}(r) = H_l(r)$$

and

$$H_{s,A}(r) = 1 - \exp\left(-\lambda_A r\left(\frac{4}{\varrho} + \pi r\right)\right) .$$

Also the spatial and planar average porosities $\langle\phi\rangle$ and $\langle\phi_A\rangle$ are equal.

Using these formulae one can estimate the parameters λ and ϱ from planar sections. Finally, N_V is given by

$$N_V = \lambda(1 - \langle\phi\rangle)(1 - 3x + x^2)$$

with $x = 6\lambda/\pi\varrho^3$. That means, for values of x between 0.382 and 2.618 (or $\langle\phi\rangle$ between 0.317 and 0.927) N_V is negative. However, for these values the connectivity of $\mathbb{P}$ differs from that of the other two models discussed above since there are isolated pores.

All these more or less plausible models make it sure that one can expect only negative values of N_V for natural permeable porous media. Any model with a positive N_V should be thus considered as hardly acceptable.

2.3. Characteristics for dilated and eroded pores

The dilated set $\mathbb{P}(d)$ is the set of all points of the space which are in a distance less or equal to d from any point of $\mathbb{P}$. It is $\mathbb{P}(0) = \mathbb{P}$ and one writes

$$\mathbb{P}(d) = \mathbb{P} \oplus B(o, d),$$

where o is the origin and $B(\mathbf{x}, r)$ the sphere of radius r centred at $\mathbf{x}$. (For two subsets X and Y of $\mathbb{R}^3$, $X \oplus Y$ is the set of all $x + y$ with $x \in X$ and $y \in Y$.) It is $\mathbb{P}(d) \supset \mathbb{P}$ and the corresponding average porosities satisfy $\langle\phi(d)\rangle > \langle\phi\rangle$.

The eroded set $\mathbb{P}(-d)$ is the set of all points $\mathbf{x}$ of $\mathbb{P}$ with the property that the sphere $B(\mathbf{x}, d)$ is completely contained in $\mathbb{P}$. One writes

$$\mathbb{P}(-d) = \mathbb{P} \ominus B(\mathbf{o}, d),$$

where $\ominus$ denotes Minkowski subtraction; for X and Y as above, $X \ominus Y = (X^c \oplus Y)^c$, where Z^c denotes the complement of the set Z.

It is $\mathbb{P}(-d) \subset \mathbb{P}$ and the corresponding average porosities satisfy $\langle\phi(-d)\rangle < \langle\phi\rangle$ for $d > 0$.

The opening and closing of $\mathbb{P}$ by $B(\mathbf{o}, d)$ are the sets

$$\mathbb{P} \circ B(\mathbf{o}, d) = (\mathbb{P} \oplus B(\mathbf{o}, d)) \ominus B(\mathbf{o}, d)$$

and

$$\mathbb{P} \bullet B(\mathbf{o}, d) = (\mathbb{P} \ominus B(\mathbf{o}, d)) \oplus B(\mathbf{o}, d)$$

which are more similar to $\mathbb{P}$ than $\mathbb{P}(d)$ or $\mathbb{P}(-d)$.

The characteristics described above become functions of d if applied to the members of the family $\{\mathbb{P}(d)\}$. In the following we describe some of these functions.

We begin with the simplest case, $\langle\phi(d)\rangle$. The average porosity $\langle\phi(d)\rangle$ of $\mathbb{P}(d)$ is monotonously increasing as a function of d. For very large positive d it tends to one, for very large negative d to zero, and it is $\langle\phi(0)\rangle = \langle\phi\rangle$. If the boundary of $\mathbb{P}$ is rough then $\langle\phi(d)\rangle$ is greater than the value for a structure with the same porosity but with a smooth boundary, particularly for small positive and negative d. Incidentally, $\langle\phi(d)\rangle$ is closely related to the spherical contact distribution function

$$1 - H_s(r) = \frac{\langle\phi(-r)\rangle}{\langle\phi\rangle} \quad \text{for } r \geqslant 0\,.$$

Mecke [10] demonstrates the use of surface area content, mean curvature and connectivity number per unit volume for positive d. Particularly interesting is the behaviour of the specific connectivity number $N_V(d)$ as a function of d. (The idea to study $N_V(d)$ for positive d goes back to Shehata [20]. For negative d, $N_V(d)$ has been used in the context of particle counting.) As explained above, for typical pore systems one expects negative values for $N_V = N_V(0)$. It is possible that for increasing d the function $N_V(d)$ decreases (becomes more negative) and then tends to zero for large positive d. For decreasing negative d it will increase and may become even positive. This results from eroding the pores, which deletes very thin pore pieces and generates dispersed small relicts of thick pore pieces. For very large negative d, $N_V(d)$ tends to zero as for large positive d. The behaviour of $N_V(d)$ for small d is of particular interest, since here big changes are possible. For example, for a dense packing of hard spheres, a small erosion changes the specific connectivity number from a negative value of $N_V(0)$ to a positive value $N_V(-d)$; conversely, a small dilation may change a system of isolated spheres with positive $N_V(0)$ to a connected system of touching bodies with a negative $N_V(d)$.

The spherical contact distribution function $H_s(r;d)$ of $\mathbb{P}(d)$ is the distribution function of the random distance from a randomly chosen point in $\mathbb{P}(d)$ to its nearest neighbour outside of $\mathbb{P}(d)$. If $\mathbb{P}$ has a rough surface (i.e., the interface between $\mathbb{P}$ and $\mathbb{M}$ is rough), then that of $\mathbb{P}(d)$ tends to be smoother. Consequently, the proportion of randomly chosen points in $\mathbb{P}(d)$ close to its boundary is smaller than for $\mathbb{P}$.

Note that for negative d one can calculate $H_s(r;d)$ using $H_s(r)$. It is

$$\langle\phi\rangle(1 - H_s(r-d)) = \langle\phi(d)\rangle(1 - H_s(r;d)),$$

$$\langle\phi(d)\rangle = \langle\phi\rangle(1 - H_s(-d))$$

and, consequently,

$$1 - H_s(r;d) = \frac{1 - H_s(r-d)}{1 - H_s(-d)}.$$

Finally, the correlation function and related characteristics can be considered in dependence of the parameter d. For purposes of comparison, we define $G_p(r;d)$ by

$$G_p(r;d) = \frac{\text{Prob}\{\text{there is any path in } \mathbb{P}(d) \text{ from } \mathbf{o} \text{ to } \mathbf{r}\} - \langle\phi\rangle^2}{\langle\phi\rangle(1 - \langle\phi\rangle)}.$$

The qualitative behaviour of G_p may heavily depend on d.

We note that also the LPT characteristics can be defined and analysed for the family $\{\mathbb{P}(d)\}$.

3. Computational aspects

This section contains some remarks on the estimation and calculation of some of the characteristics introduced in Section 2.

First two remarks on geometrical problems in the context of lattice data. Since usually the data are given in lattice data form, the calculations have to be performed in the 3D lattice. However, there are no spheres in the Euclidean sense. Instead, for the dilations and erosions 'spheres' in the lattice are used which more or less approximate true Euclidean spheres. We recommend to define their radii as the radii of Euclidean spheres of equal volume, where each voxel has unit volume. Furthermore, for lattice spheres it is not true that

$$B(\mathbf{o}, r_1) \oplus B(\mathbf{o}, r_2) = B(\mathbf{o}, r_1 + r_2)$$

and

$$B(\mathbf{o}, r_1) \ominus B(\mathbf{o}, r_2) = B(\mathbf{o}, r_1 - r_2) \quad \text{if } r_1 > r_2 .$$

This should be observed in the determination of characteristics such as $G(r; d)$ and $H_s(r; d)$.

As $G(r)$ and $H_s(r)$ (see Ref. [6]), these characteristics are estimated by ratio estimators, where numerator and denominator are porosities of $\mathbb{P}$ and transformed variants of $\mathbb{P}$ such as $\mathbb{P}(d)$. We recommend to construct first $\mathbb{P}(d)$ by dilation or erosion, respectively, and then to determine numerator and denominator based on further transformed $\mathbb{P}(d)$.

Second a remark on boundary effects is appropriate. In the estimation of $G(r)$ based on lattice data usually all pairs of lattice-points with distance r are considered. The number of those pairs where both points are in $\mathbb{P}$ divided by the number of all possible pairs is then an estimator of $\text{Prob}\{\mathbf{o} \in \mathbb{P}, \mathbf{r} \in \mathbb{P}\}$ in (2.2). As shown in Ref. [21], it improves the quality of the estimation of $G(r)$ if $\langle \phi \rangle$ is estimated based on these point pairs and not by the 'natural' estimator $n_{\mathbb{P}}/n$, where n is the number of all lattice points and $n_{\mathbb{P}}$ that of the lattice points in $\mathbb{P}$. The same idea is useful in the estimation of $G(r; d), G_p(r; d), H_s(r)$, and $H_s(r; d)$.

Finally, a remark on a stereological aspect. ('Stereology' is the art of making inference for spatial structures based on planar or linear sections [6, Chapter 11].) Often porous materials are investigated by planar sections. Then three of the characteristics introduced in Section 2.1 can be directly estimated from planar section images. If $\langle \phi_A \rangle$ is the average porosity in the section plane then clearly

$$\langle \phi \rangle = \langle \phi_A \rangle .$$

If the medium is not only homogeneous but also isotropic then also $G(r)$ and $H_l(r)$ can be estimated from the section plane. Otherwise, this is at least possible for vectors $\mathbf{r}$ parallel to the section plane.

Further characteristics can be determined by means of suitable models. The papers [14,22,23] are examples for such an approach using models which are (perhaps) physically realistic but difficult for analytical calculations. An alternative is the Boolean model as described in Section 2.3. Finally, we remark that we calculated N_V by means of the method described in Ref. [17].

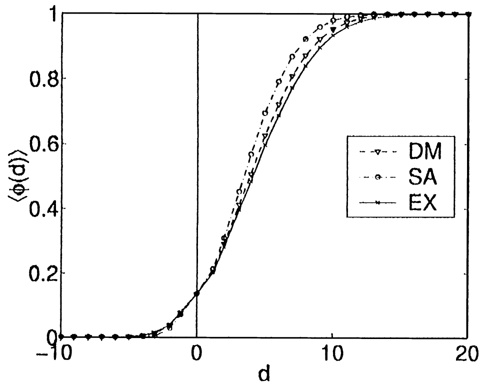

Fig. 1. $\langle\phi(d)\rangle$ for eroded (negative d) and dilated (positive d) pores.

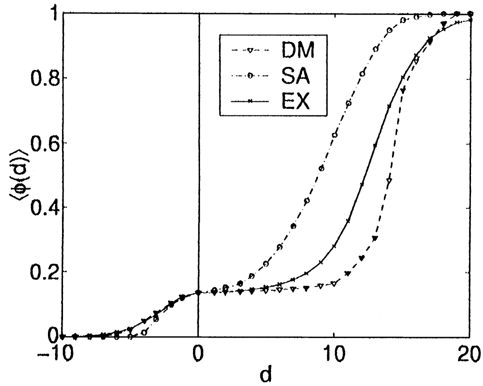

Fig. 2. $\langle\phi(d)\rangle$ for closed (negative d) and opened (positive d) pores.

4. Results and discussion

4.1. Porosity of $\mathbb{P}(d)$

Fig. 1 shows $\langle\phi(d)\rangle$ for the sample of Fontainebleau sandstone (=EX), for the diagenetic model (=DM) and for the simulated annealing (=SA) model, all described in Ref. [1]. The three curves go through the point (0; 0.135) corresponding to $d=0$ and $\langle\phi\rangle=0.135$; the simulations were carried out such that the porosity is practically equal in all three cases. For positive d (dilation of $\mathbb{P}$) we see considerable differences. For all positive values of d, the value of $\langle\phi(d)\rangle$ for SA is greater than those for EX and DM. An explanation is the greater roughness of the pore surfaces in SA in comparison to EX and DM. Clear geometrical differences become obvious in Fig. 2 which shows $\langle\phi(d)\rangle$ for opened and closed $\mathbb{P}$: Since the opening of a sphere is again a sphere, it is natural that $\langle\phi(d)\rangle$ is nearly constant in a large interval of positive d for DM, a model constructed by nonintersecting spheres.

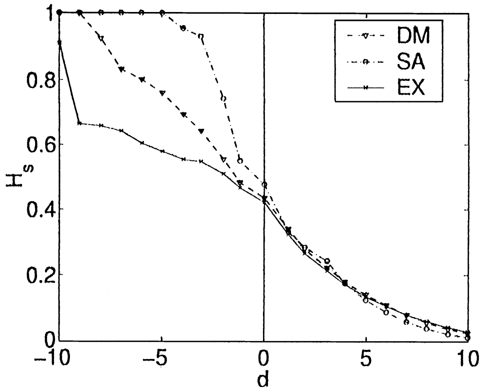

Fig. 3. Spherical contact distribution function values $H_s(r;d)$ for $r = 1.2$.

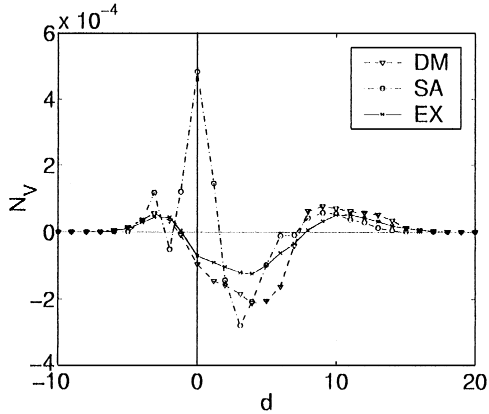

Fig. 4. $N_V(d)$ for eroded (negative d) and dilated (positive d) pores.

4.2. Spherical contact distributions

Fig. 3 shows the values of the spherical contact distribution function $H_s(r;d)$ for $r = 1.2$ for the samples EX, DM and SA. The value $r = 1.2$ is the smallest possible positive value of r in the lattice geometry. This small value is helpful for detecting micro-irregularities and roughnesses of the surface of $\mathbb{P}$. The three curves show big differences for negative d (erosion of $\mathbb{P}$ or dilation of $\mathbb{M}$) for the three samples EX, DM and SA. Obviously, they result from the different degrees of roughness of the surface of $\mathbb{P}$. EX has the smoothest surface leading to the smallest values of $H_s(1;d)$.

4.3. Specific Euler characteristic $N_V(d)$

Fig. 4 shows the curves for $N_V(d)$ for the three samples. Already for $d = 0$, i.e., for the connectivity numbers N_V of the original structures, big differences are visible. N_V is

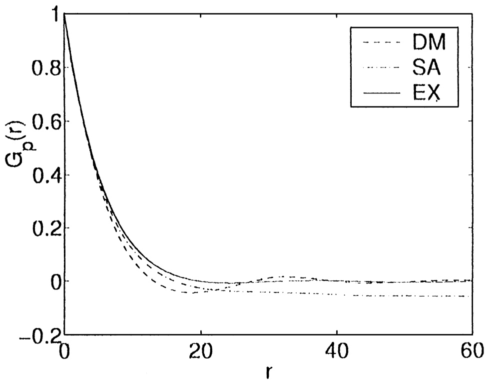

Fig. 5. Percolation correlation functions $G_p(r)$ for all three samples.

clearly positive for SA, what points to the existence of many isolated islands or holes of $\mathbb{M}$ and to a low degree of connectivity in comparison to EX; the positive value of N_V for SA shows that this model is topologically unrealistic. The curve for SA shows for increasing positive d (dilation of $\mathbb{P}$) a decrease of $N_V(d)$ towards negative values. This results probably from annihilating of small islands of $\mathbb{M}$. Also for increasing negative d (erosion of $\mathbb{P}$) $N_V(d)$ decreases. This points to vanishing small and thin pores.

The curve for DM is smoother than the curve for SA. It starts at $d = 0$ with a negative value close to zero. For increasing positive d, $N_V(d)$ decreases and takes values 'very negative' probably because of the generation of 'holes' in the set of pores in the vicinity of pores close together. For increasing negative d, $N_V(d)$ increases and becomes positive because some parts of $\mathbb{P}$ become isolated.

The curve for EX shows that erosion of $\mathbb{P}$ eliminates some small pores, while dilation closes matrix regions between pores close together. The three curves emphasize the differences in the topological structure which are already expressed by the different values of $N_V = N_V(0)$. By the way, for the fourth stochastic model considered in Ref. [1], Gaussian field, we obtained a positive value of N_V as for SA. Therefore, we did not further analyse this sample.

4.4. Correlation functions of $\mathbb{P}(d)$

Fig. 5 shows the pair connectedness functions for the three samples. The numerical differences are not big but clear enough. In particular, the curve for DM indicates a certain correlation structure which is related to the construction of this sample as a packing of hard spheres. For the dilated and eroded pores the differences in the correlation functions become still clearer (see Fig. 6).

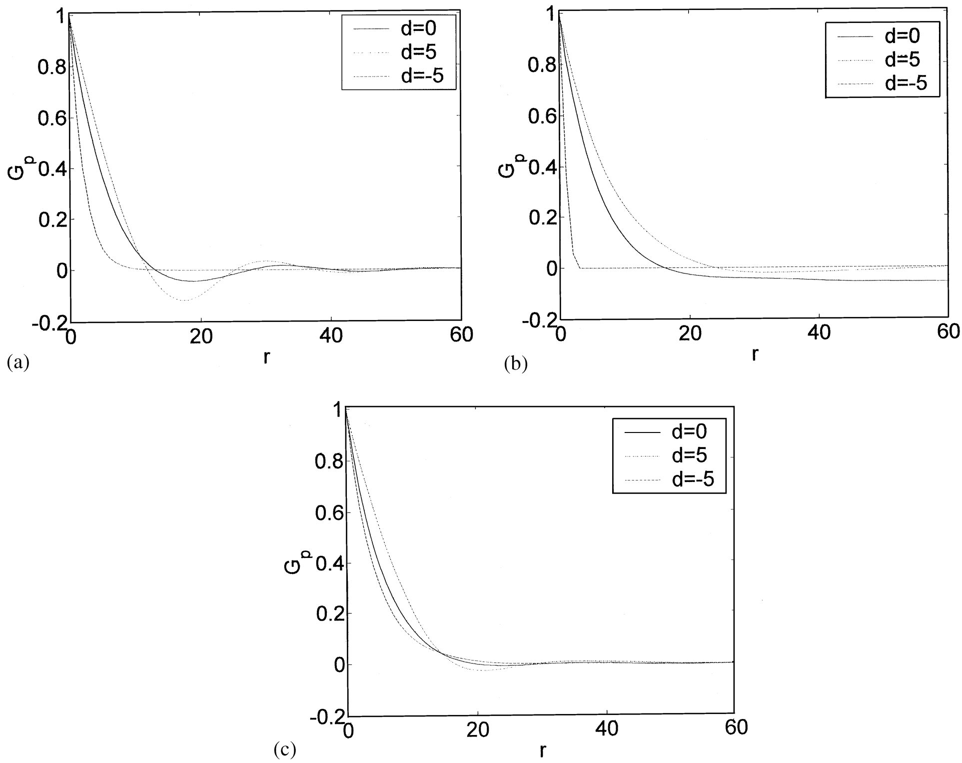

Fig. 6. Percolation correlation functions $G_p(r;d)$ for eroded (negative d) and dilated (positive d) pores for all three samples DM, SA and EX.

4.5. Poisson polyhedra formulas

The linear contact distribution function of EX is quite similar to an exponential distribution function (with parameter 0.147), as expected for any Boolean model with convex grains. Therefore, we estimated the parameters λ and ϱ from the formulae for porosity and $H_l(r)$. We obtained

$$0.135 = 1 - \exp(-6\lambda/(\pi\varrho^3))$$

and $6\lambda/\pi\varrho^2$=0.147, which yields λ=0.0785 und ϱ=1.01. Unfortunately, these parameters are not in agreement with our estimates of $G(r)$ and $H_s(r)$. We conclude that the Boolean model is not a good model for EX.

5. Conclusions

The statistical analysis of the present paper has shown clear differences in topology and structure for the three samples in agreement with the findings in Ref. [1]. These differences are particularly obvious for the specific connectivity numbers and the spherical contact distribution functions. This shows that the stochastic models yielding DM and SA need further improvement.

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