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Exact and approximate calculations for the conductivity of sandstones

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Abstract

We analyze a three-dimensional pore space reconstruction of Fontainebleau sandstone and calculate from it the effective conductivity using local porosity theory. We compare this result with an exact calculation of the effective conductivity that solves directly the disordered Laplace equation. The prediction of local porosity theory is in good quantitative agreement with the exact result.

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1. Introduction

A quantitative prediction of the effective conductivity for heterogeneous disordered media from incomplete knowledge of the microstructure is a difficult and unsolved problem dating back to early work by Maxwell and others [1–3]. Advances in this direction would be of great importance for many applications ranging from destruction free testing to tomographic imaging methods. The primary obstacle is the fact that the complexity of the microstructure cannot be captured by one or two macroscopic geometric observables such as volume fraction or specific internal surface area (see Ref. [4] for a discussion of these matters). In recent years a novel characterization for the microgeometry of arbitrary porous media, called Local Porosity Theory (LPT), was proposed [4–7]. The key quantities in LPT are local porosity distributions and

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local percolation probabilities. Local porosity distributions provide information about porosity fluctuations, and local percolation probabilities describe connectivity fluctuations.

LPT can be conveniently used to obtain transport properties. To test this method of the calculation of transport parameters we have carried out a detailed analysis on Fontainebleau sandstone.

2. Geometrical characterization

In this section we briefly introduce the functions needed for calculating the effective conductivity, namely, the local porosity distribution $\mu(\phi, L)$ and the local percolation probability $\lambda(\phi, L)$. For a more detailed discussion the reader is referred to [4].

The local porosity $\phi(\mathbf{r}, L)$ of a two phase porous medium is defined as the volume fraction of pore space in a cubic subsample $\mathbb{K}$ of sidelength L of the total sample. The subsample $\mathbb{K}$ is called a measurement cell. In a discretized sample with dimensions $M_1 \times M_2 \times M_3$ there are $m = \prod_i M_i - L + 1$ possibilities of placing this measurement cell within the sample. Ideally the measurement cells should be nonoverlapping [5,7] but here we use overlapping cells to improve the statistics. Evaluating the local porosity for each possible placement, we can define an approximate local porosity distribution $\mu(\phi, L)$ in the following way

$$\mu(\phi, L) = \frac{1}{m} \sum_{\mathbf{r}} \delta(\phi - \phi(\mathbf{r}, L)) . \tag{1}$$

The second important quantity characterizing the geometry of porous media is the connectivity. We introduce $\Lambda(\mathbf{r}, L)$ as an indicator for percolation, *i.e.* its value is 1 if all sides of the cubic measurement cell at point $\mathbf{r}$ are connected by a closed path inside the pore space with their opposite side, and it is 0 otherwise. From this we define the approximate local percolation probability as

$$\lambda(\phi, L) = \frac{\sum_{\mathbf{r}} \Lambda(\mathbf{r}, L) \delta_{\phi\phi(\mathbf{r}, L)}}{\sum_{\mathbf{r}} \delta_{\phi\phi(\mathbf{r}, L)}} . \tag{2}$$

3. Effective conductivity

In an heterogeneous medium, the electrical conductivity problem is described by the disordered Laplace equation

$$\nabla \cdot \sigma(\mathbf{r}) \nabla U(\mathbf{r}) = 0, \quad \mathbf{r} \in \mathbb{G} \ \ (\mathbb{G} = \mathbb{P},\ \mathbb{M}) , \tag{3}$$

where $\sigma(\mathbf{r}) = \sigma_{\mathbb{P}} \chi_{\mathbb{P}}(\mathbf{r}) + \sigma_{\mathbb{M}} \chi_{\mathbb{M}}(\mathbf{r})$. Here $\sigma_{\mathbb{P}}$ and $\sigma_{\mathbb{M}}$ indicate the conductivity of pore space and matrix space, respectively. $U(\mathbf{r})$ is the electrical potential at position $\mathbf{r}$.

The isotropic effective conductivity σ_{eff} of an inhomogeneous medium is defined through an averaged constitutive relation

$$\langle \mathbf{J}(\mathbf{r}) \rangle = \sigma_{\mathit{eff}} \langle \mathbf{E}(\mathbf{r}) \rangle , \tag{4}$$

where $\mathbf{J}(\mathbf{r})$ denotes the current density, and $\mathbf{E}(\mathbf{r}) = -\nabla U(\mathbf{r})$ is the electric field. The angular brackets indicate an ensemble average over the microstructure.

We have determined effective electrical conductivities σ_{eff} in two ways. First, we solve Eq. (3) numerically [8], and calculate $\mathbf{E}(\mathbf{r}) = -\nabla U(\mathbf{r})$. On the stochastic internal boundary $\partial\mathbb{P}$ we impose continuity for normal components of the current density $\mathbf{J}(\mathbf{r})$ and also for tangential components of the electric field $\mathbf{E}(\mathbf{r})$. We apply a potential gradient across the sample and apply no-flow boundary conditions on the faces parallel to the applied gradient. Then, σ_{eff} is obtained from Eq. (4) (see also [9,10]).

Second, we determine the effective conductivities of heterogeneous media applying LPT [4–6]. For a cubic lattice with unit length L the LPT result for the effective conductivity σ_{eff} reads [9,10]

$$\int_0^1 \mu(\phi, L)\lambda(\phi, L)\frac{\sigma_{CS}(\sigma_{\mathbb{P}}, \sigma_{\mathbb{M}}; 1-\phi) - \sigma_{eff}}{\sigma_{CS}(\sigma_{\mathbb{P}}, \sigma_{\mathbb{M}}; 1-\phi) + 2\sigma_{eff}}d\phi + \int_0^1 \mu(\phi, L)[1-\lambda(\phi, L)]\frac{\sigma_{CS}(\sigma_{\mathbb{M}}, \sigma_{\mathbb{P}}; \phi) - \sigma_{eff}}{\sigma_{CS}(\sigma_{\mathbb{M}}, \sigma_{\mathbb{P}}; \phi) + 2\sigma_{eff}}d\phi = 0\,, \tag{5}$$

where

$$\sigma_{CS}(\sigma_{\mathbb{M}}, \sigma_{\mathbb{P}}; \phi) = \sigma_{\mathbb{M}}\left[\frac{\sigma_{\mathbb{P}} + 2\sigma_{\mathbb{M}} + 2\,\phi(\sigma_{\mathbb{P}} - \sigma_{\mathbb{M}})}{\sigma_{\mathbb{P}} + 2\sigma_{\mathbb{M}} - \phi(\sigma_{\mathbb{P}} - \sigma_{\mathbb{M}})}\right] \tag{6}$$

denotes the conductivity of the rock-coated spherical water pore grain, and $\sigma_{CS}(\sigma_{\mathbb{P}}, \sigma_{\mathbb{M}}; 1-\phi)$ is the conductivity of the water-coated spherical rock grain.

4. Results

Fig. 1 shows a cross section through the Fontainebleau sandstone. Pore space is indicated by black, and matrix space by grey. The total sample has dimensions $128 \times 128 \times 128$ and the resolution is $a = 7.5$ µm.

The local porosity distributions $\mu(\phi, L)$ for three different measurement cells with sidelength $L = 18a, 28a, 40a$ are plotted in Fig. 2. The curve for $L = 18a$ shows a large peak at $\phi = 0$ whose value is $\mu(0, 18) = 23.24$ indicating that there are still measurement cells lying completely in matrix space. The smallest value of L for which $\mu(0, L)$ and $\mu(1, L)$ both become zero is called $L^\star$ and it is given by $L^\star = 28a$. $L^\star$ may be considered as a measure for the size of the largest grain. For values of L larger than $L^\star$ the curves become more and more Gaussian with the maximum at the mean value of the porosity $\bar{\phi} = 0.1208$. The standard deviation decreases with increasing L.

In Fig. 3 the local percolation probabilities $\lambda(\phi, L)$ are plotted for the same values L as in the above plot of μ. With increasing L the curves tend to a step function. For $L = 40a$ already 50% of the measurement cells with $\phi \approx 0.1$ are percolating. In comparison with the percolation threshold of $\phi_c = 0.31$ for a three dimensional cubic grid this shows the high degree of connectivity in natural sandstones [11].

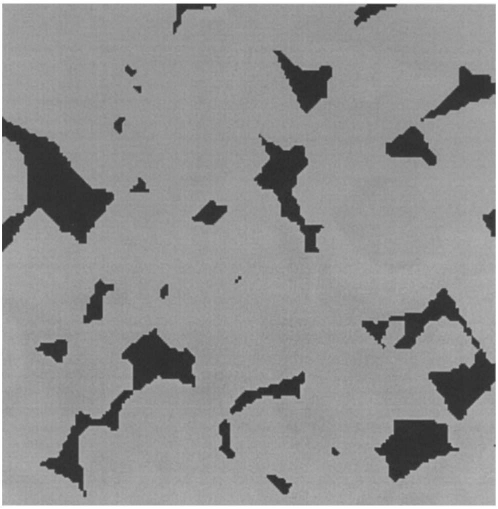

Fig. 1. Cross section through a Fontainebleau sandstone. The resolution of the image is $a = 7.5$ μm. Pore space is coloured black, matrix space grey.

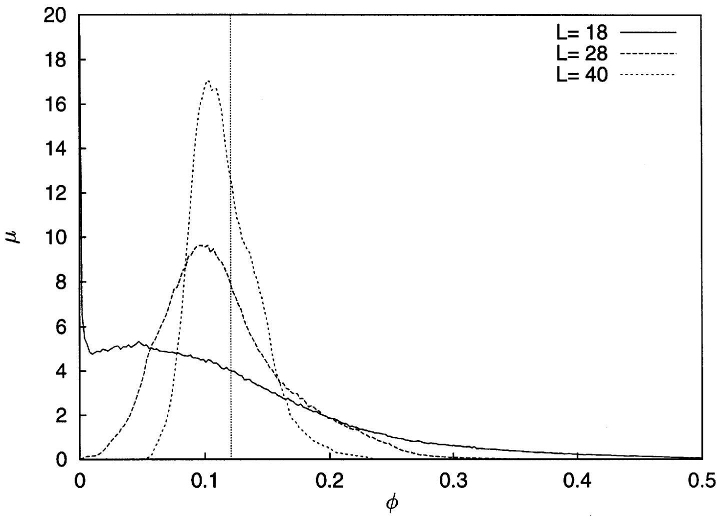

Fig. 2. Local porosity distribution of a Fontainebleau sandstone for three different values of L. $L = L^{\star} = 28a$ is a characteristic length of the local porosity analysis at which the singular component of the local porosity distribution vanishes. The dotted line at $\phi = 0.121$ indicates the mean value of the porosity.

In Fig. 4 we present the effective electrical conductivities σ_{eff} obtained by LPT (solid line with circle) and the result of the exact calculation (horizontal dotted line) for given conductivities $\sigma_{\mathbb{P}} = 1.0$ S/m and $\sigma_{\mathbb{M}} = 0$. It can be seen in Fig. 4 that the exact result of σ_{eff} intersects the prediction of σ_{eff} obtained by Eq. (5) at $L_e = 215$ μm. This length $L_e = 215$ μm was first introduced in [12] as the so called experimental length.

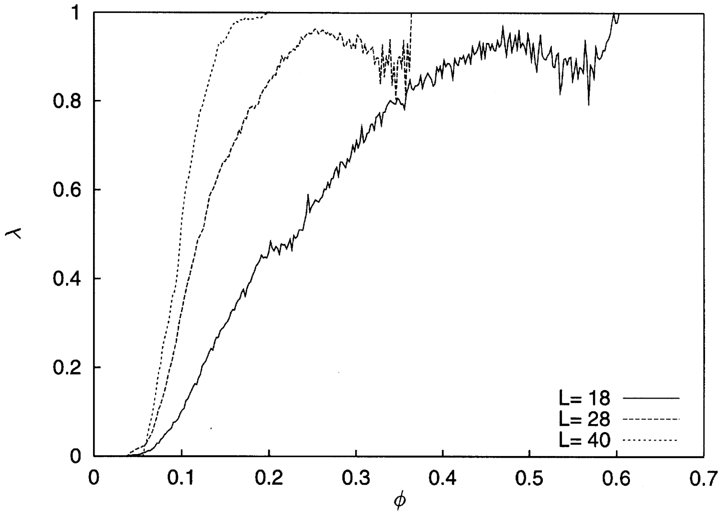

Fig. 3. Local percolation probability for the same values of L as in Fig. 2.

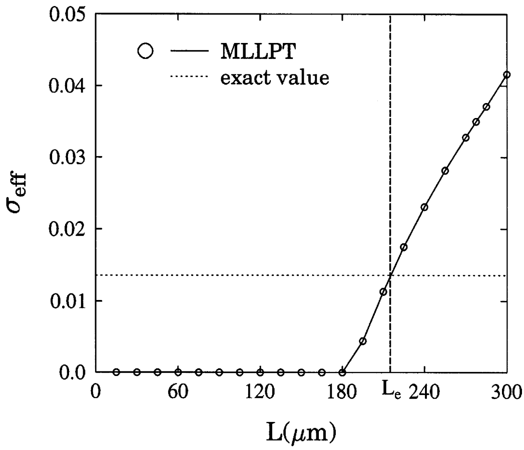

Fig. 4. Comparison between σ_{eff} predicted by Eq. (5) and the exact solution. The circle represent the solution of Eq. (5) for different L, the solid line is a guide to the eye. The solid line approaches $\sigma_{eff}(\infty) = 0.0839$ asymptotically for large L. The horizontal line represents the exact value of σ_{eff} obtained by solving Eq. (3) directly. The vertical line indicates the experimental length $L_e = 215$ μm.

In the present case it is approximately two times the correlation length $\xi \approx 100$ μm. This result ensures that the local geometries are statistically independent as required in mean field approximations. L_e may be interpreted as a characteristic length for the so called representative elementary volume in continuum theories [13]. It is remarkable that $L_e \approx L^\star = 210$ μm, but we believe this to be a coincidence.

It should be emphasized that the prediction of the effective electrical conductivity using the conventional Bruggeman's effective medium approximation [14,15] or the Maxwell–Garnett approximation [15,16] are both zero. This follows from the fact that the porosity $\bar{\phi}=0.1208$ is below the mean field percolation threshold $\bar{\phi}_{cBr}=1/3$ for the Bruggeman's effective medium equation, and the effective electrical conductivity σ_{eff} reflects the property of the matrix phase (as host medium) for the Maxwell-Garnett theory. (See also [9].)

Bruggeman's effective medium approximation (EMA) is obtained from Eq. (5) in the limit $L \to 0$. In this limit the cells become strongly correlated and hence the assumption of statistical independence underlying the EMA becomes invalid. On the other hand the limit $L \to \infty$ yields $\sigma_{eff}(\infty) = 2\bar{\phi}\sigma_{\mathbb{P}}/(3 - \bar{\phi}) = 0.0839$ and this is expected to be too high. It follows that the exact solution should fall within the bounds $\sigma_{eff}(0) = 0 \leqslant \sigma_{exact} \leqslant \sigma_{eff}(\infty) = 0.0839$. Fig. 4 shows that this expectation is indeed correct.

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