Fractals, Vol. 3, No. 1 (1995) 211–216

EXACT SOLUTIONS FOR A CLASS OF FRACTAL TIME RANDOM WALKS

R. HILFER*
International School for Advanced Studies, Via Beirut 2–4, 34013 Triest, Italy
Institut für Physik, Universität Mainz, 55099 Mainz, Germany

Received October 20, 1994; Accepted November 10, 1994

Abstract

Fractal time random walks with generalized Mittag-Leffler functions as waiting time densities are studied. This class of fractal time processes is characterized by a dynamical critical exponent $0 < \omega \leq 1$, and is equivalently described by a fractional master equation with time derivative of noninteger order ω. Exact Greens functions corresponding to fractional diffusion are obtained using Mellin transform techniques. The Greens functions are expressible in terms of general H-functions. For $\omega < 1$ they are singular at the origin and exhibit a stretched Gaussian form at infinity. Changing the order ω interpolates smoothly between ordinary diffusion $\omega = 1$ and completely localized behavior in the $\omega \to 0$ limit.

1. INTRODUCTION AND DEFINITION OF THE RANDOM WALKS

A large number of physical processes display algebraic or fractal time behavior (see Refs. 1–7 for reviews). Recently it was shown that fractal time behavior follows very generally from the theory of critical phenomena[8,9] as well as from ergodic theory.[10] It was demonstrated in these papers that time derivatives of noninteger order arise necessarily when discussing renormalized time evolutions. Fractional time derivatives had previously been introduced heuristically in a variety of different contexts.[11–20] A connection between fractional time derivatives and the continuous time random walk theory of fractal time behavior[21–25] was established in Ref. 26.

My objective in this paper is to find the fundamental solutions for fractional master equations

*Present address: Institute of Physics, University of Oslo, P. O. Box 1048, 0316 Oslo, Norway.

corresponding to fractal time random walks with finite spatial moments.[24–26] The purpose of the paper is to show that the special class of random walks treated here plays a particularly important role in the theory of fractal time processes by virtue of their universality[8–10] as well as their exact solvability.

Define $G(\mathbf{r}, t)$ to be the conditional probability density to find a random walker at the position $\mathbf{r} \in \mathbb{R}^d$ at time t if it started from the origin $\mathbf{r} = 0$ at time $t = 0$. The basic equation for a space time decoupled continuous time random walk reads[25]:

$$G(\mathbf{r}, t) = \delta_{\mathbf{r}0}\Phi(t) + \int_0^t \psi(t-t') \times \sum_{\mathbf{r}'} \lambda(\mathbf{r}-\mathbf{r}')G(\mathbf{r}', t')dt' \tag{1.1}$$

where $\lambda(\mathbf{r})$ is the probability density for a displacement of the walker by a vector $\mathbf{r}$ in each single step, and $\psi(t)$ is the waiting time distribution giving the probability density for the time interval t between two consecutive steps. The transition probabilities obey $\sum_{\mathbf{r}} \lambda(\mathbf{r}) = 1$. The function $\Phi(t)$ is the survival probability at the initial position which is related to the waiting time distribution through:

$$\Phi(t) = 1 - \int_0^t \psi(t')dt' . \tag{1.2}$$

The class of fractal time random walks discussed in this paper is defined by the waiting time densities[26]:

$$\psi(t) = \frac{t^{\omega-1}}{\tau^{\omega}} E_{\omega,\omega}(-(t/\tau)^{\omega}) . \tag{1.3}$$

where (with $\alpha, \beta > 0$)

$$E_{\alpha,\beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)} \tag{1.4}$$

is the generalized Mittag-Leffler function[27] and $\Gamma(x)$ denotes the Gamma function. The constant ω was shown to obey $0 < \omega \leq 1$ on very general grounds,[8–10] and it plays the role of a characteristic dynamical exponent, while τ is a characteristic time constant.

Given a continuous time random walk defined by a Mittag-Leffler waiting time density and transition probabilities $\lambda(\mathbf{r})$ it was shown recently[26] that $G(\mathbf{r}, t)$ solves simultaneously a fractional master equation written formally as:

$$\frac{\partial^{\omega}}{\partial t^{\omega}} G(\mathbf{r}, t) = \sum_{\mathbf{r}'} w(\mathbf{r}-\mathbf{r}')G(\mathbf{r}', t) \tag{1.5}$$

with initial condition $G(\mathbf{r}, 0) = \delta_{\mathbf{r}0}$. The relation with Eq. (1.1) emerges if the fractional differential equation (1.5) is rewritten as a fractional integral equation:

$$G(\mathbf{r}, t) = \delta_{\mathbf{r}0} + \frac{1}{\Gamma(\omega)} \int_0^t (t-t')^{\omega-1} \times \sum_{\mathbf{r}'} w(\mathbf{r}-\mathbf{r}')G(\mathbf{r}', t')dt' \tag{1.6}$$

where the initial condition $G(\mathbf{r}, 0) = \delta_{\mathbf{r}0}$ has been incorporated. The fractional transition rates $w(\mathbf{r})$ measure the propensity for a displacement $\mathbf{r}$ in units of $(1/\text{time})^{\omega}$. Their Fourier transform defined as $w(\mathbf{q}) = \sum_{\mathbf{r}} e^{i\mathbf{q}\cdot\mathbf{r}} w(\mathbf{r})$ is related to the Fourier transform of the transition probabilities by:

$$\lambda(\mathbf{q}) = 1 + \tau^{\omega} w(\mathbf{q}) \tag{1.7}$$

where τ and ω are the same constants as in Eq. (1.3).

Let the spatial transition probabilities $\lambda(\mathbf{q})$ have a finite variance σ^2 characterizing the spatial extent of a single step. Thus

$$\lambda(\mathbf{q}) = 1 - \sigma^2 \mathbf{q}^2/2 \tag{1.8}$$

which implies $w(\mathbf{q}) = -(\sigma^2\mathbf{q}^2)/(2\tau^{\omega})$ for the fractional transition rates and identifies

$$D = \frac{\sigma^2}{2\tau^{\omega}} \tag{1.9}$$

as the fractional diffusion constant. Equation (1.8) completes the definition of the class of fractal time random walks considered in this paper. Each random walk in this class is characterized by two numbers $D \geq 0$ and $0 < \omega \leq 1$ corresponding respectively to a generalized fractional diffusion constant and a dynamical critical exponent.

2. FUNDAMENTAL SOLUTIONS

With the definitions and notations given in the introduction, the Fourier-Laplace transform of $G(\mathbf{r}, t)$ for the class of fractal time processes described equivalently by Eq. (1.1) or Eq. (1.6) is obtained as:

$$G(\mathbf{q}, u) = \frac{u^{\omega-1}}{D\mathbf{q}^2 + u^{\omega}} \tag{2.1}$$

where D is the fractional diffusion constant and $\mathbf{q} = (q_1, \ldots, q_d) \in \mathbb{R}^d$. This form of $G(\mathbf{q}, u)$ arises asymptotically for continuous time random walks whose waiting time densities have long time tails $\psi(t) \propto t^{-\omega-1}$ and finite spatial moments.[24,25] For Mittag-Leffler densities $\psi(t)$ as in Eq. (1.3) the form (2.1) is not only asymptotic but exact. Using the notation $r = |\mathbf{x}| = \sqrt{x_1^2 + \cdots + x_d^2}$ Fourier inversion results in:

$$G(r, u) = (2\pi)^{-d/2} r^{1-(d/2)} D^{-1/2} \times u^{(d\omega/4)+(\omega/2)-1} K_{\frac{d-2}{2}}\left(\frac{u^{\omega/2} r}{D^{1/2}}\right) \tag{2.2}$$

where $K_\nu(x)$ denotes the modified Bessel function. The Laplace transform $\mathcal{L}\{f(t)\}(u)$ of $f(t)$ is related to its Mellin transform $\mathcal{M}\{f(t)\}(s)$ through:

$$\mathcal{M}\{\mathcal{L}\{f(t)\}(u)\}(s) = \Gamma(s)\mathcal{M}\{f(t)\}(1-s) \tag{2.3}$$

where $\Gamma(x)$ is the Gamma function. Mellin transformation gives:

$$\begin{aligned}\mathcal{M}\{G(r, u)\}(s) &= (2\pi)^{-d/2} r^{1-(d/2)} \mathcal{M}\{u^a K_\nu(ru^b)\}(s) \\ &= (2\pi)^{-d/2} r^{1-(d/2)} \frac{2}{\omega} r^{-(s+a)/b} \\ &\quad \times \mathcal{M}\{K_\nu(u)\}((s+a)/b)\end{aligned} \tag{2.4}$$

where $a = \frac{\omega}{2}(\frac{d}{2}+1-\frac{2}{\omega})$, $\nu = (d-2)/2$, $b = \omega/2$, and D has been set to unity temporarily. Employing the fact that:

$$\mathcal{M}\{K_\nu(x)\}(s) = s^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right) \tag{2.5}$$

yields the result:

$$\begin{aligned}G(r, s) &= \mathcal{M}\{G(r, t)\}(s) \\ &= (r^2\pi)^{-d/2} \frac{1}{\omega}\left(\frac{2}{r}\right)^{-2s/\omega} \\ &\quad \times \frac{\Gamma\left(\frac{d}{2} - \frac{s}{\omega}\right)\Gamma\left(1 - \frac{s}{\omega}\right)}{\Gamma(1-s)}\end{aligned} \tag{2.6}$$

for the Mellin transform. Comparison with the definition of the H-function in the appendix gives:

$$G(r, t) = (r^2\pi)^{-d/2} \times H_{21}^{02}\left(\frac{4Dt^\omega}{r^2}\,\middle|\, \begin{matrix}\left(1-\frac{d}{2}, 1\right) & (0, 1) \\ (0, \omega) & \end{matrix}\right) \tag{2.7}$$

which by virtue of general H-function relations may be rewritten as:

$$G(r, t) = (r^2\pi)^{-d/2} H_{12}^{20}\left(\frac{r^2}{4Dt^\omega}\,\middle|\, \begin{matrix} (1, \omega) & \\ (d/2, 1) & (1, 1)\end{matrix}\right) \tag{2.8}$$

or equivalently

$$G(r, t) = (4\pi Dt^\omega)^{-d/2} H_{12}^{20} \times \left(\frac{r^2}{4Dt^\omega}\,\middle|\, \begin{matrix}\left(1-\frac{d\omega}{2}, \omega\right) & \\ (0, 1) & \left(1-\frac{d}{2}, 1\right)\end{matrix}\right) \tag{2.9}$$

where the fractional diffusion constant D has been restored.

Using similar methods as above it is possible to invert the Laplace transformation in Eq. (2.1) directly to obtain the result:

$$G(\mathbf{q}, t) = E_\omega(-Dt^\omega \mathbf{q}^2) \tag{2.10}$$

where $E_\omega(x) = E_{\omega,1}(x)$ is the Mittag-Leffler function.[27] Differentiation now yields the mean square displacement:

$$\begin{aligned}\langle \mathbf{r}^2 \rangle(t) &= \sum_{\mathbf{r}} \mathbf{r}^2 G(\mathbf{r}, t) \\ &= -(\nabla_\mathbf{q}^2 G(\mathbf{q}, t)|_{\mathbf{q}=0} = \frac{2dDt^\omega}{\Gamma(\omega+1)}\end{aligned} \tag{2.11}$$

which exhibits subdiffusive behavior for $\omega < 1$.

3. DISCUSSION

The results of the previous section are best discussed by comparing the cases $\omega = 1$, $0 < \omega < 1$ and $\omega \to 0$ with the help of Table 1. The first row in Table 1 recalls the different character of the infinitesimal generator $\mathbb{A}$ of time transformations entering the fractional master equation.[10] In particular in the limit $\omega \to 0$, the generator becomes the identity. It must be emphasized that the limit $\omega \to 0$ is rather singular as indicated by the fact that $\psi(t)$ approaches a nonnormalizable function. The results for G in the last row are obtained by performing the limit in Fourier-Laplace space. The waiting time density $\psi(t)$ is deformed from an exponential form for $\omega = 1$ into t^{-1} for $\omega \to 0$. The function $G(\mathbf{q}, t)$ crosses over from a Gaussian to a Lorentzian, while in direct space $G(\mathbf{r}, t)$ changes

Table 1 Overview of results for the quantities $\psi(t)$, D, $\langle \mathbf{r}^2\rangle(t)$ and $G(r, t)$ defined in the text characterizing a class of fractal time random walks in d dimensions. The case $\omega = 1$ corresponds to ordinary diffusion or propagation, the case $0 < \omega < 1$ to fractional propagation or fractional localization, and the limit $\omega \to 0$ to complete localization. $\mathbb{A}$ denotes the infinitesimal generator of time transformations and $\mathbb{1}$ is the identity. The rows labeled $\frac{r^2}{t^\omega} \to 0$ and $\frac{r^2}{t^\omega} \to \infty$ give the asymptotic behavior of $G(r, t)$ for the indicated limits of the scaling variable $\frac{r^2}{t^\omega}$

	Propagation $\omega = 1$	Fractional Propagation/Localization $0 < \omega < 1$	Localization $\omega \to 0$
$\mathbb{A}$	$\frac{d}{dt}$	$\frac{d^\omega}{dt^\omega}$	$\mathbb{A} \to \mathbb{1}$
$\psi(t)$	$\tau^{-1}\exp(-t/\tau)$	$\tau^{-\omega}t^{\omega-1}E_{\omega,\omega}(-(t/\tau)^\omega)$	$\psi(t) \to t^{-1}$
D	σ^2/τ	σ^2/τ^ω	$D \to \sigma^2$
$G(\mathbf{q}, t)$	$\exp(-Dt\mathbf{q}^2)$	$E_\omega(-Dt\mathbf{q}^2)$	$G(\mathbf{q}, t) \to (1 + D\mathbf{q}^2)^{-1}$
$\langle \mathbf{r}^2\rangle(t)$	$2dDt$	$\frac{2dDt^\omega}{\Gamma(\omega+1)}$	$\langle \mathbf{r}^2\rangle(t) \to 2dD$
$G(r, t)$	$\frac{\exp(-r^2/(4Dt))}{(4\pi Dt)^{-d/2}}$	$(r^2\pi)^{-d/2}H^{20}_{12}\left(\frac{r^2}{4Dt^\omega}\middle\vert\begin{matrix}(1,\omega) & \\ (d/2,1) & (1,1)\end{matrix}\right)$	$\frac{r^{1-(d/2)}}{D^{1/2}(2\pi)^{d/2}}K_{\frac{d-2}{d}}\left(\frac{r}{D^{1/2}}\right)$
$\frac{r^2}{t^\omega} \to 0$	$t^{-d/2}$	$t^{-\omega}r^{2-d}$ $(d > 2)$	$r^{(2/d)-(d/2)}$ $(d > 2)$
$\frac{r^2}{t^\omega} \to \infty$	$\exp\left(-\frac{r^2}{4Dt}\right)$	$\exp\left(-(2-\omega)\left(\frac{\omega^\omega r^2}{4Dt^\omega}\right)^{1/(2-\omega)}\right)$	$\exp\left(-\frac{r}{\sqrt{D}}\right)$

from a Gaussian to a Bessel function. As ω is varied from 1 to 0, the random walk changes from freely propagating to subdiffusive and finally to completely localized behavior as evidenced by the mean square displacement. The same conclusion is obtained from the last two rows giving the asymptotic behavior of $G(r, t)$ for small and large values of the scaling variable $x = r^2/t^\omega$. For large values of x the Gaussian behavior changes into a stretched Gaussian with stretching exponent $1/(2-\omega)$. The stretched Gaussian finally becomes simply exponential as $\omega \to 0$ is approached. Together with the disappearance of t from all formulas and the constancy of the mean square displacement, this implies that the random walk becomes exponentially localized. The localization length $\xi = \sqrt{D} = \sigma$ is the standard deviation of a single step. The localization is also apparent in the limit $x \to 0$. This limit arises for example as the limit $r \to 0$ at fixed time t. In this limit the probability density $G(r, t)$ diverges for $\omega < 1$ and $d > 2$.

Note that the stretched Gaussian behavior in the limit $x \to \infty$ is obtained also from the known asymptotic expressions in terms of stable laws[21–24] which are valid for an even wider class of fractal time walks.

In summary the class of fractal time random walks considered in this paper appears universally as shown in Refs. 8–10 and is exactly solvable as was shown here. As the dynamical exponent ω is varied from 1 to 0, the qualitative behavior of the random walk changes from propagating ($\omega = 1$), through a regime of fractional propagation or fractional localization ($0 < \omega < 1$), to being completely localized in the $\omega \to 0$ limit.

ACKNOWLEDGMENTS

The author is grateful to Prof. Dr. E. Tosatti for his hospitality in Trieste, and to the Commission of the

European Communities (ERBCHBGCT920180) for financial support.

REFERENCES

1. E. Montroll and H. Scher, "Random walks on lattices IV. Continuous-time walks and influence of absorbing boundaries," *J. Stat. Phys.* **9**, 101 (1973).
2. E. Montroll and B. West, "On an enriched collection of stochastic processes," in *Fluctuation Phenomena*, eds. E. Montroll and J. Lebowitz (North Holland Publ. Co., Amsterdam, 1979), p. 61.
3. B. Mandelbrot, *The Fractal Geometry of Nature* (Freeman, San Francisco, 1982).
4. G. Weiss and R. Rubin, "Random walks: Theory and selected applications," *Adv. Chem. Phys.* **52**, 363 (1983).
5. A. Jonscher, *Dielectric Relaxation in Solids* (Chelsea Dielectrics Press, London, 1983).
6. A. Blumen, J. Klafter and G. Zumofen, "Models of reaction dynamics in glasses," in *Optical Spectroscopy of Glasses*, ed. I. Zschokke (Riedel, Dordrecht, 1986).
7. M. Shlesinger, "Fractal time in condensed matter," *Ann. Rev. Phys. Chem.* **39**, 269 (1988).
8. R. Hilfer, "Classification theory for anequilibrium phase transitions," *Phys. Rev.* **E48**, 2466 (1993).
9. R. Hilfer, "On a new class of phase transitions," in *Random Magnetism and High Temperature Superconductivity*, ed. W. Beyermann et al. (World Scientific Publ. Co., Singapore, 1994) p. 85.
10. R. Hilfer, "Fractional dynamics, irreversibility and ergodicity breaking," *Chaos, Solitons & Fractals*, p. in print, 1995.
11. A. Gemant, "A method of analyzing experimental results obtained from elastoviscous bodies," *Physics* **7**, 311 (1936).
12. G. Scott-Blair and J. Caffyn, "An application of the theory of quasi-properties to the treatment of anomalous stress-strain relations," *Phil. Mag.* **40**, 80 (1949).
13. K. Oldham and J. Spanier, "The replacement of Fick's law by a formulation involving semidifferentiation," *J. Electroanal. Chem. Interfacial Electrochem.* **26**, 331 (1970).
14. K. Oldham and J. Spanier, *The Fractional Calculus* (Academic Press, New York, 1974).
15. R. Nigmatullin, "The realization of the generalized transfer equation in a medium with fractal geometry," *Phys. Stat. Sol. B* **133**, 425 (1986).
16. T. Nonnenmacher, "Fractional integral and differential equations for a class of Levy-type probability densities," *J. Phys. A: Math. Gen.* **23**, L697 (1990).
17. C. Friederich, "Relaxation functions of rheological constitutive equations with fractional derivatives: Thermodynamical constraints," in *Rheological Modeling: Thermodynamic and Statistical Approaches*, eds. J. Casas-Vasquez and D. Jou (Springer, Berlin, 1991), p. 309.
18. T. Nonnenmacher, "Fractional relaxation equations for viscoelasticity and related phenomena," in *Rheological Modeling: Thermodynamic and Statistical Apporaoches*, eds. J. Casas-Vasquez and D. Jou (Springer, Berlin, 1991), p. 309.
19. T. Nonnenmacher and W. Glöckle, "A fractional model for mechanical stress relaxation," *Phil. Mag. Lett.* **64**, 89 (1991).
20. H. Schiessel and A. Blumen, "Hierarchical analogues to fractional relaxation equations," *J. Phys. A: Math. Gen.* **26**, 5057 (1993).
21. M. Shlesinger, "Asymptotic solutions of continuous time random walks," *J. Stat. Phys.* **10**, 421 (1974).
22. J. Tunaley, "Asymptotic solutions of the continuous time random walk model of diffusion." *J. Stat. Phys.* **11**, 397 (1974).
23. J. Tunaley, "Some properties of the asymptotic solutions of the Montroll-Weiss equation," *J. Stat. Phys.* **12**, 1 (1975).
24. M. Shlesinger, J. Klafter and Y. Wong, "Random walks with infinite spatial and temporal moments," *J. Stat. Phys.* **27**, 499 (1982).
25. J. Klafter, A. Blumen and M. Shlesinger, "Stochastic pathway to anomalous diffusion," *Phys. Rev.* **A35**, 3081 (1987).
26. R. Hilfer and L. Anton, "Fractional master equations and fractal time random walks," *Phys. Rev. E*, Rapid Commun., (1995); in print.
27. A. Erdelyi (et al.), *Higher Transcendental Functions*, Vol. III (R. E. Krieger Publ. Co., Malabar, 1981).
28. A. Prudnikov, Y. Brychkov and O. Marichev, *Integrals and Series*, Vol. 3 (Gordon and Breach, New York, 1990).

APPENDIX A

DEFINITION OF *H*-FUNCTIONS

The general H-function is defined as the inverse Mellin transform[28]:

$$H_{PQ}^{mn}\left(z\,\middle|\,\begin{matrix}(a_1,\,A_1)\cdots(a_P,\,A_P)\\(b_1,\,B_1)\cdots(b_Q,\,B_Q)\end{matrix}\right)=\frac{1}{2\pi i}\int_C$$

$$\times\frac{\prod_{j=1}^{m}\Gamma(b_j-B_js)\prod_{j=1}^{n}\Gamma(1-a_j+A_js)}{\prod_{j=m+1}^{Q}\Gamma(1-b_j+B_js)\prod_{j=n+1}^{P}\Gamma(a_j-A_js)}z^s ds \quad \text{(A1)}$$

where the contour C runs from $c-i\infty$ to $c+i\infty$ separating the poles of $\Gamma(b_j-B_js)$, $(j=1,\ldots,m)$

from those of $\Gamma(1-a_j+A_js)$, $(j=1,\dots,n)$. Empty products are interpreted as unity. The integers m, n, P, Q satisfy $0 \le m \le Q$ and $0 \le n \le P$. The coefficients A_j and B_j are positive real numbers and the complex parameters a_j, b_j are such that no poles in the integrand coincide. If

$$\Omega = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{P} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{Q} B_j > 0 \tag{A2}$$

then the integral converges absolutely and defines the H-function in the sector $|\arg z| < \Omega\pi/2$. The H-function is also well defined when either:

$$\delta = \sum_{j=1}^{Q} B_j - \sum_{j=1}^{P} A_j > 0 \quad \text{with } 0 < |z| < \infty \tag{A3}$$

or:

$$\delta = 0 \quad \text{and} \quad 0 < |z| < R = \prod_{j=1}^{P} A_j^{-A_j} \prod_{j=1}^{Q} B_j^{B_j}\,. \tag{A4}$$

The H-function is a generalization of Meijers G-function and many of the known special functions are special cases of it.