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FLUCTUATION AND DISSIPATION ON FRACTALS: A PROBABILISTIC APPROACH

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The analogies between the diffusion problem and the resistor network problem as witnessed by the Einstein relation have been very important for analytical and numerical investigations of linear problems in disordered geometries (e.g. percolating clusters)[1]. This raises the question whether the resistor problem can be identified in a purely probabilistic context. An affirmative answer has recently been given and it was shown that the Einstein relation follows from a simple probabilistic argument[2,3]. Here we present the results of a more general treatment.

We begin by considering conditional first passage probabilities in a Markov chain. From a relation for the corresponding generating functions we obtain once more the probabilistic analogue of the Einstein relation. We develop its interpretation and conclude by connecting it to the relation between conductivity and diffusion exponents[1].

Consider a homogeneous Markov chain with denumerable state space. Such a chain can be visualized as a walker (or particle) moving randomly between a countable number of states (sites). Homogeneous here means that the transitions of the walker from site i to site j are governed by single step transition probabilities which do not change with time. We will be interested in the first passage probability $F_{ij}^{(n)}$ that the walker will reach site j for the first time after n steps, given that he is at site i at time 0. In addition we introduce the conditional first passage probabilities $G_{ij}^{(n)}$ for starting at i at time 0 and reaching j for the first time at step n, conditioned on not having visited the elements of a given set S during the walk. The so called taboo set[4] S is restricted to be a finite subset of the state space. We define the generating functions

$$F_{ij}(z) = \sum_{n=1}^{\infty} F_{ij}^{(n)} z^n$$

and analogously $G_{ij}(z)$ for the conditional probabilities $G_{ij}^{(n)}$.

Let us consider the simplified case in which S={a} consists only of a single point. Then for i≠j the relation

$$G_{ij}(z) = \frac{F_{ij}(z) - F_{ia}(z)F_{aj}(z)}{1 - F_{ja}(z)F_{aj}(z)} \tag{1}$$

can be derived from a straightforward probabilistic argument[5]. Note that $G_{ij}(1)$ is the conditional probability that the walker reaches j in one or more steps after starting from i at time 0. Thus Eq. (1) gives an explicit formula for this probability in the limit $z \to 1$. To take the limit we write $F_{ij}(z) = 1-(1-z)f_{ij}(z)$ with $f_{ij}(z) = \langle T_{ij} \rangle + o(1-z)$ where we have assumed that the mean first passage times $\langle T_{ij} \rangle$ between i and j are finite. One obtains

$$G_{ij}(z) = \frac{f_{ia}(z) + f_{aj}(z) - f_{ij}(z) - (1-z)f_{ia}(z)f_{aj}(z)}{f_{ja}(z) + f_{aj}(z) - (1-z)f_{ja}(z)f_{aj}(z)} \tag{2}$$

and thence

$$q := G_{ij}(1) = \frac{\langle T_{ia} \rangle + \langle T_{aj} \rangle - \langle T_{ij} \rangle}{\langle T_{ja} \rangle + \langle T_{aj} \rangle} \tag{3}$$

For i=a=0 this implies

$$\langle T_{00} \rangle = q \; [\; \langle T_{0j} \rangle + \langle T_{j0} \rangle \;] \tag{4}$$

We argue that Eq. (4) is indeed a generalized analogue of the Einstein relation in a purely probabilistic context.

To identify the diffusion constant we write the relation $\langle r^2(t) \rangle \propto t$ as $\langle t(r) \rangle \propto r^2$ which can be justified using the invariance of Brownian motion under the transformation $t \to b^2 t$ and $r \to br$.[6] Here $\langle t(r) \rangle$ is the mean first exit time for the walker to leave a sphere of radius r around its starting point. For an inhomogeneous structure we then define a generalized r-dependent diffusion coefficient as $D(r) = r^2/\langle t(r) \rangle$. To identify the conductivity we introduce an external potential by assuming that the walker is absorbed with probability ρ at some boundary point B and subsequently replaced at 0. If N walkers start from 0 then Nq of them will reach B without having returned to 0. On the average $n = Nq\rho$ walkers will flow from 0 to B. If we identify q as the conductance and n/N as the probability current this is a statement of Ohms law. For a system of linear size L and cross section A we define the conductivity as $\sigma = qL/A$. Returning to the pure random walk picture we assume that the points 0 and B are a distance L apart and that $\langle T_{0B} \rangle \approx \langle T_{0B} \rangle$. We then get from Eq. (4) $\langle T_{00} \rangle = 2\sigma V \langle T_{0B} \rangle / L^2 \propto 2\sigma V/D$ where V denotes the corresponding volume. Hence we arrive at the Einstein relation $\sigma \propto D$. Taking ratios of the quantities in Eq. (4) for two systems whose linear sizes L,L' are scaled by a factor b, i.e L'=bL, and assuming that the limit $L \to \infty$ exists we obtain[2,3] from Eq. (4) the well known relation between diffusion and conductivity exponents[1]. In conclusion we remark that our results involve only quantities that are readily measured in simulations of diffusion in disordered media regardless of whether the systems behave fractally or not.

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REFERENCES

1. Y.Gefen, A.Aharony, S.Alexander, Phys.Rev.Lett. 50 (1983), 77 ; C.K.Harris and R.B.Stinchcombe, Phys.Rev.Lett. 50 (1983), 1399; for a review see e.g. the contributions of H.E. Stanley or A.Aharony in: Scaling Phenomena in Disordered Systems, R.Pynn and A.Skjeltorp, Plenum Press, New York 1985
2. R.Hilfer, Renormierungsansätze in der Theorie ungeordneter Systeme, Verlag Harri Deutsch, Frankfurt a.M. 1986
3. R. Hilfer and A. Blumen, preprint
4. K.L. Chung, Markov Chains, Springer, Berlin 1967
5. R. Hilfer and A. Blumen, to be published
6. E.B.Dynkin and A.A.Juschkewitsch,Markoffsche Prozesse,Springer,Berlin 1969