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FOUNDATIONS OF FRACTIONAL DYNAMICS*

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Abstract

Time flow in dynamical systems is reconsidered in the ultralong time limit. The ultralong time limit is a limit in which a discretized time flow is iterated infinitely often and the discretization time step is infinite. The new limit is used to study induced flows in ergodic theory, in particular for subsets of measure zero. Induced flows on subsets of measure zero require an infinite renormalization of time in the ultralong time limit. It is found that induced flows are given generically by stable convolution semigroups and not by the conventional translation groups. This could give new insight into the origin of macroscopic irreversibility. Moreover, the induced semigroups are generated by fractional time derivatives of orders less than unity, and not by a first order time derivative. Invariance under the induced semiflows therefore leads to a new form of stationarity, called fractional stationarity. Fractionally stationary states are dissipative. Fractional stationarity also provides the dynamical foundation for a previously proposed generalized equilibrium concept.

1. INTRODUCTION

A large number of authors have recently and in the past proposed to use fractional time derivatives on heuristic or aesthetic grounds as phenomenological models for various natural

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processes.[1–16] Can such use of fractional time derivatives in physics be justified from first principles? The traditional answer to this question is a firm "No", because fractional time derivatives, contrary to integer ones, are *nonlocal* operators, and their use would contradict the deeply rooted principle of locality in physics.[17] The first indications that fractional time derivatives have a deeper and more fundamental significance for physics than merely that of a convenient phenomenological modeling tool appeared in recent work of the present author on the classification of phase transitions.[18–21]

My objective in this paper is to further investigate the origin of fractional time derivatives in physics, and to show that they appear generically in coarse grained descriptions of dynamical behavior in the ultra-long-time limit.[21] I shall call into question the applicability of the traditional concepts of stationarity and equilibrium in this limit. The ultralong time limit is a limit in which a discretized time evolution is iterated infinitely often and the discretization time step becomes simultaneously infinite.

Dynamical descriptions of macroscopic (coarse grained) nonequilibrium phenomena typically involve a reduction in the number of underlying microscopic dynamical degrees of freedom. This reduction or coarse graining amounts to a restriction of the microscopic dynamics to a subspace (i.e., a subset of measure zero) of the microscopic phase space. Simultaneously the characteristic time scale of the reduced or coarse grained description is often so much longer than that of the underlying microscopic dynamics, that it may be idealized as infinite.

Given these general ideas the present paper employs concepts from abstract ergodic theory to show that fractional time derivatives appear as the infinitesimal generators of reduced or coarse grained dynamical descriptions in the ultralong time limit. The results of the present paper are direct consequences of a recent classification of phase transitions in statistical mechanics, and the ultralong time limit is a version of the ensemble limit.[18,19]

2. TIME FLOW AND INDUCED TRANSFORMATIONS

Let Γ be the phase or state space of a dynamical system, let $\mathcal{G}$ be a σ-algebra of measurable subsets of Γ, and μ a measure on $\mathcal{G}$ such that $\mu(\Gamma) = 1$. The triple $(\Gamma, \mathcal{G}, \mu)$ forms a probability measure space. In general the time evolution of the system is given as a *flow* (or semiflow) on $(\Gamma, \mathcal{G}, \mu)$, defined as a one-parameter family of maps $\widetilde{T}^t : \Gamma \to \Gamma$ such that $\widetilde{T}^0 = I$ is the identity, $\widetilde{T}^{s+t} = \widetilde{T}^s\widetilde{T}^t$ for all $t, s \in \mathbb{R}$ and such that for every measurable function f, the function $f(\widetilde{T}^t x)$ is measurable on the direct product $\Gamma \times \mathbb{R}$. For every $G \in \mathcal{G}$ also $\widetilde{T}G, \widetilde{T}^{-1}G \in \mathcal{G}$ holds. The measure μ is called *invariant under the flow* $\widetilde{T}^t$ if $\mu(G) = \mu(\widetilde{T}^t G) = \mu((\widetilde{T}^t)^{-1}G)$ for all $t \in \mathbb{R}$, $G \in \mathcal{G}$. An invariant measure is called *ergodic* if it cannot be decomposed into a convex combination of invariant measures, i.e., if $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ with μ_1, μ_2 invariant and $0 \le \lambda \le 1$ implies $\lambda = 1$, $\mu_1 = \mu$ or $\lambda = 0$, $\mu_2 = \mu$.

The flow $\widetilde{T}^t$ defines the time evolution of measures through $T^t\mu(G) = \mu(\widetilde{T}^t G)$ as a map $T^t : \Gamma' \to \Gamma'$ on the space Γ' of measures on Γ. Defining as usual[22,23] $\mu(G, t) = \mu((\widetilde{T}^t)^{-1}G)$ shows that:

$$T^t\mu(G, t_0) = \mu(G, t_0 - t)\,, \tag{1}$$

and thus the flow T^t acts on measures as a right translation in time. The existence of the inverse $(T^t)^{-1} = T^{-t}$ for a flow expresses microscopic reversibility. The *infinitesimal* *generator* of T^t is defined (assuming all the necessary structure for Γ' and T^t) as the strong limit:

$$A = \lim_{t \to 0+} \frac{T^t - I}{t}, \tag{2}$$

where $I = T^0$ denotes the identity, and one has $A = -d/dt$ for right translations. The invariance of the measure μ can be expressed as $A\mu = -d\mu/dt = 0$ and it implies that for given $t_0 \in \mathbb{R}$:

$$T^t \mu(G, t_0) = \mu(G, t_0), \tag{3}$$

for all $G \in \mathcal{G}$, $t \in \mathbb{R}$.

The continuous time evolution $\widetilde{T}^t$ with $t \in \mathbb{R}$ may be discretized into the discrete time evolution $\widetilde{T}^k$ with $k \in \mathbb{Z}$ generated by the map $\widetilde{T} = \widetilde{T}^{\Delta t}$ with discretization time step Δt. Consider an arbitrary subset $G \subset \Gamma$ corresponding to a physically interesting reduced or coarse grained description of the original dynamical system. Not all choices of G correspond to a physically interesting situation, and the choice of G reflects physical modeling or insight. A point $x \in G$ is called *recurrent with respect to* G if there exists a $k \geq 1$ for which $\widetilde{T}^k x \in G$. The Poincarè recurrence theorem asserts that if μ is invariant under $\widetilde{T}$ and $G \in \mathcal{G}$ then almost every point of G is recurrent with respect to G. A set $G \in \mathcal{G}$ is called a *μ-recurrent set* if μ-almost every $x \in G$ is recurrent with respect to G. By virtue of Poincarè's recurrence theorem the transformation $\widetilde{T}$ defines an *induced transformation* $\widetilde{S}_G$ on subsets G of positive measure, $\mu(G) > 0$, through:

$$\widetilde{S}_G x(t_0) = \widetilde{T}^{\tau_G(x)} x(t_0) = x(t_0 + \tau_G(x)), \tag{4}$$

for almost every $x \in G$. The *recurrence time* $\tau_G(x)$ of the point x, defined as:

$$\tau_G(x) = \Delta t \, \min\{k \geq 1 : \widetilde{T}^k x \in G\}, \tag{5}$$

is positive and finite for almost every point $x \in G$. Because G has positive measure it becomes a probability measure space with the induced measure $\nu = \mu/\mu(G)$. If μ was invariant under $\widetilde{T}$ then ν is invariant under $\widetilde{S}_G$, and ergodicity of μ implies ergodicity also for ν.[22]

The induced transformation $\widetilde{S}_G : G \to G$ exists for μ-almost every $x \in G$ with $\mu(G) > 0$ by virtue of the Poincarè recurrence theorem. To extend the definition to the case $\mu(G) = 0$ let $(G, \mathfrak{G}, \nu)$ denote a subspace $G \subset \Gamma$ of measure $\mu(G) = 0$ with σ-algebra $\mathfrak{G}$ contained in $\mathcal{G}$, $\mathfrak{G} \subset \mathcal{G}$, in the sense that $B \in \mathfrak{G}$ for all $B \in \mathcal{G}$. $\mu(B) = 0$ for all $B \in \mathfrak{G}$ while $\nu(B) = \infty$ for all sets $B \in \mathcal{G}$ with $\mu(B) > 0$. Let $0 < \nu(G) < \infty$. If G is ν-recurrent under $\widetilde{T}$ in the sense that ν-almost every point (rather than μ) is recurrent with respect to G then the recurrence time $\tau_G(x)$ and the map $\widetilde{S}_G$ are defined for ν-almost every point $x \in G$. Throughout the following it will be assumed that G is ν-recurrent under $\widetilde{T}$, and that $\nu(G \setminus \widetilde{S}_G G) = 0$. An example is given by solidification where Γ represents the high temperature phase space, while G corresponds to the phase space at low temperatures when a large number of nuclear translational degrees of freedom is frozen out.

The pointwise definition of $\widetilde{S}_G$ can be extended to a transformation on measures by averaging over the recurrence times. This extension was first given in Ref. 21. Let:

$$G_k = \{x \in G : \tau(x) = k\Delta t\}, \tag{6}$$

be the set of points whose recurrence time is $k\Delta t$. The number:

$$p(k) = \frac{\nu(G_k)}{\nu(G)}, \tag{7}$$

is the probability to find a recurrence time $k\Delta t$ with $k \in \mathbb{N}$. The numbers $p(k)$ define a discrete (lattice) probability density $p(k)\delta(t-k\Delta t)$ concentrated on the arithmetic progression $k\Delta t$, $k \in \mathbb{N}$. The induced transformation S_G acting on a measure ϱ on G is defined as the mathematical expectation:

$$S_G\varrho(B, t_0) = \langle T^{\tau_G}\varrho(B, t_0)\rangle = \sum_{k=1}^{\infty} \varrho(B, t_0 - k\Delta t)p(k)\,, \tag{8}$$

where $B \subset G$, and T^t was given in (1). This defines a transformation $S_G : G' \to G'$ on the space G' of measures on G. The next section discusses the iterated transformation S_G^N and the long time limit $N \to \infty$.

3. AVERAGED INDUCED DYNAMICS IN THE ULTRALONG TIME LIMIT

The induced transformations $\widetilde{S}_G$ and S_G were defined for discrete time, and it is of interest to remove the discretization to obtain the induced dynamics in continuous time. The conventional view on discrete vs. continuous time in ergodic theory assumes $0 < \Delta t < \infty$ for the discretization time step, and holds that "*there is no essential difference between discrete-time and continuous-time systems*[a]" (see Ref. 24, page 51). Obviously, this equivalence between discrete and continuous time breaks down for induced dynamics because the continuous flow of time within G is interrupted by time periods of fluctuating length during which the trajectory wanders outside G. These interruptions produce a discontinuous (fluctuating) flow of time.

There are three possibilities for removing the discretization using a *long time limit.* Only one of these employs the conventional assumption $0 < \Delta t < \infty$ (or $\Delta t = 1$). The two other alternatives are $\Delta t \to 0$ and $\Delta t \to \infty$. The first alternative considers the limit $\lim_{\Delta t\to 0, k\to\infty} \widetilde{S}^{k\Delta t}$ in which the discretization step becomes small. This possibility may be called the *short-long-time limit* or *continuous time limit*, and it was discussed in Ref. 21. The second alternative is to consider the limit $\lim_{\Delta t\to\infty, k\to\infty} \widetilde{S}^{k\Delta t}$ in which the discretization step diverges $\Delta t \to \infty$. This will be considered in this paper, and it is called the *long-long-time limit* or the *ultralong-time limit.* These limits are analogous to the ensemble limit.[18–21]

According to its definition (8) the induced time transformation S_G acts as a convolution operator in time:

$$S_G\varrho(B) = \varrho(B) * p\,. \tag{9}$$

Applying the transformation N times yields:

$$S_G^N\varrho(B) = (S_G^{N-1}\varrho(B)) * p = \varrho(B) * \underbrace{p * \cdots * p}_{N \text{ factors}} = \varrho(B) * p_N\,, \tag{10}$$

[a]It is argued that one can always write $t \in \mathbb{R}$ as $t = \epsilon + n\Delta t$ where $\epsilon = t - n\Delta t$ is small and $n = [t/\Delta t]$ is the largest integer not larger than $t/\Delta t$. As long as $0 < \Delta t < \infty$ the continuous long time limit $\lim_{t\to\infty} \widetilde{T}^t$ corresponds to the discrete long time limit $\lim_{k\to\infty} \widetilde{T}^{k\Delta t}$.

where the last equation defines the N-fold convolution $p_N(k)$. If $p_\infty = \lim_{N\to\infty} p_N$ exists this defines also S_G^N in the $N \to \infty$ long time limit.

To determine whether a limiting density p_∞ exists, note that the N-fold convolution $p_N(k) = p(k) * \cdots * p(k)$ gives the probability density $p_N(k) = \text{Prob}\{\mathcal{T}_N = k\Delta t\}$ of the random variable $\mathcal{T}_N = \tau_1 + \cdots + \tau_N$ representing the sum of N independent and identically distributed random recurrence times τ_j with common lattice distribution $p(k) = p_1(k)$. A necessary and sufficient condition for the existence of a limiting density p_∞ for suitably renormalized recurrence times is that the discrete lattice probability density $p(k)$ belongs to the domain of attraction of a stable density.[25,26] Then, because Δt is defined as the maximal value such that all the τ_i are concentrated on the arithmetic progression $k\Delta t$, it follows that for a suitable choice of renormalization constants C_N, D_N:

$$\lim_{N\to\infty} \sup_k \left| \frac{D_N}{\Delta t} p_N(k) - h\left(\frac{k\Delta t - C_N}{D_N}; \varpi, \zeta, C, D\right)\right| = 0\,, \tag{11}$$

where $h(x; \varpi, \zeta, C, D)$ is a limiting stable density whose parameters obey $0 < \varpi \le 2$, $-1 \le \zeta \le 1$, $-\infty < C < \infty$, and $D \ge 0$.[25–27] If $D = 0$ then the limiting distribution is degenerate, $h(x; \varpi, \zeta, C, 0) = \delta(x - C)$ for all values of ϖ, ζ.

The positivity of the recurrence times $\tau_i \ge 0$ for all $i \in \mathbb{N}$ implies that the renormalized recurrence times $\mathcal{T}_N$ are bounded below, and this gives rise to the constraint $P_\infty(t) = 0$ for $t \le C$ on the possible limiting distributions. The limiting stable distributions compatible with this constraint are given by those with parameters $0 < \varpi \le 1$ and $\zeta = -1$. For $0 < \varpi < 1$ the limiting densities may be abbreviated as:

$$h(x; \varpi, -1, C, D) = \frac{1}{D^{1/\varpi}} h_\varpi\left(\frac{t - C}{D^{1/\varpi}}\right)\,, \tag{12}$$

which expresses the well known scaling relations for stable distributions.[18,20,25,26] The scaling function $h_\varpi(x)$ can be expressed explicitly as:

$$h_\varpi(x) = \frac{1}{x\varpi} H_{11}^{10}\left(\frac{1}{x}\middle| \begin{matrix}(0, 1)\\(0, 1/\varpi)\end{matrix}\right)\,, \tag{13}$$

in terms of general H-functions whose definition may be found in Ref. 28 or in Refs. 18 and 20. For $\varpi = 1$ one finds:

$$h_1(x) = \lim_{\varpi\to 1^-} h_\varpi(x) = \delta(x - 1)\,, \tag{14}$$

the Dirac distribution concentrated at $x = 1$ as the limiting density. If the limit exists and is nondegenerate, i.e., $D \neq 0$, the renormalization constants D_N must have the form:

$$D_N = (N\Lambda(N))^{1/\varpi x}\,, \tag{15}$$

where $\Lambda(N)$ is a slowly varying function,[26] defined by the condition that:

$$\lim_{x\to\infty} \frac{\Lambda(bx)}{\Lambda(x)} = 1\,, \tag{16}$$

for all $b > 0$.

Using Eqs. (11) and (12) one has for $N \to \infty$:

$$p_N(k) \approx \frac{\Delta t}{D_N} h\left(\frac{k\Delta t - C_N}{D_N}; \varpi, -1, C, D\right) = \frac{\Delta t}{D_N D^{1/\varpi}} h_\varpi\left(\frac{k\Delta t}{D_N D^{1/\varpi}}\right)\,, \tag{17}$$

where the centering constants have been chosen conveniently as $C_N = -CD_N$. From this it is clear that the traditional long time limit $N \to \infty$ keeping $0 < \Delta t < \infty$ finite produces $\lim_{\substack{N\to\infty \\ 0<\Delta t<\infty}} k\Delta t/(DN\Lambda(N))^{1/\varpi} = 0$ for k finite, and thus $\lim_{\substack{N\to\infty \\ 0<\Delta t<\infty}} p_N(k) = 0$, unless $D = 0$. Therefore the conventional long time limit produces a degenerate limiting distribution if it exists. The ultralong time limit on the other hand allows Δt to become infinite. If Δt diverges such that:

$$\lim_{\substack{N\to\infty \\ \Delta t\to\infty}} \frac{k\Delta t}{D_N} = t\,, \tag{18}$$

exists, then this defines a *renormalized ultralong continuous time*, $0 < t < \infty$. In this case $D > 0$ contrary to the conventional limit. It follows that $\lim_{\substack{N\to\infty \\ \Delta t\to\infty}} kp_N(k) = th_\varpi(t/D^{1/\varpi})/D^{1/\varpi}$ and thus from Eq. (10) that:

$$\begin{aligned} S_\varpi^{t^*}\varrho(B,\, t_0^*) &= \int_0^\infty \varrho(B,\, t_0^* - t)h_\varpi\left(\frac{t}{t^*}\right)\frac{dt}{t^*} \\ &= \frac{1}{t^*}\int_0^\infty T^t\varrho(B,\, t_0^*)h_\varpi\left(\frac{t}{t^*}\right)dt\,, \end{aligned} \tag{19}$$

where the *ultralong time parameter* t^* was identified as:

$$t^* = D^{1/\varpi} > 0\,. \tag{20}$$

The identification of t^* is justified for two reasons. On the one hand $D \propto \langle|\tau - \tau'|^\sigma\rangle^{\varpi/\sigma}$ for all $\sigma < \varpi$, where $\langle\cdots\rangle$ is the expectation with respect to the limiting distribution, and τ, τ' are two independent random recurrence times. This shows that $D^{1/\varpi}$ has dimensions of time. Secondly for $\varpi = 1$ it follows from (14) that:

$$S_1^{t^*}\varrho(B,\, t_0^*) = \int_{-\infty}^\infty \varrho(B,\, t_0^* - t)\delta\left(\frac{t}{t^*} - 1\right)\frac{dt}{t^*} = \varrho(B,\, t_0^* - t^*) = T^{t^*}\varrho(B,\, t_0^*)\,, \tag{21}$$

which again identifies $t^* = D^{1/\varpi}$ as an ultralong time parameter. Note that the results (19) and (21) imply macroscopic (= ultralong time) irreversibility by virtue of (20) even if the underlying time evolution $\widetilde{T}^t$ resp. T^t was reversible. Perhaps this could provide new insight into the longstanding irreversibility paradox. The fundamental convolution semigroup (19) was first obtained in Refs. 18, 19 and 21.

4. FRACTIONAL STATIONARITY

This section investigates the condition of invariance or stationarity for the induced ultralong time dynamics $S_\varpi^{t^*}$. Invariance of a measure ν on G under the induced dynamics $S_\varpi^{t^*}$ is defined as usual (see (3)) by requiring that:

$$S_\varpi^{t^*}\nu(B,\, t_0^*) = \nu(B,\, t_0^*)\,, \tag{22}$$

for $t > 0$ and $B \subset G$. For $0 < \varpi < 1$ (22) may be called the condition of *fractional invariance* or *fractional stationarity.* Using (2) the invariance condition becomes:

$$A_\varpi\nu(B,\, t) = 0\,, \tag{23}$$

for $t > 0$ where A_ϖ is the infinitesimal generator of the semigroup $S_\varpi^{t^*}$. For $\varpi = 1$ the relation (21) implies $A_1\nu(B, t) = -d\nu(B, t)/dt = 0$, and thus in this case invariant measures conserve volumes in phase space as usual. A very different situation arises for $\varpi < 1$.

For $0 < \varpi < 1$ the infinitesimal generators of the stable convolution semigroup $S_\varpi^{t^*}$ are obtained[26] by evaluating the generalized function[29] $s_+^{-\varpi-1}$ on the time translation group T^s:

$$A_\varpi \varrho(t) = c^+ \int_0^\infty s^{-\varpi-1}(T^s - T^0)ds\, \varrho(t) = c^+ \int_0^\infty s_+^{-\varpi-1} T^s\, ds\, \varrho(t)\,, \tag{24}$$

where $c^+ > 0$ is a constant. Comparing (24) with the Balakrishnan algorithm[30–32] for fractional powers of the generator of a semigroup T^t:

$$\begin{aligned}(-A)^\alpha \varrho(t) &= \lim_{t\to 0+}\left(\frac{I - T^t}{t}\right)^\alpha \varrho \\ &= \frac{1}{\Gamma(-\alpha)} \int_0^\infty s^{-\alpha-1}(I - T^s)\varrho(t) ds\,,\end{aligned} \tag{25}$$

shows that if $A = -d/dt$ denotes the infinitesimal generator of the original time evolution T^t then $A_\varpi = (-A)^\varpi$ is the infinitesimal generator of the induced time evolution $S_\varpi^{t^*}$. For $0 < \varpi < 1$ the generators A_ϖ for $S_\varpi^{t^*}$ are *fractional time derivatives.*[15,29,31] The differential form (23) of the fractional invariance condition for ν becomes:

$$\frac{d^\varpi}{dt^\varpi}\nu(B, t) = 0\,, \tag{26}$$

for $t > 0$ which was first derived in Refs. 18 and 19. Its solution is:

$$\nu(B, t) = C_0 t^{\varpi-1}\,, \tag{27}$$

for $t > 0$ with C_0 a constant. This shows that $\nu(B)$ for a fractional stationary dynamical state is not constant. Fractional stationarity or fractional invariance of a measure ν implies that phase space volumes $\nu(B)$ shrink with time. Thus fractional dynamics is *dissipative.* More generally (26) reads $A_\varpi \nu(B, t) = \delta(t)$ with solution $\nu(B, t) = C_0 t_+^{\varpi-1}$ for $t \geq 0$ in the sense of distributions. The stationary solution with $\varpi = 1$ has a jump discontinuity at $t = 0$, and is not simply constant.

The transition from an original invariant measure μ on Γ to a fractional invariant measure ν on a subset G of measure $\mu(G) = 0$ may be called *stationarity breaking.* It occurs spontaneously in the sense that it is generated by the dynamics itself. Stationarity breaking implies ergodicity breaking, and thus the ultralong time limit is a possible scenario for ergodicity breaking in ergodic theory.

The present paper has shown that the use of fractional time derivatives in physics is not only justified, but arises generically for induced dynamics in the ultralong time limit. This mathematical result applies to many physical situations. In the simplest case the resulting fractional differential equation (26) defines fractional stationarity which provides the dynamical basis for the equilibrium concept.[18] Recently fractional random walks were discussed[8] and solved[10] in the continuum limit.

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