



Fractional calculus for distributions

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Abstract

Fractional derivatives and integrals for measures and distributions are reviewed. The focus is on domains and co-domains for translation invariant fractional operators. Fractional derivatives and integrals interpreted as \mathcal{D}' -convolution operators with power law kernels are found to have the largest domains of definition. As a result, extending domains from functions to distributions via convolution operators contributes to far reaching unifications of many previously existing definitions of fractional integrals and derivatives. Weyl fractional operators are thereby extended to distributions using the method of adjoints. In addition, discretized fractional calculus and fractional calculus of periodic distributions can both be formulated and understood in terms of \mathcal{D}' -convolution.

Keywords Fractional calculus (primary) · distributions · convolution

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1 Introduction

A central motivation behind the development of fractional calculus has been the original idea of Leibniz to treat integrals symbolically as negative powers of differentials [1, p. 105], [2] (and [3] for more). Distribution theory [4, 5], as well as operational calculus [6–8], originated from that very same idea [6, p. 574] [5, p. 174] [8, p. 121] of justifying the “symbolic integrations” of physicists and engineers [6, 9–11], so that fractional calculus is closely related to both, operational calculus and distribution theory.

Most mathematical implementations, interpretations and investigations of fractional calculus are concerned with ordinary functions as witnessed by recent reviews [3, 12–14] and numerous articles in this journal. A fractional calculus for certain gen-

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eralized functions (distributions) appeared already with the concept of distributions itself in Schwartz' monograph [5, p. 174]. It was later discussed in [15, Sec. I.5.5] and briefly in [16, Sec. 8.3] and [17, Sec. 2.9], but the "Schwartz fractional calculus" continues to receive little or no attention in most old and new books on fractional calculus and its applications [18–24]. Our objective in this review is to draw attention to Schwartz' fractional calculus of distributions as an ideal framework for extending and unifying theory and applications of fractional calculus. Recall that fractional derivatives for functions are usually discussed only for sufficiently "good", meaning smooth and integrable, functions [19, p. 237]. Extending the domain of definition from "good" functions to less good generalized functions (distributions), however, can make an otherwise unsolvable (fractional) equation solvable. More precisely, extending domains and co-domains can lead to an extended concept of solution by the following general mechanism [25].

Define a mapping $A : X \rightarrow Y$ between two non-empty sets and consider the problem of solving the equation

$$Af = g \quad (1.1)$$

for given $g \in Y$. Equation (1.1) cannot have a solution $f \in X$, unless $g \in Y$ lies in the range of A . If, however, the domains X, Y can be extended to larger sets \tilde{X}, \tilde{Y} into which f, g are suitably embedded, and if A can be extended to a surjective mapping $\tilde{A} : \tilde{X} \rightarrow \tilde{Y}$ such that

$$\tilde{A}\tilde{f} = \tilde{A}f \quad (1.2)$$

holds for the embedding $f \rightarrow \tilde{f}$, then, if g is the right hand side in eq. (1.1), the extended equation

$$\tilde{A}u = \tilde{g} \quad (1.3)$$

has at least one (generalized or weak) solution $u \in \tilde{X}$, even when $g \in Y$ is not in the range of A .

Generalized or weak formulations in this sense sometimes lead to unification of operators whereby previously different operators A_1, A_2, \dots become restrictions to subdomains of one and the same extended operator \tilde{A} . Let $f : [a, b] \rightarrow \mathbb{R}$ be a real-valued function on an interval $[a, b] \subseteq \mathbb{R}$. Operators of interest in this review are fractional derivatives and integrals ${}_x D^\alpha f, {}_x I^\alpha f$ of order α and type X for which numerous different mathematical interpretations have been proposed in the literature [3, 18–21, 25–32]. *Riemann-Liouville fractional integrals of order $\alpha > 0$* [33, 34] for real valued functions $f : [a, b] \rightarrow \mathbb{R}$ on a closed interval $[a, b] \subset \mathbb{R}$ are a popular example of fractional integrals, and usually defined as [35, p. 566], [36, p. 181], [16,

p. 33], [37, p. 6]

$$({}_{\text{RL}}I_{a+}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - y)^{\alpha-1} f(y) \, dy \tag{1.4a}$$

$$({}_{\text{RL}}I_{b-}^{\alpha}f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y - x)^{\alpha-1} f(y) \, dy. \tag{1.4b}$$

It is common to choose

$$\text{dom}({}_{\text{RL}}I_{a+}^{\alpha}) = \text{dom}({}_{\text{RL}}I_{b-}^{\alpha}) = L^1([a, b]) \tag{1.5}$$

as the domain for these operators [16, 17, 19, 20, 37, 38] with $L^1([a, b])$ the set of integrable functions $f : [a, b] \rightarrow \mathbb{R}$. An example for fractional derivatives are the closely related *Riemann-Liouville fractional derivatives of order $\alpha > 0$* defined as

$$({}_{\text{RL}}D_{a+}^{\alpha}f)(x) = \left(\frac{d}{dx}\right)^{[\alpha]} \left({}_{\text{RL}}I_{a+}^{[\alpha]-\alpha}f\right)(x) \tag{1.6a}$$

$$({}_{\text{RL}}D_{b-}^{\alpha}f)(x) = \left(-\frac{d}{dx}\right)^{[\alpha]} \left({}_{\text{RL}}I_{b-}^{[\alpha]-\alpha}f\right)(x) \tag{1.6b}$$

where $[\alpha] = \min\{m \in \mathbb{Z} : m \geq \alpha\}$. Many domains $\text{dom}({}_{\text{RL}}D_{a+}^{\alpha})$ have been used in the literature [18, 19], a frequent, but by no means maximal, example being [16, p. 37]

$$\text{dom}({}_{\text{RL}}D_{a+}^{\alpha}) = \text{dom}({}_{\text{RL}}D_{b-}^{\alpha}) = \mathcal{AC}^{[\alpha]-1}([a, b]), \tag{1.7}$$

the space of functions such that $f^{[\alpha]}$ is absolutely continuous (see [12, 20, 21, 38]).

Riemann-Liouville fractional derivatives depend on the lower, resp. upper, limit of integration a , resp. b . Integer order derivatives, however, do not depend on such a parameter. This difference is fundamental. Integer order derivatives are translation invariant, meaning that they commute with translations, while Riemann-Liouville fractional derivatives are not.

Translation symmetry, i.e. invariance under translations in space and time, is a basic symmetry of nature [39, 40]. In theoretical physics it is closely related by Noethers theorem with the laws of energy and momentum conservation [41]. With respect to applications in physics it is therefore of interest to consider translation invariant fractional derivatives. From a purely mathematical point of view translation invariance is interesting, because it reduces the number of parameters.

Fractional calculus for distributions was introduced in a translation invariant formulation already in [5, p. 174], but has subsequently received little attention. Exceptions are [42], [15, Sec. I.5.5], [43], [29, p. 151] and [16, Sec.8.3]. Possible reasons for this negligence might be that multiplication of distributions is not defined in general

[44], and that spaces of distributions tend to be locally convex while the focus was on fractional powers of operators on normed spaces [27, 45–48]. Later, in [27], fractional powers were extended abstractly to non-negative operators on Frechet spaces, but without providing concrete spaces of distributions for applications. In recent years the extension of fractional calculus from functions to measures [49, 50] and distributions [51–53], [54], [55] has found renewed interest.

A significant number of common interpretations and definitions for translation invariant fractional integrals and derivatives are unified in this review by defining them as convolution operators with power law monomials on spaces of distributions. Thus, the convolution interpretation turns out to be the most general approach to fractional calculus at present.

The present review is concerned mainly with asymmetric convolution kernels. Symmetric kernels, arising e.g. in Riesz-Feller operators or fractional powers of the Laplacian, can often be reduced to linear combinations of asymmetric kernels [16, Sec. 12.1]. Such kernels are mentioned briefly in Section 7.6.

To help readers navigating and finding quickly material of their interest we provide a structured overview of subsequent contents:

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As seen above the review begins with operators for functions in Section 2, proceeds to measures in Section 3, then to generalized functions or “hyperfunctions” [56] in Section 4, and discusses fractional calculus for distributions in Sections 5, 6, 7, and 8. Section 7 is the centerpiece of this work. It reviews the generalization of Schwartz’s fractional calculus in [53–55] and provides new results for discrete distributional Grünwald-Letnikov fractional derivatives, for their continuum limit and for periodic distributions. Also, the fractional derivative and integral operators for Radon measures in Section 3, the Weyl fractional calculus extended to distributions in Section 6.1 and the Laplace multiplier approach in Section 8, apparently, have not been defined or discussed previously in the literature.

It seems appropriate to conclude this introduction with some remarks concerning applications, even if applications are decidedly outside the scope of this review. Distributional fractional calculus, as presented here, has recently been applied in studies of sequential generalized Riemann-Liouville fractional derivatives [57]. Generalized Riemann-Liouville fractional derivatives [58] are becoming increasingly popular for applications, not only in mathematics [59–64], but also in physics [65, 66], engineering [67] and beyond [68, 69], because their “type” parameter is crucial for formulating well-posed initial and boundary value problems. In another recent application of distributional fractional calculus to the linear response theory of dielectric relaxation in glasses [54] this “type” parameter was conjectured to cause the universal observation of nearly constant loss and high frequency excess wings in glasses [70–72]. The conjecture is relevant for the question what is the physical origin of the fractional order α . For applications in physics the interpretation and independent measurability of α is of crucial importance [38, 73]. Spatial fractional derivatives, such as powers of the Laplacian, are plagued with unphysical predictions emerging already within fractional potential theory [74]. Until recently there were, to the best of our knowledge, only two

interpretations of α , that are compatible with the fundamental laws (in particular the law of locality) in physics. Firstly, α has been linked rigorously to critical exponents and the Ehrenfest order of phase transitions in [75, 76], where locality in space and time plays no role. Secondly, fractional time derivatives of order α were related with anomalous diffusion and long time tails in continuous time random walks in [77–79] (semi-Markov processes) and with fractional time evolutions in [58, 80, 81]. The generalizations, unifications and simplifications in this review are hoped to directly generalize, unify and simplify such applications of fractional calculus.

2 Translation invariant fractional calculus for functions (\mathbb{R}^1)

2.1 Liouville-Weyl

The earliest translation invariant formulation of fractional derivatives can be traced back to Liouville's fractional derivative formula [33, p. 3,(1)]

$$\frac{d^\alpha f}{dx^\alpha} = \sum_k A_k k^\alpha e^{kx} \quad (2.1)$$

for functions representable as

$$f(x) = \sum_k A_k e^{kx}, \quad (2.2)$$

a series of exponentials. Liouville's formula was used by Weyl [82, Satz 2] to define what could be called *Liouville-Weyl fractional derivatives and integrals of trigonometric series* [82, 83] as

$$({}_{\text{LW}}D_{\pm}^\alpha f)(x) = \sum_{k=-\infty}^{\infty} (\pm ik)^\alpha f_k e^{ikx} \quad (2.3)$$

$$({}_{\text{LW}}I_{\pm}^\alpha f)(x) = \sum_{k=-\infty}^{\infty} (\pm ik)^{-\alpha} f_k e^{ikx} \quad (2.4)$$

where $\alpha > 0$ and

$$f_k = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikx} f(x) dx \quad (2.5)$$

with $k = \pm 1, \pm 2, \dots$ and $f_0 = 0$ is the Fourier series of f . The operators ${}_{\text{LW}}I^\alpha, {}_{\text{LW}}D^\alpha$ are defined for functions $f : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{C}$ on the unit circle $\mathbb{G} = \mathbb{R}/2\pi\mathbb{Z}$ (2π -periodic functions on the real line) for which $f_0 = 0$, i.e. for functions whose mean or integral over a period vanishes. Contrary to Riemann-Liouville integrals these definitions do

not depend on an extra parameter such as a lower or upper limit of integration. A possible domain of definition is

$$\text{dom} \left({}_{\text{LW}}\mathbf{I}_{\pm}^{\alpha} \right) = L_{2\pi}^p = L^p(\mathbb{G}) = \left\{ f \text{ is } 2\pi\text{-periodic on } \mathbb{R} \text{ and } \|f\|_{L_{2\pi}^p} < \infty \right\} \quad (2.6)$$

of periodic Lebesgue integrable functions with norms

$$\|f\|_{L_{2\pi}^p} = \left[\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p dx \right]^{1/p} \quad (2.7)$$

where $1 \leq p < \infty$. The domain

$$\text{dom} \left({}_{\text{LW}}\mathbf{D}_{\pm}^{\alpha} \right) = \left\{ f \in L_{2\pi}^2 : \sum_{k=-\infty}^{\infty} k^{2\alpha} |f_k|^2 < \infty \right\} \quad (2.8)$$

can be used for Liouville-Weyl derivatives.

2.2 Weyl

Recall the formula for *convolution of two 2π -periodic functions* [83, p. 36]

$$(f * g)(x) := \frac{1}{2\pi} \int_0^{2\pi} f(x - y)g(y)dy = \sum_{k=-\infty}^{\infty} f_k g_k e^{ikx} \quad (2.9)$$

which gives rise to the *Weyl fractional integral for periodic functions*

$$({}_{\text{W}}\mathbf{I}_p^{\alpha} f)(x) = (\Psi_{\alpha} * f)(x) = \frac{1}{2\pi} \int_0^{2\pi} \Psi_{\alpha}(x - y) f(y)dy \quad (2.10)$$

on $L_{2\pi}^p$ with $p \in [1, \infty[$ where

$$\Psi_{\alpha}(x) = \sum_{k \neq 0} (ik)^{-\alpha} e^{ikx} = 2 \sum_{k=1}^{\infty} \frac{\cos(kx - \alpha\pi/2)}{k^{\alpha}} \quad (2.11)$$

is Weyl’s kernel function from [82, p. 300]. The series is convergent for all $x \in (0, 2\pi)$, $\alpha > 0$, and uniformly convergent for $\varepsilon \leq x \leq 2\pi - \varepsilon$, $\varepsilon > 0$, so that it represents the Fourier series of Ψ_{α} .

It can be shown [83] that eq. (2.10) coincides with the Riemann-Liouville definition (1.4) for $a = -\infty$ resp. $b = \infty$. For this reason¹ the Riemann-Liouville fractional

¹ and because Weyl is credited for being the first to define fractional integrals for functions on the whole real axis [84].

integral ${}_{\text{RL}}I_{(-\infty)^+}^\alpha = {}_{\text{w}}I_+^\alpha$ resp. ${}_{\text{RL}}I_{\infty-}^\alpha = {}_{\text{w}}I_-^\alpha$ with infinite limits $\pm\infty$ is traditionally called a *Weyl fractional integral* [82, p. 300], [35, p. 567], [85], [36], [16, p. 353], [37, p. 7]. For $\alpha > 0$ and non-periodic functions $f : \mathbb{R} \rightarrow \mathbb{C}$ the *Weyl fractional integral* is defined as

$${}_{\text{w}}I_{\pm}^\alpha f(x) = \int_0^\infty f(x \mp s) \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds \tag{2.12}$$

where $x \in \mathbb{R}$ and Γ is the Gamma function. The Weyl integral is translation invariant. Interpreting the Weyl integral (2.12) as an absolutely convergent Lebesgue integral, the maximal domain of locally integrable functions is given by

$$\text{dom} \left({}_{\text{w}}I_+^\alpha \right) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}) : \int_0^\infty |f(-s)| \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds < \infty \right\}. \tag{2.13}$$

In the context of Weyl’s considerations the integral (2.12) is considered as an improper integral in the upper limit, resulting in the maximal domain

$$\text{dom} \left({}_{\text{w}}I_+^\alpha \right) = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}) : \lim_{a \rightarrow \infty} \int_0^a |f(-s)| \frac{s^{\alpha-1}}{\Gamma(\alpha)} ds < \infty \right\}, \tag{2.14}$$

which contains periodic functions for $0 < \alpha < 1$. This domain was explicitly considered in [86, p. 116].

Let $D^k = d^k/dx^k$ denote the derivative of integer order $k \in \mathbb{N}$. The *Weyl fractional derivative of order $\alpha > 0$* is defined by

$${}_{\text{w}}D_+^\alpha f(x) = D^{\lceil \alpha \rceil} {}_{\text{w}}I_+^{\lceil \alpha \rceil - \alpha} f(x) \tag{2.15}$$

where $\lceil \beta \rceil = \min\{m \in \mathbb{Z} : \beta \leq m\}$ is the smallest integer above $\beta \in \mathbb{R}$. Choosing the set of absolutely continuous functions $\mathcal{AE}(\mathbb{R})$ as the domain of definition for the derivative d/dx , the domain

$$\text{dom} \left({}_{\text{w}}D_+^\alpha \right) = \left\{ f \in \text{dom} \left({}_{\text{w}}I_+^{\lceil \alpha \rceil - \alpha} \right) : {}_{\text{w}}I_+^{\lceil \alpha \rceil - \alpha} f \in \text{dom} \left(D^{\lceil \alpha \rceil} \right) \right\} \tag{2.16}$$

is obtained by $\lceil \alpha \rceil$ -fold iteration as the domain for partially defined operators.² Note that for $\alpha \notin \mathbb{N}_0$ the constant function $x \mapsto c$ is not contained in the domain of the Weyl fractional derivative ${}_{\text{w}}D_+^\alpha$.

Weyl fractional derivatives and integrals have also been studied on the Hölder spaces $\mathcal{E}^{k,\beta}(\Omega)$ [16, Sec. 19.6]. Essentially, fractional integrals, resp. derivatives, of order $\alpha > 0$ increase, resp. decrease, the Hölder order β by α .

² For operators A and B whose domain and range is contained within a common space X the domain of their composition is $\text{dom}(A \circ B) = \{f \in \text{dom}(B) : Bf \in \text{dom}(A)\}$.

2.3 Marchaud

A larger domain of definition than for Weyl fractional derivatives is obtained for Marchaud fractional derivatives [87]. Let $0 < \alpha < \infty$, $\alpha \notin \mathbb{N}$, and introduce the notation $\lceil x \rceil := \min\{m \in \mathbb{Z} : x \leq m\}$ for $x \in \mathbb{R}$. The *Marchaud fractional derivative* is defined as

$$({}_M D_+^\alpha f)(t) = \frac{1}{C(\alpha)} \int_0^\infty \frac{\Delta_s^{\lceil \alpha \rceil} f(t)}{s^{\alpha+1}} ds \tag{2.17}$$

for all $t \in \mathbb{R}$, where

$$C(\alpha) = \int_0^\infty \frac{(1 - e^{-s})^{\lceil \alpha \rceil}}{s^{\alpha+1}} ds \tag{2.18}$$

and the right differences of order $m \in \mathbb{N}$ and step size $s > 0$ are

$$(\Delta_s^m f)(t) = \sum_{k=0}^m (-1)^k \binom{m}{k} f(t - ks) \tag{2.19}$$

for all $t \in \mathbb{R}$.

A possible choice for the domain is

$$\text{dom}({}_M D_+^\alpha) = \left\{ f \in \mathcal{C}^{\lceil \alpha \rceil} : \int_1^\infty \frac{|f(-s)|}{s^{\alpha+1}} ds < \infty \right\} \tag{2.20}$$

and it depends explicitly on α . This domain contains the constant function.

2.4 Grünwald-Letnikov

For $\alpha > 0$ the *Grünwald-Letnikov fractional derivative* [88, 89] of a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is defined as

$$({}_{GL} D_+^\alpha f)(t) = \lim_{h \rightarrow 0} \frac{\Delta_h^\alpha f(t)}{h^\alpha} = \lim_{h \rightarrow 0} \frac{1}{h^\alpha} \sum_{k=0}^\infty (-1)^k \binom{\alpha}{k} f(t - kh) \tag{2.21}$$

by generalizing the integer order difference quotients from eq. (2.19) to fractional orders. A suitable domain is [16, p. 382]

$$\text{dom}({}_{GL} D_+^\alpha) = L^p(\mathbb{R}) \tag{2.22}$$

for any $1 \leq p < \infty$. For $0 < \alpha < 1$ it was shown in [16, Thm. 20.4] that on this domain the Grünwald-Letnikov fractional derivative ${}_{GL} D_+^\alpha$ coincides with the

Marchaud derivative ${}_M D_+^\alpha$. An analogous result is obtained in [90] for the Hölder spaces $\mathcal{C}^{k,\alpha}(\mathbb{R})$.

Here, in Section 7.4 below, these results for functions are extended and generalized to distributions. It is found that the maximal domains of Marchaud derivatives and Grünwald-Letnikov derivatives for distributions nearly coincide. As shown in Section 9.3, where a weak formulation for fractional derivatives of distributions is introduced, differences may only arise from the fact that ${}_{GL} D_+^\alpha$ is defined as a pointwise limit, while ${}_M D_+^\alpha$ is defined by a Lebesgue integral.

Similar to the Marchaud case $\text{dom}({}_{GL} D_+^\alpha)$ contains the constant function, but does not contain the Heaviside step function or the Dirac distribution.

2.5 Liouville-Grünwald

The Weyl calculus for periodic functions has been related rigorously in [37, 91] to the Grünwald-Letnikov approach [88, 89] on the Lebesgue spaces $L_{2\pi}^p$ of periodic functions defined above. These Banach spaces, as well as the space

$$\mathcal{C}_{2\pi} := \{f \text{ is } 2\pi\text{-periodic on } \mathbb{R} \text{ and } \|f\|_{\mathcal{C}_{2\pi}} < \infty\} \tag{2.23}$$

of continuous 2π -periodic functions with norm

$$\|f\|_{\mathcal{C}_{2\pi}} := \sup_{t \in [0, 2\pi]} |f(t)| \tag{2.24}$$

were used in [37, 91] to characterize the strong Liouville-Grünwald fractional derivatives. Let $1 \leq p < \infty$ and $\alpha > 0$. The *strong Liouville-Grünwald fractional derivative of order α* with domain

$$\text{dom}({}_{LG} D_p^\alpha) = L_{2\pi}^p, \tag{2.25}$$

resp.

$$\text{dom}({}_{LG} D^\alpha) = \mathcal{C}_{2\pi} \tag{2.26}$$

is defined as that function

$$g =: {}_{LG} D_p^\alpha f \quad \text{resp.} \quad g =: {}_{LG} D^\alpha f \tag{2.27}$$

for which the limit

$$\lim_{h \rightarrow 0^+} \left\| \frac{\Delta_h^\alpha f}{h^\alpha} - g \right\|_{L_{2\pi}^p} = 0 \quad \text{resp.} \quad \lim_{h \rightarrow 0^+} \left\| \frac{\Delta_h^\alpha f}{h^\alpha} - g \right\|_{\mathcal{C}_{2\pi}} = 0 \tag{2.28}$$

exists. Because $\Psi_\alpha \in L^1_{2\pi}$ it can be used to define the *Liouville-Grünwald fractional integral*

$$({}_{\text{LG}}\mathbf{I}_p^\alpha f)(x) := (\Psi_\alpha * f)(x) = \frac{1}{2\pi} \int_0^{2\pi} \Psi_\alpha(x - y) f(y) dy \tag{2.29}$$

on $\text{dom}({}_{\text{LG}}\mathbf{I}_p^\alpha) = L^p_{2\pi}$ with $p \in [1, \infty[$ such that

$$\|{}_{\text{LG}}\mathbf{I}_p^\alpha f\|_{L^p_{2\pi}} \leq \|\Psi_\alpha\|_{L^1_{2\pi}} \|f\|_{L^p_{2\pi}} \tag{2.30}$$

for all $f \in L^p_{2\pi}$. Note that ${}_{\text{LG}}\mathbf{I}_p^\alpha$ has a convolution kernel Ψ_α .

Theorem 1 (Thm 3.4, [37]) *The following three assertions are equivalent for $f \in L^p_{2\pi}$, $1 \leq p < \infty$ and $\alpha > 0$:*

- a) ${}_{\text{LG}}\mathbf{D}_p^\alpha f \in L^p_{2\pi}$.
- b) *There exists $g \in L^p_{2\pi}$ such that $(ik)^\alpha f_k = g_k, k \in \mathbb{Z}$.*
- c) *There exists $g \in L^p_{2\pi}$ such that $f - f_0 = {}_{\text{LG}}\mathbf{I}_p^\alpha g$ almost everywhere.*

Here f_k, g_k are the Fourier coefficients of f, g as defined in (2.5). The following three properties hold for $f \in L^p_{2\pi}, \alpha, \beta > 0$:

A) *(Monotonicity)*

If ${}_{\text{LG}}\mathbf{D}_p^\alpha f \in L^p_{2\pi}$, then ${}_{\text{LG}}\mathbf{D}_p^\beta f \in L^p_{2\pi}$ for any $0 < \beta < \alpha$.

B) *(Additivity)*

${}_{\text{LG}}\mathbf{D}_p^\alpha {}_{\text{LG}}\mathbf{D}_p^\beta f = {}_{\text{LG}}\mathbf{D}_p^{\alpha+\beta} f$ whenever one of the two sides is meaningful.

C) *(Fundamental theorem of fractional calculus)*

${}_{\text{LG}}\mathbf{D}_p^\alpha ({}_{\text{LG}}\mathbf{I}_p^\alpha f) = f - f_0 = {}_{\text{LG}}\mathbf{I}_p^\alpha ({}_{\text{LG}}\mathbf{D}_p^\alpha f)$ where the second equality holds if ${}_{\text{LG}}\mathbf{D}_p^\alpha f \in L^p_{2\pi}$.

The Liouville-Grünwald fractional calculus and its applications are further developed in [37, 92–94] (see [37] and references therein).

3 Translation invariant fractional calculus for measures (\mathbb{R}^d)

Common domains of definition for translation invariant fractional integrals and derivatives of functions are often small and their intersection may be too small for applications. For this reason an extension of translation invariant fractional calculus from functions to measures was initiated in [49, 50] for the case of Weyl fractional integrals. It was later superseded and unified with the distributional fractional calculus of Section 7 below. The present section reviews some results and ideas from this initial approach in [49, 50] by placing them into the distributional context.

3.1 Radon measures

Let $1 \leq p \leq \infty$ and let \mathfrak{K} denote the family of compact subsets of \mathbb{R}^d . The indicator function $1_\Omega : \Omega \rightarrow \mathbb{R}$ of a set $\Omega \subseteq \mathbb{R}^d$ is defined by

$$1_\Omega(x) := \begin{cases} 1 & \text{for } x \in \Omega, \\ 0 & \text{for } x \notin \Omega. \end{cases} \tag{3.1}$$

The notation \mathbb{R}^Ω is used for the set of real (or sometimes complex) valued functions $f : \Omega \rightarrow \mathbb{R}$. The Lebesgue space of locally p -integrable functions $L^p_{\text{loc}}(\mathbb{R}^d)$ consists of the measurable functions $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that $f \cdot 1_K \in L^p(\mathbb{R}^d)$ for all $K \in \mathfrak{K}$. It is endowed with the topology generated by the seminorms $f \mapsto \|f \cdot 1_K\|_p$ with $K \in \mathfrak{K}$. The space $\mathcal{C}(\mathbb{R}^d)$ of continuous functions is endowed with the subspace topology inherited from $L^p_{\text{loc}}(\mathbb{R}^d)$. The spaces of continuous functions with compact support are defined by

$$\mathcal{C}_c(\mathbb{R}^d) := \mathcal{K}(\mathbb{R}^d) = \bigcup_{K \in \mathfrak{K}} \mathcal{K}_K(\mathbb{R}^d) \tag{3.2a}$$

$$\mathcal{C}_{c,K}(\mathbb{R}^d) := \mathcal{K}_K(\mathbb{R}^d) = \left\{ \varphi \in \mathcal{C}(\mathbb{R}^d) : \text{supp } \varphi \subseteq K \right\} \tag{3.2b}$$

for any compact set $K \subseteq \mathbb{R}^d$. The topology on every $\mathcal{K}_K(\mathbb{R}^d)$ is the subspace topology induced from $\mathcal{C}(\mathbb{R}^d)$. The locally convex topology on the union $\mathcal{K}(\mathbb{R}^d)$ is generated by all seminorms on $\mathcal{K}(\mathbb{R}^d)$ that have a continuous restriction to $\mathcal{K}_K(\mathbb{R}^d)$.

The spaces $\mathcal{C}(\mathbb{R}^d)$, $\mathcal{K}(\mathbb{R}^d)$ and $L^p_{\text{loc}}(\mathbb{R}^d)$ are vector lattices (or Riesz spaces) with respect to pointwise (almost everywhere) ordering [95, 96]. Lattice orderings \leq are characterized by the existence of supremum and infimum for any two of its elements. For any x in a vector lattice its absolute value is defined as $|x| := \sup\{x, -x\}$.

The locally convex space of Radon measures is defined as the topological dual space

$$\mathcal{M}(\mathbb{R}^d) := \mathcal{K}'(\mathbb{R}^d) = \left(\mathcal{K}(\mathbb{R}^d) \right)' \tag{3.3}$$

endowed with the $\beta(\mathcal{K}', \mathcal{K})$ -topology (see Section 10.2). Because the dual of a locally convex vector lattice is again a vector lattice (see Proposition 4.17 in [95, p. 108]) the space \mathcal{M} of Radon measures is again a locally convex vector lattice under the dual ordering, i.e. $\mu, \nu \in \mathcal{M}$ satisfy $\mu \leq \nu$ iff $\mu(\varphi) \leq \nu(\varphi)$ for all $\varphi \geq 0$. Thus, $|\mu|$ exists for all $\mu \in \mathcal{M}$ and the topology on \mathcal{M} is generated by the seminorms $\mu \mapsto |\mu(\varphi)|$ with $\mathcal{K} \ni \varphi \geq 0$. The set of real valued Radon measures is a real vector lattice. From this one obtains the Riesz decomposition theorem $\mathcal{M} = \mathcal{M}_+ - \mathcal{M}_+$ where $\mathcal{M}_+ := \{\mu \in \mathcal{M} : \mu \geq 0\}$.

3.2 Integration with respect to Radon measures

The evaluation of a Radon measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ on a function $\varphi \in \mathcal{K}(\mathbb{R}^d)$ is written as *integration of φ with respect to μ*

$$\int \varphi(x) d\mu(x) := {}_M\langle \mu, \varphi \rangle_K = \mu(\varphi) \tag{3.4}$$

and this continuous linear functional can be extended uniquely to more general real valued functions using the “upper” and “lower” integrals from [97], [98], [99, Ch.XI], [100, Sec. 3.2]. Recall, that a function $f : \mathbb{R}^d \rightarrow [0, +\infty]$ is *lower semicontinuous*, if the set $\{f \geq a\}$ is closed for every $a \in [0, \infty]$. The *set of lower semicontinuous functions* is denoted as $\mathcal{F}_+(\mathbb{R}^d)$. Let

$$\mathcal{F}_+(\mathbb{R}^d) + \mathcal{K}(\mathbb{R}^d) := \left\{ \sup \mathcal{A} : \mathcal{A} \subseteq \mathcal{K}(\mathbb{R}^d), \mathcal{A} \neq \emptyset \right\} \tag{3.5a}$$

$$= \left\{ \lim_{n \rightarrow \infty} \varphi_n : \varphi_n \in \mathcal{K}(\mathbb{R}^d), \varphi_n \leq \varphi_{n+1}, n \in \mathbb{N} \right\} \tag{3.5b}$$

denote the *set of upper envelopes*, where “sup” is the pointwise supremum of $[-\infty, +\infty]$ -valued functions. The last equality then allows to extend μ from \mathcal{K} to upper envelopes by

$$\mu(f) := \sup \left\{ \mu(\varphi) : \varphi \in \mathcal{K}(\mathbb{R}^d) \text{ with } \varphi \leq f \right\} \tag{3.6}$$

for all $f \in \mathcal{F}_+(\mathbb{R}^d) + \mathcal{K}(\mathbb{R}^d)$. The *upper μ -integral $\mu^*(f)$* and the *lower μ -integral $\mu_*(f)$* of a function $f : \mathbb{R}^d \rightarrow [-\infty, +\infty]$ are defined as

$$\mu^*(f) := \inf \left\{ \mu(g) : g \in \mathcal{F}_+(\mathbb{R}^d) + \mathcal{K}(\mathbb{R}^d) \text{ with } f \leq g \right\} \tag{3.7a}$$

$$\mu_*(f) := -\mu^*(-f) \tag{3.7b}$$

and f is called *μ -integrable* or *integrable with respect to μ* if

$$\mu(f) := \mu^*(f) = \mu_*(f), \tag{3.7c}$$

that is, the upper and lower integrals agree.

Accordingly, the domain for integration with respect to μ is the *space of μ -integrable real valued functions*, defined as

$$\mathcal{L}^1(\mathbb{R}^d, \mu) := \left\{ f : \mathbb{R}^d \rightarrow \mathbb{R} : -\infty < \mu_*(f) = \mu^*(f) < \infty \right\}. \tag{3.8}$$

Let

$$\mathcal{N}^1(\mathbb{R}^d, \mu) := \left\{ f : \mathbb{R}^d \rightarrow \mathbb{R} : \mu^*(|f|) = 0 \right\} \tag{3.9}$$

be the space of functions that vanish μ -almost everywhere. Equipped with the norm $f \mapsto \mu(|f|)$ the quotient space

$$L^1(\mathbb{R}^d, \mu) := \mathcal{L}^1(\mathbb{R}^d, \mu) / \mathcal{N}^1(\mathbb{R}^d, \mu) \tag{3.10}$$

becomes a Banach space. The definition can be extended to signed Radon measures and complex functions.

The space $L^1_{\text{loc}}(\mathbb{R}^d)$ can be identified with a closed subspace of $\mathcal{M}(\mathbb{R}^d)$ by associating to every $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ the measure μ_f defined by

$$\mu_f(\varphi) := \int_{\mathbb{R}^d} \varphi(x) f(x) dx \tag{3.11}$$

for all $\varphi \in \mathcal{K}(\mathbb{R}^d)$. It is equipped with the subspace topology inherited from $\mathcal{M}(\mathbb{R}^d)$ via the map $f \mapsto \mu_f$. The introduced spaces are then related by a chain

$$\mathcal{C}(\mathbb{R}^d) \subseteq L^\infty_{\text{loc}}(\mathbb{R}^d) \subseteq L^q_{\text{loc}}(\mathbb{R}^d) \subseteq L^p_{\text{loc}}(\mathbb{R}^d) \subseteq L^1_{\text{loc}}(\mathbb{R}^d) \subseteq \mathcal{M}(\mathbb{R}^d) \tag{3.12}$$

of continuous inclusions where $1 \leq p \leq q \leq \infty$.

3.3 Convolution of Radon measures

The *convolution of two locally integrable functions* $f, g \in L^1_{\text{loc}}(\mathbb{R}^d)$ is commonly defined pointwise in the almost everywhere sense as

$$(f * g)(x) = \int_{\mathbb{R}^d} f(x - y)g(y)dy \tag{3.13}$$

for $x \in \mathbb{R}^d$, if it exists. This is guaranteed in the most common case $f, g \in L^1(\mathbb{R}^d)$. Reformulated weakly for the corresponding measures μ_f, μ_g from eq. (3.11) and $\varphi \in \mathcal{K}(\mathbb{R}^d)$ this becomes with Fubini's theorem

$$\begin{aligned} (\mu_f * \mu_g)(\varphi) &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x - y)g(y)\varphi(x)dydx \\ &= \int_{\mathbb{R}^{2d}} f(x - y)g(y)\varphi(x)d(x, y) = (\mu_f \otimes \mu_g)(\varphi^\Delta) \end{aligned} \tag{3.14}$$

where $\mu_f \otimes \mu_g$ is the product measure, $\varphi^\Delta \in \mathcal{K}(\mathbb{R}^{2d})$ is the *codiagonal function* defined as

$$\varphi^\Delta(x, y) := \varphi(x + y) = \varphi(\Delta(x, y)) = (\varphi \circ \Delta)(x, y), \tag{3.15}$$

and the mapping $\Delta : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ with

$$\Delta(x, y) := x + y \tag{3.16}$$

denotes addition in \mathbb{R}^d .

The \mathcal{M} -convolution of two Radon measures $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ is defined as the image of their product measure under Δ ,

$$\mu * \nu := \Delta[\mu \otimes \nu] \tag{3.17a}$$

whenever μ, ν are convolvable or, equivalently, whenever

$$\mathcal{M}\langle \mu * \nu, \varphi \rangle_{\mathcal{X}} := \int_{\mathbb{R}^{2d}} \varphi(x + y) d(\mu \otimes \nu)(x, y) = (\mu \otimes \nu)(\varphi^\Delta) \tag{3.17b}$$

for $\varphi \in \mathcal{X}(\mathbb{R}^d)$. Two measures $\mu, \nu \in \mathcal{M}(\mathbb{R}^d)$ are called *convolvable* if and only if $(\mu \otimes \nu)\varphi^\Delta \in \mathcal{M}(\mathbb{R}^d)$ or, equivalently,

$$\int_{\mathbb{R}^{2d}} \varphi(x + y) d(|\mu| \otimes |\nu|)(x, y) < \infty \tag{3.18}$$

for all $\varphi \in \mathcal{X}(\mathbb{R}^d)$ with $\varphi \geq 0$. The *image of a measure* under a continuous mapping is defined generally in [97, Ch. V, §6, No. 4, Def. 2]. Here, for $\mu \in \mathcal{M}(\mathbb{R}^{2d})$ and a continuous mapping $m : \mathbb{R}^{2d} \rightarrow \mathbb{R}^d$ it is $(m\mu)(\varphi) := \mu(\varphi \circ m)$, $\varphi \in \mathcal{X}(\mathbb{R}^d)$ whenever $|\mu|(\varphi \circ m) < \infty$ for all $\varphi \geq 0$.

Proposition 1 *Let $f, g \in L^1_{loc}(\mathbb{R}^d)$. Then μ_f and μ_g are \mathcal{M} -convolvable if and only if*

- 1) $(f * g)(x)$ is well defined by equation (3.13) for almost all $x \in \mathbb{R}^d$ in terms of a Lebesgue integral and
- 2) $f * g \in L^1_{loc}(\mathbb{R}^d)$.

*If μ_f and μ_g are \mathcal{M} -convolvable, then one has $\mu_f * \mu_g = \mu_{f * g}$.*

Proof Follows from eq. (3.14). □

3.4 Convolution duals and extremal domains

In [49, 50] a constructive method was introduced to define weighted measure spaces on which a given set of convolution operators acts as an equicontinuous family of endomorphisms. The constructive method was later extended to distributions. For $d = 1$ this method was applied in [50, Sec. 9] to linear combinations of fractional Weyl integrals and derivatives with orders and coefficients from a bounded set.

The constructive method is based on the *convolution dual* $(F)_{\mathcal{M}}^*$ of a set $F \subseteq \mathcal{M}$ of measures defined as [49, 101, 102]

$$(F)_{\mathcal{M}}^* := \left\{ g \in \mathcal{M}(\mathbb{R}^d) : g \text{ is } \mathcal{M}\text{-convolvable with all } f \in F \right\}. \tag{3.19}$$

By construction $(F)_{\mathcal{M}}^*$ is a *polarity* in the sense of [103, V. 7, p. 122]. Thus the *convolution perfection operator*

$$F \mapsto (F)_{\mathcal{M}}^{**} := ((F)_{\mathcal{M}}^*)_{\mathcal{M}}^* \tag{3.20}$$

is a closure operator [103, p. 105, p. 116–120]. The corresponding closures are called *convolution perfect*. It follows from the definitions that

$$* : (F)_{\mathcal{M}}^{**} \times (F)_{\mathcal{M}}^* \rightarrow ((F)_{\mathcal{M}}^* * (F)_{\mathcal{M}}^*)_{\mathcal{M}}^{**} \tag{3.21}$$

is a well defined bilinear mapping. Within the class of convolution perfect spaces, convolution operators with kernels from F have maximal domain $(F)_{\mathcal{M}}^*$ and are embedded in the minimal space $(F)_{\mathcal{M}}^{**}$.

The construction (3.21) does not necessarily give domains on which the convolution operators with kernels from F map $\text{dom}((\cdot) * (F)_{\mathcal{M}}^{**}) = (F)_{\mathcal{M}}^*$ into itself. To construct endomorphic domains the concepts of simultaneous convolution of p -tuples and maximal convolution modules were introduced.

Definition 1 For $F \subseteq \mathcal{M}(\mathbb{R}^d)$ the *maximal convolution module associated with F* is defined as

$$(F)_{\mathcal{M}}^{*M} := \left\{ g \in \mathcal{M} : f_1 * \dots * f_p * g \text{ exists for all } f_k \in F, p \in \mathbb{N} \right\}. \tag{3.22}$$

Systematic investigation of $(F)_{\mathcal{M}}^*$ and $(F)_{\mathcal{M}}^{*M}$ was carried out in [55] for the more general case of distributions. It resulted in a calculus for convolution perfect spaces as extremal domains of definition (or operation) for convolution operators. The main problem solved by the calculus for convolution perfect spaces is to find extremal and convolution perfect domains $F_1 = (F_1)_{\mathcal{M}}^{**}$, $F_2 = (F_2)_{\mathcal{M}}^{**}$, and $F_3 = (F_3)_{\mathcal{M}}^{**}$ for inclusion relations such as

$$F_1 * F_2 \subseteq F_3. \tag{3.23}$$

A detailed description in this review would lead too far afield. Instead we mention only Theorem 5 from [49, p. 1561] in the next paragraph.

Let $u, v, w : \mathbb{R}^d \rightarrow \mathbb{R}_+$ be three upper semicontinuous and positive weight functions. Given a weight w define two weighted balls

$$w[\mathcal{M}] := \left\{ \mu \in \mathcal{M}(\mathbb{R}^d) : (|\mu|/w) \leq 1 \right\} \tag{3.24}$$

$$\mathcal{E}_v[w] := \left\{ f \in \mathcal{C}(\mathbb{R}^d) : \|fw\|_{\infty} \leq 1, f(x)w(x) \rightarrow 0 \text{ for } |x| \rightarrow \infty \right\} \tag{3.25}$$

of measures, respectively for functions. Let \otimes denote *supremal convolution* defined for $0 \leq f \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ and $0 \leq g \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ pointwise as

$$(f \otimes g)(x) := \|f \cdot T_x \check{g}\|_\infty \tag{3.26}$$

with $(T_x f)(y) = f(y - x)$ defining translation by x . Theorem 5 in [49, p. 1561] characterizes inclusion relations

$$w[\mathcal{M}] * v[\mathcal{M}] \subseteq u[\mathcal{M}] \iff w[\mathcal{M}] * \mathcal{C}_v[v] \subseteq \mathcal{C}_v[u] \iff w \otimes v \leq u \tag{3.27}$$

between weighted balls in terms of an inequality between the weight functions, that is important for the investigations described in Section 6.1. The proof uses the triangle inequality for Radon measures: Two Radon measures μ, ν are \mathcal{M} -convolvable if and only if their absolute values are convolvable. And in that case $|\mu * \nu| \leq |\mu| * |\nu|$ holds true.

3.5 Fractional derivatives and integrals for Radon measures

For $\alpha \in \mathbb{H}$ let $\nu_\alpha \in \mathcal{M}(\mathbb{R})$ denote the *monomial measure* with Lebesgue density

$$\frac{d\nu_\alpha}{dx} = \begin{cases} \frac{x^{\alpha-1}}{\Gamma(\alpha)} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases} \tag{3.28}$$

Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ and $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{C}^d$. The *fractional integral of a Radon measure* μ is defined as the convolution

$${}_m I_+^\alpha \mu := (\check{\nu}_{\alpha_1} \otimes \dots \otimes \check{\nu}_{\alpha_d}) * \mu \tag{3.29}$$

where the $\check{\mu}(x) := \check{\mu}(-x)$ denotes the reflected measure. Thus

$$\text{dom} \left({}_m I_+^\alpha \right) = (\nu_\alpha)^* \mathcal{M} \tag{3.30}$$

for $\alpha \in \mathbb{C}$.

4 Operational Calculus (\mathbb{R}^1)

Operational calculus was initiated by Leibniz [6] and investigated in the 18th and 19th century by Lagrange, Bernoulli, Laplace, Lorgna, Gruson, Arbogast, Francais, Servois, Oltramare, Liouville, Cauchy, Boole and many others (see [104] for a historical account). Symbolic computation with operational symbols was also practiced by physicists and engineers in the early 20th century [6, 9–11, 105] and this motivated Mikusinski [7] and Schwartz [4]. For reviews of operational calculus see [8, 56, 106–110].

Operational calculus is usually formulated for continuous real- or complex-valued continuous functions $f \in \mathcal{C}(\mathbb{R}_{0+})$ on the half-axis $\mathbb{R}_{0+} = [0, \infty[$. Let $a \in \mathbb{C}$, $t \in \mathbb{R}_{0+}$ and $f, g \in \mathcal{C}(\mathbb{R}_{0+})$. Then $(\mathcal{C}(\mathbb{R}_{0+}), +, *)$ becomes a commutative ring with the operations

$$(af)(t) = af(t) \quad (0 \leq t < \infty) \tag{4.1a}$$

$$(f + g)(t) = f(t) + g(t) \quad (0 \leq t < \infty) \tag{4.1b}$$

$$(f * g)(t) = \int_0^t f(t - s)g(s)ds \quad (0 \leq t < \infty) \tag{4.1c}$$

of scalar multiplication, addition, and convolution $* : \mathcal{C}(\mathbb{R}_{0+}) \times \mathcal{C}(\mathbb{R}_{0+}) \rightarrow \mathcal{C}(\mathbb{R}_{0+})$ of f and g .

The ring $\mathcal{C}(\mathbb{R}_{0+})$ has no divisors of zero due to a theorem by Titchmarsh [111, Thm 152, p.325] : If f, g are locally integrable and $f * g = 0$, then either $f = 0$ or $g = 0$. As a consequence, $\mathcal{C}(\mathbb{R}_{0+})$ is an integral domain [112, p. 115] and can be extended to a field $(Q[\mathcal{C}(\mathbb{R}_{0+})], +, \times)$ of quotients in the same way as integers are extended to rationals [113, p. 110].

The elements of the quotient field $Q[\mathcal{C}(\mathbb{R}_{0+})]$ are equivalence classes of ordered pairs, denoted as $(f \not: g)$, representing a convolution numerator f over a convolution denominator $g \neq 0$. Two convolution quotients $(f_1 \not: g_1)$ and $(f_2 \not: g_2)$ are equivalent if the relation

$$f_1 * g_2 = f_2 * g_1 \tag{4.2}$$

holds for all $g_1, g_2 \neq 0$. The multiplication of a pair with numbers $a \in \mathbb{C}$ is defined such that

$$a(f \not: g) = (af \not: g) \tag{4.3a}$$

holds for all pairs $(f \not: g) \in Q[\mathcal{C}(\mathbb{R}_{0+})]$. The addition and multiplication of pairs is defined such that

$$(f \not: g) + (h \not: k) = ((f * k + g * h) \not: (g * k)) \tag{4.3b}$$

$$(f \not: g) \times (h \not: k) = ((f * h) \not: (g * k)) \tag{4.3c}$$

holds for all $f, g, h, k \in \mathcal{C}(\mathbb{R}_{0+})$, $g \neq 0, k \neq 0$. The neutral element for the multiplication \times of pairs is the pair $(1 \not: 1) \in Q[\mathcal{C}(\mathbb{R}_{0+})]$. It is not a continuous function and resembles the generalized δ -function. Because the elements in $Q[\mathcal{C}(\mathbb{R}_{0+})]$ are sets, they are usually called “operators” [108] and sometimes “hyperfunctions” [56]. The two mappings

$$\mathbb{C} \ni a \mapsto (a1 \not: 1) \in Q[\mathcal{C}(\mathbb{R}_{0+})] \tag{4.4a}$$

$$\mathcal{C}(\mathbb{R}_{0+}) \ni f \mapsto ((1 * f) \not: 1) \in Q[\mathcal{C}(\mathbb{R}_{0+})] \tag{4.4b}$$

are embeddings of \mathbb{C} , resp. $\mathcal{G}(\mathbb{R}_{0+})$, into $\mathcal{Q}[\mathcal{G}(\mathbb{R}_{0+})]$. Obviously, scalar multiplication $a(f \text{ :/ } g) \neq a \times (f \text{ :/ } g) = a1 \times (f \text{ :/ } g) = a(1 * f \text{ :/ } g)$ is not the same as multiplication with a constant function.

The relevance of the ring $\mathcal{G}(\mathbb{R}_{0+})$ and its quotient field $\mathcal{Q}[\mathcal{G}(\mathbb{R}_{0+})]$ for fractional calculus emerges from choosing for f and g in (4.1c) the constant function $f = 1$, defined by $f(t) = 1$ for all $t \in [0, \infty[$. One has $1 * 1 = t$ and

$$1^{*n} = \underbrace{1 * \dots * 1}_{n\text{-times}} = \frac{t^{n-1}}{\Gamma(n)}, \quad n \in \mathbb{N}, t \geq 0 \tag{4.5}$$

by iteration. Choosing $f = 1$ in (4.1c) shows

$$(1 * g)(t) = \int_0^t g(s) ds \tag{4.6}$$

which identifies convolution with $f = 1$ as the operator of integration.

The operator of fractional integration in operational calculus is then defined by extending iterated convolution with 1 to noninteger n as

$$({}_{oc}I^\alpha f)(t) = (1^{*\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(t-s) s^{\alpha-1} ds \tag{4.7}$$

with domain

$$\text{dom}({}_{oc}I^\alpha) = \mathcal{G}(\mathbb{R}_{0+}) \tag{4.8}$$

for $\text{Re}\alpha > 0$.

The definition of fractional integration can be extended to all $\alpha \in \mathbb{C}$ as convolution with the convolution quotient

$$1^{*\alpha} = (1^{*(\alpha+n)} \text{ :/ } 1^{*n}) \tag{4.9}$$

where $n = (\lceil \text{Re}\alpha \rceil - 1)$ is the smallest positive integer such that $\text{Re}\alpha + n > 1$. For $\alpha = -1$ one finds

$$D = 1^{*(-1)} = (1 \text{ :/ } 1^{*2}) \tag{4.10}$$

and this is interpreted as the operator of differentiation. The fractional derivative operators are

$${}_{oc}D^\alpha = 1^{*(-\alpha)} \tag{4.11}$$

with domain

$$\text{dom} \left({}_{\text{oc}}D^\alpha \right) = \mathcal{E}(\mathbb{R}_{0+}) \tag{4.12}$$

by virtue of the embedding (4.4a).

The construction of the quotient field $(Q[\mathcal{E}(\mathbb{R}_{0+})], +, \times)$ is quite general. It can be carried out for other spaces such as the space of locally integrable functions $L^1_{\text{loc}}(\mathbb{R}_{0+})$ or distributions $\mathcal{D}'(\mathbb{R}_{0+})$ on the half axis. One has the algebra isomorphism [114]

$$Q[\mathcal{E}(\mathbb{R}_{0+})] = Q[L^1_{\text{loc}}(\mathbb{R}_{0+})] = Q[\mathcal{D}'(\mathbb{R}_{0+})] \tag{4.13}$$

relating operational calculus with Schwartz distributional calculus. On the other hand, the construction of $(Q[\mathcal{E}(\mathbb{R}_{0+})], +, \times)$ has also a few disadvantages:

- (i) Although every function in the ring $\mathcal{E}(\mathbb{R}_{0+})$ also represents an operator (because it can be represented as $(f * b \text{ ; } b \neq 0)$ not every operator in the quotient field corresponds to a function.
- (ii) Generalization of the fundamental equation (4.1c) from $d = 1$ to higher dimension is not straightforward, because of the difficulty to specify what happens at the boundary.

5 Schwartz fractional calculus for distributions (\mathbb{R}^1)

The first fractional calculus for distributions was introduced together with the theory of distributions [5, 15, 115] as an application of defining the convolution of distributions [5, p. 174]. Let $\alpha \in \mathbb{C}$, $x \in \mathbb{R}$ and

$$\Theta(x) = 1_{\mathbb{R}_+}(x) = \begin{cases} 1, & x > 0 \\ 0, & x \leq 0 \end{cases} \tag{5.1}$$

be the *Heaviside step function*. For $x \in \mathbb{R}$ define

$$Y_\alpha(x) = \Theta(x) \frac{x^{\alpha-1}}{\Gamma(\alpha)}, \quad \text{Re}\alpha > 0 \tag{5.2}$$

and its reflection $\check{Y}_\alpha(x) = Y_\alpha(-x)$. This definition is extended as

$$Y_\alpha = D^{1+\lceil -\text{Re}\alpha \rceil} Y_{1+\lceil -\text{Re}\alpha \rceil+\alpha}, \quad \text{Re}\alpha \leq 0 \tag{5.3}$$

to $\alpha \in \mathbb{C}$. Here D^k is the distributional derivative of order k . Note that $Y_1 = \Theta$, that $Y_0 = D\Theta = \delta$ is the Dirac distribution, and that $Y_{-k} = D^{1+k}\Theta = \delta^{(k)}$ their k -th derivative for $k \in \mathbb{N}$. The abbreviation $Y = Y_1$ will also be employed. The symbol Y_α is the notation from [5, p. 43] while it is denoted as Φ_α in [15, p. 48]. The notation

χ_+^α is used in [116, p. 73] and the authors used p_+^α for Y_α and q_+^α for $Y_{-\alpha}$ in Definition 1 in [54].

The idea is now to define fractional derivatives as convolution with the monomial distribution Y_α as kernel in analogy with operational calculus. Let $\alpha \in \mathbb{C}$. The Schwartz fractional derivative is defined as [5, p. 172]

$${}_sD_+^\alpha f := f * Y_{-\alpha} \tag{5.4}$$

for distributions f in the set

$$\text{dom}({}_sD_+^\alpha) = \mathcal{D}'_+ = \{f \in \mathcal{D}' : -\infty < \inf \text{supp} f\} \tag{5.5}$$

of distributions bounded on the left, also called ‘‘causal distributions’’. Note that the domain is independent of α . Analogously, a reflected operator ${}_sD_-^\alpha f = f * \check{Y}_{-\alpha}$ is defined for anticausal distributions.

The convolution of two locally integrable functions $f, g \in L^1_{\text{loc}}(\mathbb{R})$ is given by eq. (3.13) with $d = 1$. This definition can be extended to distributions along the same lines as the extension to measures discussed above in Section 3.3.

Given two test functions $\varphi, \psi \in \mathcal{D}(\mathbb{R})$ their tensor product is the bilinear mapping $\varphi \otimes \psi : \mathcal{D}(\mathbb{R}) \times \mathcal{D}(\mathbb{R}) \rightarrow \mathcal{D}(\mathbb{R}^2)$ that assigns to every pair of functions (φ, ψ) the function $(x, y) \mapsto \varphi(x)\psi(y)$. The tensor product of two distributions $f, g \in \mathcal{D}'(\mathbb{R})$ is the unique distribution $f \otimes g \in \mathcal{D}'(\mathbb{R}^2)$ which satisfies

$$\langle f \otimes g, \varphi \otimes \psi \rangle = \langle f, \varphi \rangle \langle g, \psi \rangle \tag{5.6}$$

for all $\varphi, \psi \in \mathcal{D}(\mathbb{R})$. The problem with using eq. (3.14) as a definition for convolution of distributions is that φ^Δ is smooth, but in general not a test function, i.e. $\varphi^\Delta \notin \mathcal{D}(\mathbb{R}^2)$, because it does not have compact support. Figure 1 illustrates $\text{supp} \varphi^\Delta$ for a test function with $\text{supp} \varphi = [a, b]$.

The problem is circumvented by using a cutoff function $\chi_\varphi \in \mathcal{D}$ to cut off the integrand. In other words, given φ a cutoff function χ_φ is chosen such that $\{\chi_\varphi = 1\} \supseteq \text{supp}((f \otimes g) \cdot \varphi^\Delta)$. Note, that such a cutoff cannot be found for all pairs of distributions $f, g \in \mathcal{D}'$. If a cutoff function can be found, then f, g are called *convolvable by support*. If $f, g \in \mathcal{D}'$ are convolvable by support, then their convolution is defined by

$$\langle f * g, \varphi \rangle = \langle (f \otimes g) \cdot \varphi^\Delta, \chi_\varphi \rangle \tag{5.7}$$

for $\varphi \in \mathcal{D}$. Two distributions $f, g \in \mathcal{D}'$ are convolvable by support if and only if

$$(f \otimes g) \cdot \varphi^\Delta \in \mathcal{E}'(\mathbb{R}^2) \tag{5.8}$$

for $\varphi \in \mathcal{D}$. This is the most common definition of convolution of distributions, which can be found up to details in [116, 117]. For example, convolvability by support is guaranteed,

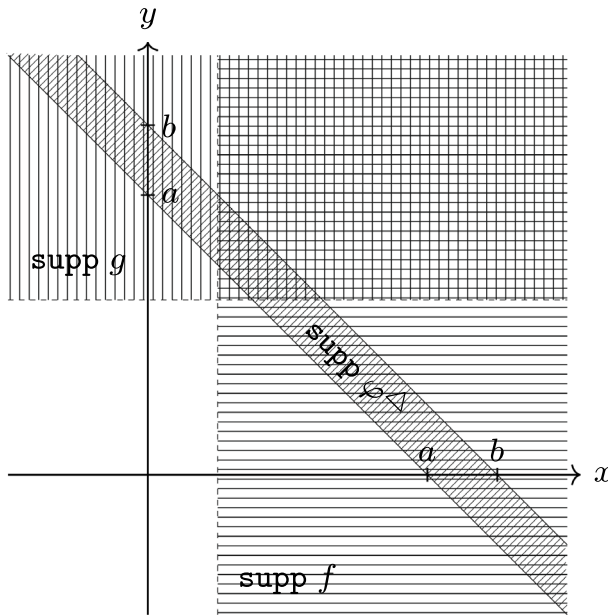


Fig. 1 Illustration of $\text{supp}\varphi^\Delta$ for $\text{supp}\varphi = [a, b]$ and of convolvability by support (condition (2) below) for distributions $f, g \in \mathcal{D}'_+$. The set $\text{supp}\varphi^\Delta$ is textured by north east lines, $\text{supp}f$ is textured by horizontal lines, and $\text{supp}g$ is textured by vertical lines

- (1) if $f \in \mathcal{E}'$ or $g \in \mathcal{E}'$, or
- (2) if $f, g \in \mathcal{D}'_+ = \{f \in \mathcal{D}' : -\infty < \inf \text{supp}f\}$.

The set \mathcal{D}'_+ with the operations of addition and convolution $(\mathcal{D}'_+, +, *)$ becomes a convolution algebra without divisors of zero [5, Thm. XIV, p. 173]. Thus, the algebra $(\mathcal{D}'_+, +, *)$ in distribution theory resembles the algebra $(\mathcal{C}(\mathbb{R}_{0+}), +, *)$ in operational calculus.

While $\text{dom}({}_sD_+^\alpha) = \mathcal{D}'_+$ contains the δ -distribution, Y_α and the Heaviside function, it does not contain the constant function or the trigonometric functions. Some authors maintain that fractional derivatives of integer powers $x \mapsto x^n, n \in \mathbb{N}$ cannot exist [118, p. 24, Sec. 3.5.2], [119, Sec. 2.3].

Schwartz calculus on the half axis, i.e. on $\mathcal{D}'(\mathbb{R}_{0+})$, was applied to a convolution operator with nonsingular kernel in [120], and has been extended to the cone \mathbb{R}_{0+}^d in [52]. The well known results from [15] were recently repeated in [121].

6 Other approaches for distributions

6.1 Method of adjoints

A useful method to define fractional derivatives for distributions is the *method of adjoints or transposition* [42, 122, 123]. Let $(X, \tau_X), (Y, \tau_Y)$ be two locally convex Hausdorff spaces endowed with topologies τ_X, τ_Y and let X', Y' be their topological

duals. Let $A : X \rightarrow Y$ be a linear operator. Its adjoint or transpose ${}^tA : Y' \rightarrow X'$ is the operator defined by the relation

$$\langle {}^tA f, \varphi \rangle = \langle f, A\varphi \rangle. \tag{6.1}$$

The vector ${}^tA f$ is defined as that linear functional on X , which assigns to each $\varphi \in X$ the same number, that the functional $f \in Y'$ assigns to $A\varphi$. If $A : (X, \tau_X) \rightarrow (Y, \tau_Y)$ is continuous, then also $A : (X, \sigma(X, X')) \rightarrow (Y, \sigma(Y, Y'))$ is continuous, i.e when the spaces are endowed with the weak topologies $\sigma(X, X')$ and $\sigma(Y, Y')$. Moreover, if A is continuous, then the adjoint operator is continuous ${}^tA : (Y', \sigma(Y', Y)) \rightarrow (X', \sigma(X', X))$ in the weak and the strong topology, i.e. as a map ${}^tA : (Y', \beta(Y', Y)) \rightarrow (X', \beta(X', X))$ [117, p. 256] [124, eq.(3.23)][25, p. 7]. When A is a fractional operator and X, Y are space of test functions, then tA is a fractional operator for distributions.

The method of adjoints (6.1) can be used to extend Weyl fractional integrals ${}_wI_+^\alpha$ with $\alpha \in \mathbb{H}$ and derivatives ${}_wD_+^\alpha$ with $\alpha \in \mathbb{C}$ to certain weighted space of distributions. This will now be illustrated in this section.

In [49, 50, 125] the Weyl operators ${}_wI_+^\alpha$ and ${}_wD_+^\alpha$ were discussed as continuous endomorphisms of certain weighted spaces of continuous or smooth functions. The results were characterized using the equivalence (3.27). For $k \in \mathbb{N}_0 \cup \{\infty\}$ and a set W of weights with $W \subseteq \mathcal{C}_+(\mathbb{R}) := \{f \in \mathcal{C}(\mathbb{R}) : f \geq 0\}$ define

$$\mathcal{C}_W^k(\mathbb{R}) := \left\{ f \in \mathcal{C}^k(\mathbb{R}) : \forall w \in W, m \in \mathbb{N}_0, m \leq k : f^{(m)} w \in \mathcal{C}_v(\mathbb{R}) \right\} \tag{6.2}$$

with $\mathcal{C}_W(\mathbb{R}) := \mathcal{C}_W^0(\mathbb{R})$ and $\mathcal{C}_W(\mathbb{R}) := \mathcal{C}_W^\infty(\mathbb{R})$. These spaces are endowed with the locally convex topology generated by the seminorms $f \mapsto \|f^{(m)} w\|_\infty$ with $w \in W, m \in \mathbb{N}_0$. Let the *supremal convolution dual* for a set $W \subseteq \mathcal{C}_+(\mathbb{R})$ of weights be denoted as (see (3.26) for the definition of \circledast)

$$(W)_{\mathcal{C}_+}^{\circledast} := \{v \in \mathcal{C}_+(\mathbb{R}) : \forall w \in W : w \circledast v \in L_{\text{loc}}^\infty(\mathbb{R})\}. \tag{6.3}$$

In the notation of [50, eqs. (2.3), (4.1), (9.1)] the set $(W)_{\mathcal{C}_+}^{\circledast}$ corresponds to the intersection of \mathcal{C}_+ with the expression “ $\mathcal{U}^+(\cdot, W; \mathcal{U}^+)$ ” in [50, p. 1246]. With notations similar to [50, pp. 1245] one denotes $P := \left\langle (1+x^2)^n : n \in \mathbb{N} \right\rangle_{\text{tici}}$ and $P_\pm := P \cap \mathcal{D}'_\pm$ (compare also (7.25a) below for $d = 1$). Then $(P)_{\mathcal{C}_+}^{\circledast} = \mathcal{C}_+(\mathbb{R}) \cap \mathcal{C}_P(\mathbb{R})$ and $(P_\pm)_{\mathcal{C}_+}^{\circledast} = (P)_{\mathcal{C}_+}^{\circledast} + [\mathcal{C}_+(\mathbb{R}) \cap \mathcal{D}'_\pm(\mathbb{R})]$.

Retracing the steps in [50, Sec. 11] it can be shown that $P_- \circledast W \subseteq W$ for all $W \in \{P_-, (P_-)_{\mathcal{C}_+}^{\circledast}, P_- + (P)_{\mathcal{C}_+}^{\circledast}, \langle e^{-\mu x} \rangle_{\text{tici}}\}, \mu > 0$. From Theorem 8.1 in [50, p. 1240] (see Theorem 6 in [49, p. 1565] for its proof) and Proposition 10.2 in [50, p. 1243] follows that Weyl fractional integrals ${}_wI_+^\alpha$ with $\alpha \in \mathbb{H}$ define continuous endomorphisms of $\mathcal{C}_W(\mathbb{R})$. Except for $W = \langle e^{-\mu x} \rangle_{\text{tici}}$ this follows also from Korollar 3.7 in [125, p. 63]. Thus Weyl fractional derivatives ${}_wD_+^\alpha$ with $\alpha \in \mathbb{C}$ are continuous endomorphisms of $\mathcal{C}_W(\mathbb{R})$.

It was mentioned in Section 8.1 of [54, p. 130] that the locally convex space $\mathcal{C}_{P_-}(\mathbb{R}) = \mathcal{S}(\mathbb{R}) + (\mathcal{C}(\mathbb{R}) \cap \mathcal{D}'_+(\mathbb{R}))$ can be understood as a translation invariant

reflected version of the “space of good functions \mathcal{E} ” from [126], which consists of rapidly decreasing smooth functions on $[0, \infty[$. In Chapter VII of [19, p. 236] this space is denoted as \mathbf{S} and in [16, p. 155] it is written as $S_+(\mathbb{R}_+^1)$. The set of weights $W = P_- + (P)_{\mathcal{E}_+}^{\otimes}$ fulfills the identity $\mathcal{E}_W(\mathbb{R}) = \mathcal{S}(\mathbb{R}) + [\mathcal{O}_M(\mathbb{R}) \cap \mathcal{D}'_+(\mathbb{R})] = \mathfrak{S}_+$ where \mathfrak{S}_+ is the space from [16, p. 146]. In [16, p. 147] it is claimed misleadingly that the “topology in \mathfrak{S}_+ is easily defined by means of a countable set of norms” although this does not hold true for the topology of $\mathcal{E}_W(\mathbb{R})$.

Consider now the set $W = (P_-)_{\mathcal{E}_+}^{\otimes}$ of weights. In this case, as mentioned in [125, p. 67], the dual space of $\mathcal{E}_W(\mathbb{R}) = \mathcal{O}_M(\mathbb{R}) + \mathcal{D}'_+(\mathbb{R}) \cap \mathcal{E}(\mathbb{R})$ is the space

$$(\mathcal{E}_W)' = \mathcal{O}'_M(\mathbb{R}) + [\mathcal{D}'^F(\mathbb{R}) \cap \mathcal{D}'_+(\mathbb{R})] \tag{6.4}$$

where $\mathcal{D}'^F(\mathbb{R})$ is the space of distributions with finite order [117, p. 339]. This permits to define *adjoint Weyl fractional integrals* ${}_{\text{aw}}\mathbf{I}_+^\alpha$ for $\alpha \in \mathbb{H}$ by

$$\langle {}_{\text{aw}}\mathbf{I}_+^\alpha f, \varphi \rangle := \langle f, {}_{\text{w}}\mathbf{I}_+^\alpha \varphi \rangle \tag{6.5}$$

and *adjoint Weyl fractional derivatives* ${}_{\text{aw}}\mathbf{D}_+^\alpha$ with $\alpha \in \mathbb{C}$ by

$$\langle {}_{\text{aw}}\mathbf{D}_+^\alpha f, \varphi \rangle := \langle f, {}_{\text{w}}\mathbf{D}_+^\alpha \varphi \rangle \tag{6.6}$$

for all $\varphi \in \mathcal{E}_W(\mathbb{R})$. Their domains are

$$\text{dom}({}_{\text{aw}}\mathbf{D}_+^\alpha) = (\mathcal{E}_W(\mathbb{R}))' = \text{dom}({}_{\text{aw}}\mathbf{I}_+^\alpha) \tag{6.7}$$

with $W = (P_-)_{\mathcal{E}_+}^{\otimes}$ and $\alpha \in \mathbb{C}$ resp. $\alpha \in \mathbb{H}$. It is known from [5, p. 245] that $\exp(ix^2) \in \mathcal{O}'_C \setminus \mathcal{O}'_M$ and thus that $\exp(ix^2) \in (\mathcal{O}'_C + \mathcal{D}'_+) \setminus (\mathcal{E}_W)'$. Comparing this with $\text{dom}(\mathbf{I}_+^\alpha)$ in (7.26) below shows that the adjoint extended Weyl operators can be extended further using Schwarz’ approach.

6.2 Lizorkin and Rubín (\mathbb{R}^1)

Interpreting fractional derivatives and integrals as Fourier multiplication operators is a common approach, which will be discussed below in Section 8. It led Lizorkin [42, 43] already in the early 1960’s to introduce the space

$$\mathcal{S}_{\text{Lz}} := \left\{ \varphi \in \mathcal{S}(\mathbb{R}) : \int_{\mathbb{R}} x^k \varphi(x) dx = 0, k \in \mathbb{N}_0 \right\} \tag{6.8}$$

named after him. The Lizorkin space \mathcal{S}_{Lz} contains those (rapidly decreasing) Schwartz test functions, that are orthogonal to all polynomials. It carries the subspace topology inherited from the Schwartz space $\mathcal{S}(\mathbb{R})$.

The *Lizorkin fractional derivative* ${}_L D_{\pm}^{\alpha}$ of order $\alpha \in \mathbb{R}$ is then defined by the method of adjoints for $f \in (\mathcal{S}_{LZ})'$ as

$$\langle {}_L D_{\pm}^{\alpha} f, \varphi \rangle := \langle f, {}_w D_{\mp}^{\alpha} \varphi \rangle \tag{6.9}$$

for all $\varphi \in \mathcal{S}_{LZ}$. The domain is

$$\text{dom} \left({}_L D_{\pm}^{\alpha} \right) = (\mathcal{S}_{LZ})' = \mathcal{S}'(\mathbb{R}) / \mathcal{P}(\mathbb{R}) \tag{6.10}$$

where $\mathcal{S}'(\mathbb{R})$ is the space of tempered distributions and $\mathcal{P}(\mathbb{R})$ the space of polynomials [16, Sec. 8.2]. The operators ${}_L D_{\pm}^{\alpha}$ obey the general semigroup or index law ${}_L D_{\pm}^{\alpha} {}_L D_{\pm}^{\beta} = {}_L D_{\pm}^{\alpha+\beta}$ for all orders $\alpha, \beta \in \mathbb{R}$.

Lizorkin fractional operators have not found much attention or application, because multiplication of $\varphi \in \mathcal{S}_{LZ}(\mathbb{R})$ with a power functions leads out of Lizorkin space. This circumstance renders \mathcal{S}_{LZ} virtually useless as a domain for fractional differential equations.

Lizorkin’s approach for fractional integration was considered in [127] for the half-axis \mathbb{R}_{0+} , and with Fourier transformation replaced by Mellin transformation. Let $\mathcal{S}(\mathbb{R}_{0+})$ be the space consisting of restrictions of Schwartz functions from the line to the half-line and define

$$\mathcal{S}_+(\mathbb{R}_{0+}) := \left\{ \varphi \in \mathcal{S}(\mathbb{R}_{0+}) : \lim_{x \rightarrow 0} x^{-k} \varphi^{(m)}(x) = 0, \forall k, m \in \mathbb{N}_0 \right\} \tag{6.11}$$

$$\Phi_+(\mathbb{R}_{0+}) := \left\{ \varphi \in \mathcal{S}_+(\mathbb{R}_{0+}) : \int_0^{\infty} x^k \varphi(x) dx = 0, \forall k \in \mathbb{N}_0 \right\} \tag{6.12}$$

$$\Phi_{\alpha}(\mathbb{R}_{0+}) := \left\{ \varphi \in \mathcal{S}_+(\mathbb{R}_{0+}) : \int_0^{\infty} x^{\alpha-k} \varphi(x) dx = 0, \forall k \in \mathbb{N}_0 \right\}, \quad \alpha \in \mathbb{H} \tag{6.13}$$

with topology generated by the seminorms

$$q_k(\varphi) := \sup_{m \leq k} \sup_{x > 0} (1+x)^k |\varphi^{(m)}(x)| \tag{6.14}$$

with $k \in \mathbb{N}_0$. The *Rubin fractional integrals of order* $\alpha \in \mathbb{H}$ are defined implicitly by demanding that

$$\left\langle {}_{\text{Rb}} I_{-}^{\alpha} f, \varphi \right\rangle := \left\langle f, {}_{\text{RL}} I_{0+}^{\alpha} \varphi \right\rangle \tag{6.15}$$

$$\left\langle {}_{\text{Rb}} I_{0+}^{\alpha} f, \varphi \right\rangle := \left\langle f, {}_w I_{-}^{\alpha} \varphi \right\rangle \tag{6.16}$$

holds for all $\varphi \in \Phi_+(\mathbb{R}_{0+})$ in the first, resp. $\varphi \in \Phi_\alpha(\mathbb{R}_{0+})$ in the second relation. The domains are

$$\text{dom} \left({}_{\text{Rb}}\mathbf{I}_-^\alpha \right) = (\Phi_+(\mathbb{R}_{0+}))' \tag{6.17}$$

$$\text{dom} \left({}_{\text{Rb}}\mathbf{I}_{0+}^\alpha \right) = (\Phi_{-\alpha}(\mathbb{R}_{0+}))' \tag{6.18}$$

the topological duals of the test function spaces $\Phi_+(\mathbb{R}_{0+})$ and $\Phi_{-\alpha}(\mathbb{R}_{0+})$.

6.3 Erdelyi-McBride (\mathbb{R}^1)

Let $\alpha \in \mathbb{C}$, $\text{Re}\alpha > 0$, $m > 0$. The Erdelyi-Kober operators are defined for functions $f :]0, \infty[\rightarrow \mathbb{R}$ by [85]

$$(\mathbf{I}_m^{\eta, \alpha} f)(x) = \frac{mx^{-m\eta-m\alpha}}{\Gamma(\alpha)} \int_0^x (x^m - y^m)^{\alpha-1} y^{m\eta+m-1} f(y) dy \tag{6.19}$$

$$(\mathbf{K}_m^{\zeta, \alpha} f)(x) = \frac{mx^{m\zeta}}{\Gamma(\alpha)} \int_x^\infty (y^m - x^m)^{\alpha-1} y^{-m\zeta-m\alpha+m-1} f(y) dy \tag{6.20}$$

and they are studied in [122, 123] with $\zeta = \eta + 1 - m^{-1}$ as $\mathbf{I}_m^{\eta, \alpha}$, $\mathbf{K}_m^{\eta+1-m^{-1}, \alpha}$ with $m > 0$, complex α and η with $\text{Re}\alpha > 0$, $\text{Re}\eta > m^{-1} - 1$ and $\text{Re}(\eta + \alpha) > m^{-1} - 1$. Although Erdelyi-Kober operators are related to Riemann-Liouville and Weyl operators as [85]

$$(\mathbf{K}_1^{0, \alpha} f)(x) = {}_{\text{w}}\mathbf{I}_-^\alpha (x^{-\alpha} f) \tag{6.21}$$

$$(\mathbf{I}_1^{0, \alpha} f)(x) = x^{-\alpha} ({}_{\text{RL}}\mathbf{I}_{0+}^\alpha f)(x) \tag{6.22}$$

they are not identical with them. The extension to spaces of distributions via the method of adjoints is based on “fractional integration by parts” [128]

$$\int_a^b ({}_{\text{RL}}\mathbf{I}_{a+}^\alpha f)(x) g(x) dx = \int_a^b f(x) ({}_{\text{RL}}\mathbf{I}_{b-}^\alpha g)(x) dx \tag{6.23}$$

which is written in [122, eq. (1.10)] as

$$\left\langle {}_{\text{RL}}\mathbf{I}_{0+}^\alpha f, g \right\rangle = \left\langle f, {}_{\text{w}}\mathbf{I}_-^\alpha g \right\rangle \tag{6.24}$$

for $a = 0$ and $b = \infty$. The idea is to define the Erdelyi-McBride fractional integral ${}_{\text{EM}}\mathbf{I}_{0+}^\alpha f$ of a distribution $f \in \mathcal{D}'_+$ by the right hand side of this relation, which remains well defined although in general ${}_{\text{w}}\mathbf{I}_-^\alpha g \notin \mathcal{D}$.

A suitable space of test functions for the Erdelyi-Kober operators is constructed in [123]. Let $a \in \mathbb{R}$. The set of test functions

$$\mathcal{F}_a := \bigcup_{\ell=1}^{\infty} \mathcal{F}_{a+\ell-1, \ell} \tag{6.25a}$$

$$\mathcal{F}_{a, \ell} := \left\{ f \in \mathcal{E}([0, \infty[) : f(x) = 0 \text{ for } x > \ell, p_{a, k}(f) < \infty, k \in \mathbb{N}_0 \right\}, \ell > 0 \tag{6.25b}$$

is equipped with seminorms

$$p_{a, k}(f) := \sup_{x > 0} \left\{ x^{1-a+k} |f^{(k)}(x)| \right\}, \quad k \in \mathbb{N}_0 \tag{6.26}$$

determining the topology on $\mathcal{F}_{a, \ell}$. The space $\mathcal{D}([0, \infty[)$ is dense in \mathcal{F}_a . Similar but somewhat larger spaces were proposed in [122] and [129]. The *Erdelyi-McBride pseudo-fractional integral* ${}_{EM}I_{0+}^{\alpha}$ of order $\alpha \in \mathbb{C}$ is defined by demanding that

$$\left\langle {}_{EM}I_{0+}^{\alpha} f, g \right\rangle := \Gamma(1 - \alpha) \left\langle f, {}_wI_{1-\alpha}^{-} \cdot g \right\rangle \tag{6.27}$$

holds true for all $g \in \mathcal{F}_a$. Its domain is the dual space

$$\text{dom} \left({}_{EM}I_{0+}^{\alpha} \right) = \mathcal{F}'_a \tag{6.28}$$

with $a < 1$. It contains distributions of order zero having limited growth at $0+$ but arbitrary growth at $+\infty$. For fixed α the operator ${}_{EM}I_{0+}^{\alpha}$ is bounded on \mathcal{F}'_a .

The Erdelyi-McBride calculus of distributions is restricted to the half axis. Hence it is not translation invariant and cannot be applied e.g. to the constant function on \mathbb{R} . Erdelyi-McBride operators for distributions combine convolution with multiplication. Because they differ in this respect from ordinary fractional integrals operators we have called them pseudo-fractional integral operators.

6.4 Braaksma-Schuitman (\mathbb{R}^1)

In [130] multiplicative convolution and Mellin transformation are employed to study the Erdelyi-Kober operators $I_m^{\eta, \alpha}$ and $K_m^{\eta, \alpha}$. Let $\lambda, \mu \in \mathbb{R} \cup \{\pm\infty\}, \lambda < \mu$. Let $(\lambda_n)_{n=0}^{\infty}$ and $(\mu_n)_{n=0}^{\infty}$ be sequences of real numbers with $\lambda_n \downarrow \lambda$ and $\mu_n \uparrow \mu$ and $\lambda_n < \mu_n$ for all $n \in \mathbb{N}$. The space $T(\lambda, \mu)$ is defined as the space of all functions $\varphi \in \mathcal{E}(0, \infty)$ with the property

$$p_n(\varphi) = \sup_{\substack{t > 0 \\ p=0, 1, \dots, n \\ \lambda_n \leq t \leq \mu_n}} \left| t^{c+p} f^{(p)}(t) \right| < \infty \tag{6.29}$$

for all $n \in \mathbb{N}$. $T(\lambda, \mu)$ is a locally convex space with topology generated by the norms p_n . The Erdelyi-Kober operators are written multiplicatively as Mellin convolutions

$$(I_m^{\eta,\alpha} f)(x) = \frac{m}{\Gamma(\alpha)} \int_0^x \left\{ \left(\frac{x}{y}\right)^m - 1 \right\}^{\alpha-1} \left(\frac{x}{y}\right)^{-m(\alpha+\eta)} f(y) \frac{dy}{y} \tag{6.30}$$

for all $f \in T(\lambda, \mu)$, $m(\operatorname{Re}\eta + 1) > \lambda$ and

$$(K_m^{\eta,\alpha} f)(x) = \frac{m}{\Gamma(\alpha)} \int_x^\infty \left\{ 1 - \left(\frac{x}{y}\right)^m \right\}^{\alpha-1} \left(\frac{x}{y}\right)^{m\eta} f(y) \frac{dy}{y} \tag{6.31}$$

for all $f \in T(\lambda, \mu)$, $\operatorname{Re}\alpha > 0$ and $m\operatorname{Re}\eta + \mu > 0$.

Given a positive number $a > 0$, let $T([0, a], \lambda)$ denote the subspace of $T(\lambda, \infty)$ of functions with support contained in $[0, a]$. Similarly, let $T([a, \infty), \mu)$ denote the subspace of $T(-\infty, \mu)$ of functions with support contained in $[a, \infty)$. It is shown in [130] that $I_m^{\eta,\alpha}$ is a continuous operator of $T([a, \infty), \mu)$ into itself, if $\mu \leq m(1 + \operatorname{Re}\eta)$, and it is an automorphism, if $\mu \leq m(1 + \operatorname{Re}\alpha + \operatorname{Re}\eta)$. Similarly, $K_m^{\eta,\alpha}$ is a continuous operator of $T([0, a], \lambda)$ into itself, if $\lambda \geq -m\operatorname{Re}\eta$ and it is an automorphism if $\lambda \geq -m\operatorname{Re}(\alpha + \eta)$.

The operators $I_m^{\eta,\alpha}$ and $K_m^{\eta,\alpha}$ are then extended by transposition to distributions as continuous operators with domains

$$\operatorname{dom}(I_m^{\eta,\alpha}) = T'([0, a], 1 - \mu), \quad \mu \leq m(1 + \operatorname{Re}\eta) \tag{6.32}$$

$$\operatorname{dom}(K_m^{\eta,\alpha}) = T'([a, \infty), 1 - \lambda), \quad \lambda \geq -m\operatorname{Re}\eta \tag{6.33}$$

where $a > 0$. Setting $\eta = 0$ and $m = 1$ leads to the definition of the *Braaksma-Schuitman pseudo-fractional integral* of order $\alpha \in \mathbb{C}$ via the relation

$$\left\langle {}_{\text{BS}}I_{0+}^\alpha f, g \right\rangle := \left\langle f, K_1^{0,\alpha} g \right\rangle \tag{6.34}$$

for all $g \in T([a, \infty), 1 - \lambda)$, $\lambda \geq 0$, $a > 0$.

As in the Erdelyi-McBride case also the Braaksma-Schuitmann approach is not translation invariant. Braaksma-Schuitmann operators cannot be compared with ordinary fractional calculus operators insofar as they involve not only convolution but also multiplication with fractional powers.

6.5 Lamb (\mathbb{R}^1)

The distributional fractional calculus developed in [131, 132] is based on fractional powers of the integral operators J and K defined as

$$(J\varphi)(x) = \int_{-\infty}^x \varphi(y)dy, \quad x \in \mathbb{R} \tag{6.35a}$$

$$(K\varphi)(x) = \int_x^{\infty} \varphi(y)dy, \quad x \in \mathbb{R} \tag{6.35b}$$

on test function spaces ($1 \leq p < \infty, \mu \in \mathbb{R}$)

$$D_{p,\mu} := \{f \in \mathcal{D}(\mathbb{R}) : e^{-\mu x} f(x) \in \mathcal{D}_{L^p}\} = \mathcal{D}_{L^p} \cdot e^{\mu x} \tag{6.36}$$

$$D_{\infty,\mu} := \{f \in \mathcal{D}(\mathbb{R}) : e^{-\mu x} f(x) \in \mathcal{C}_v^0\} = \mathcal{B} \cdot e^{\mu x} \tag{6.37}$$

where \mathcal{D}_{L^p} is the space of smooth functions such that all derivatives are in L^p (defined in [5, p. 199]). For $p = \infty$ the space $\mathcal{D}_{L^p}(\mathbb{R})$ is replaced by the subspace of those functions from \mathcal{D}_{L^p} , that also converge to zero at infinity, because \mathcal{D} is dense in the latter but not the former. Lamb defined the spaces $D_{p,\mu}$ for $\mu \in \mathbb{C}$ [131, Def. 3.1], although they do not depend on any non-zero imaginary part of μ . Hence it suffices to parametrize them with $\mu \in \mathbb{R}$ as we do it here.

The spaces $D_{p,\mu}$ are Frechet spaces equipped with the topology generated by the seminorms

$$q_k^{p,\mu}(\varphi) = \left\| \frac{d^k}{dx^k} (e^{-\mu x} f(x)) \right\|_p \tag{6.38}$$

where $\| \cdot \|_p$ is the usual L^p -norm. If $\mu > 0$, then J is a homeomorphism of $D_{p,\mu}$ onto $D_{p,\mu}$ with inverse $J^{-1} = d/dx$. If $\mu < 0$, then K is a homeomorphism of $D_{p,\mu}$ onto $D_{p,\mu}$ with inverse $K^{-1} = -d/dx$ [131, Thm 3.4].

Fractional powers of a linear and continuous operator $A = J, K$ on $D_{p,\mu}$ are defined by the formula

$$(-A)^\alpha f = -\frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{\alpha-1} \left[(\lambda 1 - A)^{-1} - \frac{\lambda}{(1+\lambda^2)} \right] A f d\lambda - \sin\left(\frac{\pi\alpha}{2}\right) A f \tag{6.39a}$$

$$(-A)^\alpha f = (-A)^{\alpha-n} (-A)^n f \tag{6.39b}$$

for the α -th power of a linear and continuous operator A on X with continuous inverse A^{-1} . Here $0 < \text{Re}\alpha < 2$ in eq. (6.39a) and $n < \text{Re}\alpha < n + 2, n = \pm 1, \pm 2, \dots$ with

$(-A)^n = (-A^{-1})^m$ when $n = -m, m > 0$ in eq. (6.39b). The class of admissible operators is restricted by the conditions:

- (1) $(0, \infty) \subset \rho(A)$, where $\rho(A)$ is the resolvent set of A .
- (2) For each $q_k \in \{q_k\}_{k=0}^\infty$ there exists a seminorm $p \in \{q_k\}_{k=0}^\infty$ such that

$$q_k \left(\left[C\lambda(\lambda 1 - A)^{-1} \right]^n f \right) \leq p(f) \tag{6.40}$$

for all $f \in X$ where $\lambda > 0, n = 1, 2, \dots$ and the constant $C > 0$ is independent of f, λ and n .

These conditions ensure convergence of the integral in (6.39a), interpreted as an improper Riemann integral. It is shown in [131] that on $D_{p,\mu}$ the fractional powers of J and K are the Weyl fractional integrals $(J)^\alpha = {}_wI_+^\alpha$ and $(K)^\alpha = {}_wI_-^\alpha$.

The integrals J and K are then extended to operators \tilde{J} and \tilde{K} on the dual spaces $D'_{p,\mu}$ by

$$\langle \tilde{J}f, \varphi \rangle = \langle f, K\varphi \rangle, \quad \langle \tilde{K}f, \varphi \rangle = \langle f, J\varphi \rangle \tag{6.41}$$

where $f \in D'_{p,\mu}, \varphi \in D_{p,\mu}$ and $\langle \cdot, \cdot \rangle$ is the bilinear pairing of the dual system $(D'_{p,\mu}, D_{p,\mu})$. Let $\alpha \in \mathbb{C}$ with $\text{Re}\alpha > 0$. The *distributional Lamb fractional derivatives and integrals* are defined as the fractional powers of the extended Weyl fractional derivatives and integrals

$$\text{Lb}I_+^\alpha := (\tilde{J})^\alpha \tag{6.42}$$

$$\text{Lb}I_-^\alpha := (\tilde{K})^\alpha \tag{6.43}$$

with

$$\text{dom} \left(\text{Lb}I_+^\alpha \right) = D'_{p,\mu} = \mathcal{D}'_{L^q} \cdot e^{-\mu x}, \quad \mu < 0 \tag{6.44}$$

$$\text{dom} \left(\text{Lb}I_-^\alpha \right) = D'_{p,\mu} = \mathcal{D}'_{L^q} \cdot e^{-\mu x}, \quad \mu > 0 \tag{6.45}$$

on $D'_{p,\mu}$ where $(1/p) + (1/q) = 1$.

Although it is defined for functions on the full real axis the Lamb-calculus does not apply to $\exp(i\omega x)$ or trigonometric functions.

6.6 Khan-Lamb-McBride (\mathbb{R}^1)

Periodic distributions were considered in [133] with the objective to extend earlier work of Butzer and Westphal [37] on periodic functions to distributions. In [133] $d = 1$ and $\mathbb{G} = \mathbb{R}/2\pi\mathbb{Z}$. The space $\mathcal{A}_{2\pi}$ of test functions is defined as

$$\mathcal{A}_{2\pi} := \{ \varphi \in \mathcal{C}^\infty(\mathbb{R}); \varphi \text{ is } 2\pi\text{-periodic} \} \tag{6.46}$$

the vector space of smooth and 2π -periodic test functions equipped with the topology generated by the seminorms

$$p_k(\varphi) := \|D^k\varphi\|_{L^2_{2\pi}} \tag{6.47}$$

with $k \in \mathbb{N}_0$. The direction reversed Liouville-Weyl fractional operators

$$({}_{\text{LW}}D_-^\alpha\varphi)(x) = \sum_{k=-\infty}^{\infty} (-ik)^\alpha \varphi_k e^{ikx} \tag{6.48}$$

$$({}_{\text{LW}}I_-^\alpha\varphi)(x) = \sum_{k=-\infty}^{\infty} (-ik)^{-\alpha} \varphi_k e^{ikx} \tag{6.49}$$

are defined for all test functions $\varphi \in \mathcal{A}_{2\pi}$ and $\alpha > 0$.

The dual space $\mathcal{A}'_{2\pi}$ of $\mathcal{A}_{2\pi}$ is equipped with the weak*-topology [133, p. 267]. The *periodic fractional derivative and integral* of periodic distributions $f \in \mathcal{A}'_{2\pi}$ are then defined for $\alpha > 0$ as

$$\langle {}_pD_+^\alpha f, \varphi \rangle := \langle f, {}_{\text{LW}}D_-^\alpha\varphi \rangle \tag{6.50}$$

$$\langle {}_pI_+^\alpha f, \varphi \rangle := \langle f, {}_{\text{LW}}I_-^\alpha\varphi \rangle \tag{6.51}$$

Both operators are endomorphisms on the distributional domain

$$\text{dom}({}_pD_+^\alpha f) = \text{dom}({}_pI_+^\alpha f) = \mathcal{A}'_{2\pi} \tag{6.52}$$

as shown in [133].

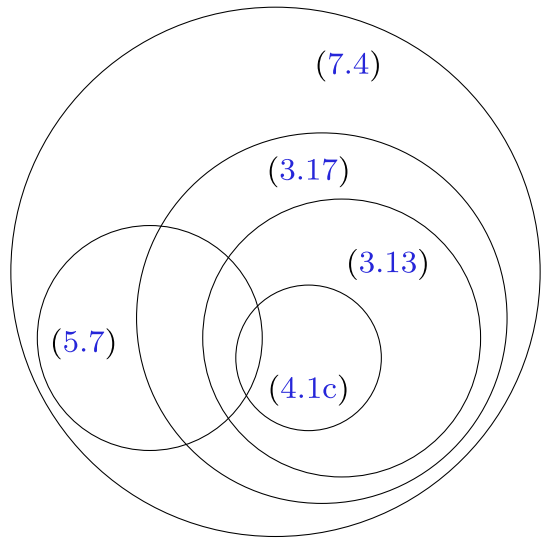
7 Generalized Schwartz fractional calculus for distributions

Schwartz' fractional calculus was significantly extended in [53–55]. The extension is based on generalizing the standard definition (5.7) of convolvability by support. The generalized convolution was first introduced already in [134] but has remained relatively unknown (it was rediscovered in [135], see also [136]). It is not usually discussed in books on distributions. Notable exceptions are [137–139], but even these books do not give the full picture. Additional information can be found in [135, 136, 140–144]. The generalized concept of convolution allows to extend and unify several translation invariant definitions of fractional derivatives [55].

7.1 \mathcal{D}' -convolution

The generalization of convolution proceeds in two steps. The first step is to define the distributional integral in a way analogous to eq. (3.4) for measures. Because $\mathcal{D}_{L^\infty}(\mathbb{R}^d)$,

Fig. 2 Schematic set-theoretic illustration of inclusion and subset relations for the various domains of definition for convolution discussed in the text. The equation numbers refer to the definitions of \mathcal{D}' -convolution (7.4), \mathcal{E}' -convolution (5.7), \mathcal{M} -convolution (3.17), L^1_{loc} -convolution (3.13), $\mathcal{E}(\mathbb{R}_{0+})$ -convolution (4.1c), and $d = 1$ in (3.13) and (4.1c)



when equipped with a suitable topology [138, p. 11], is dual to $\mathcal{D}'_{L^1}(\mathbb{R}^d)$, the *distributional integral of a distribution* $f \in \mathcal{D}'_{L^1}(\mathbb{R}^d)$ is defined as

$$\int f := \langle f, 1_{\mathbb{R}^d} \rangle \tag{7.1}$$

where $1_{\mathbb{R}^d}$ is the constant function on \mathbb{R}^d . The second step is to generalize convolvability by support (5.8) (or \mathcal{E}' -convolvability) to \mathcal{D}' -convolvability by replacing \mathcal{E}' with \mathcal{D}'_{L^1} in (5.8). For $x, y \in \mathbb{R}^d$ one has $\varphi^\Delta(x, y) = \varphi(x + y) \in \mathcal{E}(\mathbb{R}^{2d})$ and (5.8) becomes

$$(f \otimes g) \cdot \varphi^\Delta \in \mathcal{E}'(\mathbb{R}^{2d}) \tag{7.2}$$

when $f, g \in \mathcal{D}'(\mathbb{R}^d)$. Two distributions $f, g \in \mathcal{D}'(\mathbb{R}^d)$ are called *\mathcal{D}' -convolvable* if and only if

$$(f \otimes g) \cdot \varphi^\Delta \in \mathcal{D}'_{L^1}(\mathbb{R}^{2d}) \tag{7.3}$$

for $\varphi \in \mathcal{D}(\mathbb{R}^d)$. This condition is always fulfilled if f, g fulfill (5.8), because $\mathcal{E}'(\mathbb{R}^{2d}) \subset \mathcal{D}'_{L^1}(\mathbb{R}^{2d})$. Finally, *\mathcal{D}' -convolution of two \mathcal{D}' -convolvable distributions* f, g is defined as

$$\langle f * g, \varphi \rangle = \int (f \otimes g) \cdot \varphi^\Delta \tag{7.4}$$

for $\varphi \in \mathcal{D}(\mathbb{R}^d)$. The various definitions of convolution are related as illustrated in Figure 2.

\mathcal{D}' -convolution generalizes also the convolution formula (2.9) for periodic functions as shown in Section 7.5 below. In addition, it generalizes the convolution of Radon measures [98], [138, p. 7] defined as continuous linear functionals on the space $\mathcal{K}(\mathbb{R}^d)$ of continuous functions with compact support. Two Radon measures $\mu, \nu \in \mathcal{M}(\mathbb{R}^d) = (\mathcal{K}(\mathbb{R}^d))'$ are convolvable if and only if $\varphi^\Delta(\mu \otimes \nu)$ is an integrable Radon measure on \mathbb{R}^{2d} for each $\varphi \in \mathcal{E}_c(\mathbb{R}^d)$. The space $\mathcal{M}^1(\mathbb{R}^d)$ of integrable measures is defined as the dual of $\mathcal{E}_v(\mathbb{R}^d)$. Because Radon measures are convolutions [145, p. 300] and \mathcal{D}' -convolution generalizes the convolution of Radon measures, several results of [49, 50] for Weyl fractional integration of measures are contained in the more general results for distributions.

Let $L_c^1(\mathbb{R}^d)$ and $\mathcal{M}_c(\mathbb{R}^d)$ denote the space of integrable functions with compact support and the space of compactly supported measures. Then all spaces in the diagram

$$\begin{array}{ccccc} L_c^1(\mathbb{R}^d) & \subset & \mathcal{M}_c(\mathbb{R}^d) & \subset & \mathcal{E}'(\mathbb{R}^d) \\ \cap & & \cap & & \cap \\ L^1(\mathbb{R}^d) & \subset & \mathcal{M}^1(\mathbb{R}^d) & \subset & \mathcal{D}'_{L^1}(\mathbb{R}^d) \end{array} \tag{7.5}$$

are \mathcal{D}' -convolution algebras [139, p. 69].

7.2 Associativity of \mathcal{D}' -convolution

Let $f_1, \dots, f_p \in \mathcal{D}'(\mathbb{R}^d)$, $d \in \mathbb{N}$, $p \in \mathbb{N}$, $p \geq 2$. The simultaneous \mathcal{D}' -convolution is defined as

$$(f_1 * \dots * f_p)(\varphi) = \int (f_1 \otimes \dots \otimes f_p) \cdot \varphi^{p\Delta} \tag{7.6}$$

whenever (f_1, \dots, f_p) is \mathcal{D}' -convolvable, i.e. whenever

$$(f_1 \otimes \dots \otimes f_p) \cdot \varphi^{p\Delta} \in \mathcal{D}'_{L^1}(\mathbb{R}^{dp}) \tag{7.7}$$

for all $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Here $\varphi^{p\Delta}(x_1, \dots, x_p) = \varphi(x_1 + \dots + x_p)$. This definition follows Definition 5 in [136, p. 373] and remarks from [138, Sec. 1.3]. Convolution is linear and convolvability is preserved under addition in each factor. For the construction of convolution modules the following law of associativity is fundamental.

Proposition 2 (Thm 2 in [143]) *If the p -tuple (f_1, \dots, f_p) , $p \in \mathbb{N}$, $p \geq 2$ with $f_i \in \mathcal{D}' \setminus \{0\}$ is \mathcal{D}' -convolvable, then*

$$f_1 * \dots * f_p = f_1 * \dots * f_q * (f_{q+1} * \dots * f_p) \tag{7.8}$$

for $q \in \{1, \dots, p - 2\}$, where the right hand side is well defined in the sense of definition (7.6)

The set of lower semicontinuous functions $f : \mathbb{R}^d \rightarrow [0, \infty]$ becomes a semigroup for convolution as group operation, if convolution is pointwise defined. This follows

from Fubini’s theorem [99, p. 55] and allows to formulate the following convenient convolvability criterion:

Proposition 3 *A p -tuple (f_1, \dots, f_p) , $p \geq 2$, $f_i \in \mathcal{D}'$ is convolvable if and only if for all test functions $\varphi_k \in \mathcal{D}$ the convolution of the regularisations*

$$|\varphi_1 * f_1| * \dots * |\varphi_p * f_p| \tag{7.9}$$

is a finite-valued function.

Proof The criterion is equivalent to the criterion given in [140, p. 19] after a linear transformation of \mathbb{R}^{dp} . □

The construction of convolution duals $(F)_{\mathcal{M}}^*$, perfection $(F)_{\mathcal{M}}^{**}$ and $(F)_{\mathcal{M}}^{*M}$ for measures can be carried over to distributions and \mathcal{D}' -convolution by replacing \mathcal{M} with \mathcal{D}' . Let $\langle F \rangle_{\mathbb{C},+}$ (resp. $\langle F \rangle_{\mathbb{C},+,*}$) denote the *complex linear span* (resp. *convolution algebra*) generated by $F \subseteq \mathcal{D}'$ (resp. by $F \subseteq \mathcal{D}'$ with $(F)_{\mathcal{D}'}^{*M} \neq \{0\}$).

Theorem 2 (Thm 8, [54]) *Let $F \subseteq \mathcal{D}'$. If $(F)_{\mathcal{D}'}^{*M} \neq \{0\}$, then every tuple from F is convolvable and $(F)_{\mathcal{D}'}^{*M}$ is a convolution module over $\langle F \rangle_{\mathbb{C},+,*}$.*

Denote as \mathcal{S}_+ the space of lower semicontinuous functions $f : \mathbb{R}^d \rightarrow [0, \infty]$, and let $W \subseteq \mathcal{S}_+$. The W -weighted L^1 -space of distributions is defined as

$$\mathcal{D}'_{L^1,W} = \{f \in \mathcal{D}' : \forall w \in W : \|(\varphi * f)w\|_1 < \infty\}. \tag{7.10}$$

Its definition depends only on the *translation invariant cone ideal generated by W* , which is defined as

$$\langle W \rangle_{\text{tici}} = \left\{ v \in \mathcal{S}_+ : \exists w_k \in W, x_k \in \mathbb{R}^d : v \leq T_{x_1} w_1 + \dots + T_{x_n} w_n \right\}, \tag{7.11}$$

where $(T_x f)(y) = f(y - x)$ is translation by x .

Proposition 4 *Let $F \subseteq \mathcal{D}'(\mathbb{R}^d)$. The convolution dual $(F)_{\mathcal{D}'}^*$ and the maximal module $(F)_{\mathcal{D}'}^{*M}$ can be represented as weighted L^1 -spaces of distributions*

$$(F)_{\mathcal{D}'}^* = \mathcal{D}'_{L^1,W}, \quad \text{with } W = \left\{ |\varphi * \check{f}| : \varphi \in \mathcal{D}(\mathbb{R}^d), f \in F \right\} \tag{7.12}$$

$$(F)_{\mathcal{D}'}^{*M} = \mathcal{D}'_{L^1,V}, \quad \text{with } V = (W)_* \tag{7.13}$$

where $(W)_*$ is the set of simultaneous convolution products formed with elements in W , and \check{f} is the reflection of f defined in (10.37).

Proof With the notation $w = |\psi_1 * f_1| * \dots * |\psi_p * f_p|$ one obtains

$$\|(\varphi * g)\check{w}\|_1 = (|\varphi * g| * |\psi_1 * f_1| * \dots * |\psi_p * f_p|)(0) \tag{7.14}$$

for all $g, f_k \in \mathcal{D}'$ and $\varphi, \psi_k \in \mathcal{D}$. Applying Proposition 3 concludes the proof. □

7.3 Generalized Schwartz fractional derivative

To formulate the extension of Schwartz' fractional derivatives ${}_sD_+^\alpha$ and integrals ${}_sI_+^\alpha$ succinctly the following discussion is presented for $d = 1$. The extension to $d \neq 1$ is considered at the end.

The *generalized Schwartz fractional derivative* D_+^α is defined exactly as the Schwartz derivative in eq. (5.4) as

$$D_+^\alpha f := f * Y_{-\alpha} \tag{7.15}$$

for $\alpha \in \mathbb{C}$ but now using the \mathcal{D}' -convolution (7.4) resulting in the significantly enlarged domain

$$\text{dom}(D_+^\alpha) = (Y_{-\alpha})_{\mathcal{D}'}^* \tag{7.16}$$

for $\alpha \in \mathbb{C}$. The *generalized Schwartz fractional integral* is defined as a derivative of negative order $I_+^\alpha := D_+^{-\alpha}$ for $\alpha \in \mathbb{C}$.

Using (7.12) and some simple calculations the domain (7.16) can be described more explicitly. Let $\varphi \in \mathcal{D}(\mathbb{R})$. For $\alpha \in \mathbb{C} \setminus (-\mathbb{N}_0)$ there exists $C < \infty$ such that

$$\left| \left(1 - \frac{y}{x}\right)^{\alpha-1} - 1 \right| \leq C \frac{y}{x} \tag{7.17}$$

for all $x \geq 2y > 0$. Using this inequality gives the asymptotic behaviour

$$\begin{aligned} (Y_\alpha * \varphi)(x) &= \frac{x^{\alpha-1}}{\Gamma(\alpha)} \int \varphi(y) \left(1 - \frac{y}{x}\right)^{\alpha-1} dy \\ &= Y_\alpha(x) \int \varphi(y) dy + \mathcal{O}(x^{\alpha-2}) \end{aligned} \tag{7.18}$$

for $x \rightarrow \infty$. The extended domains can now be described as weighted $L^1(\mathbb{R})$ -spaces. Define the power logarithmic weights

$$w^\mu(x) := w^{\mu,0}(x) \tag{7.19}$$

$$w^{\mu,n}(x) := (1+x^2)^{\mu/2} \left[1 + \log(1+x^2)\right]^n \tag{7.20}$$

for $\mu, x \in \mathbb{R}, n \in \mathbb{N}_0$. Using the shorthand notation $f_+ := f \cdot \Theta$ and $f_- := f \cdot \check{\Theta}$, an asymptotic expansion shows that

$$\langle |Y_\alpha * \mathcal{D}| \rangle_{\text{tici}} = \left\langle w_+^{\text{Re}\alpha-1} \right\rangle_{\text{tici}} \tag{7.21}$$

for $\alpha \in \mathbb{C} \setminus (-\mathbb{N}_0)$. Thus, the enlarged domain $\text{dom}(D_+^\alpha)$ is

$$\begin{aligned} (Y_{-\alpha})_{\mathcal{D}'}^* &= \mathcal{D}'_{L^1, w_-^{\text{Re}\alpha-1}} \\ &= \left\{ f \in \mathcal{D}'(\mathbb{R}); \forall \varphi \in \mathcal{D}(\mathbb{R}) : \int_1^\infty \frac{|(f * \varphi)(-x)|}{x^{\text{Re}\alpha+1}} dx < \infty \right\} \end{aligned} \tag{7.22}$$

for $\alpha \in \mathbb{C} \setminus \mathbb{N}_0$ by virtue of Proposition 4.

Theorem 3 *The index law*

$$I_+^\alpha(I_+^\beta f) = I_+^{\alpha+\beta} f \tag{7.23}$$

holds if $\{\alpha, \beta\} \cap (-\mathbb{N}_0) \neq \emptyset$, or if $f \in \mathcal{D}'_{L^1, v_{\alpha, \beta}}$ with $v_{\alpha, \beta} := w_-^{\text{Re}\alpha-1} * w_-^{\text{Re}\beta-1}$.

Proof Apply Propositions 2 and 3 to the simultaneous convolution $Y_\alpha * Y_\beta * f$. \square

Proposition 5 *Convolution of power logarithmic weights satisfy*

$$\left\langle w_+^{p-1} * w_+^{q-1} \right\rangle_{tici} = \left\langle w_+^{p+q-1} \right\rangle_{tici} \quad \text{for } p, q > 0, \tag{7.24a}$$

$$\left\langle w_+^{-1} * w_+^{-1, n} \right\rangle_{tici} = \left\langle w_+^{-1, n+1} \right\rangle_{tici} \quad \text{for } n \in \mathbb{N}_0, \tag{7.24b}$$

$$\left\langle w_+^{p-1} * w_+^{q-1} \right\rangle_{tici} = \left\langle w_+^{\max\{p, q\}-1} \right\rangle_{tici} \quad \text{for } p < 0 \text{ or } q < 0. \tag{7.24c}$$

Proof Equations (7.24a) and (7.24b) are contained in [146, Lemma 2.2]. Equation (7.24c) follows from [147, VIII.8]. \square

Let $\mathbb{H} := \{z \in \mathbb{C} : \text{Re}z > 0\}$, $\overline{\mathbb{H}} := \{z \in \mathbb{C} : \text{Re}z \geq 0\}$ and $\mathbb{H}_0 := \mathbb{H} \cup \{0\}$ and define the following sets

$$P_\pm := \{w_\pm^n : n \in \mathbb{N}\}, \tag{7.25a}$$

$$Q_\pm := \{w_\pm^q : q < -1\}, \tag{7.25b}$$

$$R_\pm := \{w_\pm^{-1, n} : n \in \mathbb{N}_0\} \tag{7.25c}$$

of weights. Then $\mathcal{D}'_{L^1, P_-} = \mathcal{O}'_C + \mathcal{D}'_+$.

Theorem 4 *Let $\alpha \in \mathbb{C}$. The maximal invariant domain of the Schwartz fractional integral is given by*

$$\text{dom}(I_+^\alpha) = (Y_\alpha)_{\mathcal{D}'}^{*M} = \begin{cases} \mathcal{D}'_{L^1, P_-} & \text{if } \text{Re}\alpha > 0, \\ \mathcal{D}'_{L^1, R_-} & \text{if } \text{Re}\alpha = 0 \text{ and } \alpha \neq 0, \\ (Y_\alpha)_{\mathcal{D}'}^* & \text{if } \text{Re}\alpha < 0 \text{ or } \alpha = 0. \end{cases} \tag{7.26}$$

Fractional integrals with orders $\alpha \in \mathbb{C}$ define a group operating bijectively on \mathcal{D}'_{L^1, P_-} . Fractional integrals with orders $\alpha \in i\mathbb{R}$ define a group operating bijectively on \mathcal{D}'_{L^1, R_-} . Fractional integrals with orders $\alpha \in -\mathbb{H}$ define a semigroup operating on \mathcal{D}'_{L^1, R_-} . Fractional integrals with orders $\alpha \in -\mathbb{H}$ define a semigroup operating on \mathcal{D}'_{L^1, Q_-} .

Proof Apply Proposition 4, Theorem 2 and Proposition 5. □

In [119, p. 126] it has been argued at length that fractional derivatives of the power function $x \mapsto x^n, n \in \mathbb{N}$ do not exist in general, not even in the sense of distributions. The generalized Schwartz derivative $D_+^\alpha(x^n)$ defined above, however, exists if and only if $\alpha \in \mathbb{N}_0$ or $\alpha \in \mathbb{C}$ with $\text{Re}\alpha > n$.

7.4 Discrete Grünwald-Letnikov calculus and the continuum limit

This section discusses discretized fractional integrals and derivatives on domains of sequences and distributional domains in analogy with the continuous case. Discretized functions are sequences, i.e. functions defined on $\text{dom}(f) = h\mathbb{Z}$ instead of \mathbb{R} for some step size $h > 0$. Let $\mathbb{C}^{h\mathbb{Z}}$ be the space of all discrete functions $h\mathbb{Z} \rightarrow \mathbb{C}$. Analogous to the monomials Y_α from (5.2) define the *discrete power function* as

$$y_{\alpha,h}(hk) := \begin{cases} (-1)^k \binom{\alpha}{k} h^\alpha & k \in \mathbb{N}_0 \\ 0 & k \in \mathbb{Z} \setminus \mathbb{N}_0 \end{cases} \tag{7.27}$$

for $\alpha \in \mathbb{C}$ and $h > 0$. Define the associated *discrete Grünwald-Letnikov fractional derivative* as

$${}_{\text{dGL}}D_{+,h}^\alpha f(hk) := \sum_{l \in \mathbb{Z}} f(h(k-l)) y_{\alpha,h}(hl) \tag{7.28}$$

for $k \in \mathbb{Z}$ and $f \in \text{dom}({}_{\text{dGL}}D_{+,h}^\alpha)$ with the natural domain

$$\text{dom}({}_{\text{dGL}}D_{+,h}^\alpha) := \left\{ f \in \mathbb{C}^{h\mathbb{Z}} : \sum_{k \in \mathbb{N}} \frac{|f(-hk)|}{k^{\text{Re}\alpha+1}} < \infty \right\} \tag{7.29}$$

for $\alpha \in \mathbb{C} \setminus \mathbb{N}_0$ and $\text{dom}({}_{\text{dGL}}D_{+,h}^\alpha) = \mathbb{C}^{h\mathbb{Z}}$ for $\alpha \in \mathbb{N}_0$. Thus, ${}_{\text{dGL}}D_{+,h}^\alpha$ is the discrete convolution operator $(\cdot) * y_{\alpha,h}$ on the discrete convolution dual of $\{y_{\alpha,h}\}$. The operators (7.28) were considered, for example, in [148]. With Equation (7.29) we provide a precise definition of their natural domain.

Every discrete function $f \in \mathbb{C}^{h\mathbb{Z}}$ can be identified with a distribution $\delta_{h\mathbb{Z}}(f) \in \mathcal{D}'(\mathbb{R})$ (even with a Radon measure) via

$$\delta_{h\mathbb{Z}}(f) := \sum_{k \in \mathbb{Z}} f(hk)\delta_{kh} \tag{7.30}$$

where δ_x is the Dirac distribution at $x \in \mathbb{R}$ and $h > 0$ the step size. The distribution corresponding to $y_{\alpha,h}$ is

$$Y_{\alpha,h} := \delta_{h\mathbb{Z}}(y_{\alpha,h}) = \sum_{k \in \mathbb{Z}} y_{\alpha,h}(hk)\delta_{kh}. \tag{7.31}$$

The *generalized Grünwald-Letnikov fractional derivative of order $\alpha \in \mathbb{C}$ with step $h > 0$* is then defined as

$${}_{\text{GL}}D_h^\alpha f := f * Y_{-\alpha,h} = \sum_{k \in \mathbb{N}} y_{-\alpha,h}(hk)T_{kh}f \tag{7.32}$$

where $(T_x f)(y) := f(y - x)$ is the translation operator. The domain is

$$\text{dom}({}_{\text{GL}}D_h^\alpha) = (Y_{-\alpha,h})_{\mathcal{D}'}^* = (Y_{-\alpha})_{\mathcal{D}'}^*. \tag{7.33}$$

The definition (7.32) can be understood in terms of \mathcal{D}' -convolution or in terms of the series on the right hand side, that is absolutely convergent in \mathcal{D}' . The second equality in (7.33) is proved analogous to (7.22).

Proposition 6 *Let $\alpha \in \mathbb{C} \setminus -\mathbb{N}_0$ and $\varphi \in \mathcal{D}(\mathbb{R})$. There exist $C, x \in \mathbb{R}_+$ such that*

$$|(Y_\alpha - Y_{\alpha,h}) * \varphi| \leq h \cdot C \cdot T_{-x}w_+^{\text{Re}\alpha-1} \quad \text{for all } 0 < h \leq 1. \tag{7.34}$$

Thus, $Y_{\alpha,h} \rightarrow Y_\alpha$ with respect to the normal topology $\mathfrak{T}^*((Y_\alpha)_{\mathcal{D}'}^{**})$ for all $\alpha \in \mathbb{C}$.

Proof The estimate (7.34) is proved similar to Theorem 1.3 in [90]. The second statement is clear from the definition of the normal topology $\mathfrak{T}*((Y_\alpha)_{\mathcal{D}'}^{**})$ in Definition 8 from [55] and Theorem 3.11 in [149]. □

Proposition 6 and Theorem 8 in [55, p. 143] imply that

$$\lim_{h \searrow 0} {}_{\text{dGL}}D_{+,h}^\alpha f = f * \left(\lim_{h \searrow 0} Y_{-\alpha,h} \right) = f * Y_{-\alpha} = D_{+,f}^\alpha \tag{7.35}$$

for all $f \in (Y_{-\alpha})_{\mathcal{D}'}^*$.

7.5 Periodic distributions

7.5.1 Induced kernels for convolution with periodic distributions

Fractional derivatives and integrals for periodic functions and distributions were already discussed in Sections 2.1, 2.2, 2.5 and 6.6. They are naturally induced by their counterparts on the real line or on the integers.

Periodic functions can be identified with functions on the unit circle understood as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. More generally, consider two commutative Lie groups G, H . Recall, that every commutative Lie group is isomorphic to $\mathbb{R}^a \times \mathbb{Z}^b \times (\mathbb{R}/\mathbb{Z})^c \times D$ for some $a, b, c \in \mathbb{N}_0$ and a finite Abelian group D . Below, $G = \mathbb{R}$ or $G = h\mathbb{Z}$ and $H = 2\pi\mathbb{Z}$ with $h = 2\pi/\Lambda, \Lambda \in \mathbb{N}$ will be considered.

Let $A : G \rightarrow H$ be an affine mapping, i.e. a homomorphism followed by a shift y with $y \in H$. Define the *image* $A(f)$ of a distribution $f \in \mathcal{D}'(G)$ under A as

$$\langle A(f), \varphi \rangle := \langle f \cdot (\varphi \circ A), 1_G \rangle \tag{7.36}$$

for all $\varphi \in \mathcal{D}(H)$, whenever this is well defined. The image of a distribution under a mapping is analogous to the image of a Radon measure under a mapping, see [97, Ch. V, §6, No. 4, Def. 2] or Subsection 3.3. On the other hand, the *pullback of a distribution* $f \in \mathcal{D}'(H)$ under A is defined by (the arrow under the limit sign means convergence in $\mathcal{E}'(H)$)

$$\langle A^*(f), \varphi \rangle := \lim_{\psi \rightarrow \delta} \langle (f * \psi) \circ A, \varphi \rangle \tag{7.37}$$

for all $\varphi \in \mathcal{D}(G)$. This extends the classical definition of $A^*(f) := f \circ A$ for functions $f \in \mathcal{C}(H)$. See [116, Ch. VI] for a more general discussion of pullbacks.

Let $H \hookrightarrow G \rightarrow G/H$ be a short exact sequence of commutative Lie groups and denote the projection as $P_H : G \rightarrow G/H$. The pullback P_H^* is injective on $\mathcal{D}'(G/H)$ and induces a bijection on its range $\mathcal{D}'_H(G)$, where $\mathcal{D}'_H(G)$ denotes the set of distributions from $\mathcal{D}'(G)$ that are invariant under H . The convolution of a distribution $g \in \mathcal{D}'_{L^\infty}(G/H)$ with the projected kernel $P_H(f)$ of a distribution $f \in \mathcal{D}'_{L^1}(G)$ is related to convolution with the original kernel f via

$$P_H^* [P_H(f) *_{G/H} g] = f *_G P_H^*(g). \tag{7.38}$$

This can now be applied to the continuous case where $G = \mathbb{R}$ with $H = 2\pi\mathbb{Z}$ and to the discrete case where $G = h\mathbb{Z}$ with $H = 2\pi\mathbb{Z}$.

In the special case $G = \mathbb{R}$ and $H = \mathbb{Z}$ equation (7.38) follows from

$$\int_{\mathbb{R}/\mathbb{Z}} P(f)(\tilde{y})g(x - \tilde{y})d\tilde{y} = \int_{\mathbb{R}/\mathbb{Z}} \sum_{z \in \tilde{y}} f(z)g(x - \tilde{y})d\tilde{y} = \int_{\mathbb{R}} f(y)P_{\mathbb{Z}}^*(g)(x - y)dy \tag{7.39}$$

for $x \in \mathbb{R}$, $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R}/\mathbb{Z})$ via Fubini's Theorem and a unique continuous extension argument using the continuity of $*$: $\mathcal{D}'_{L^1} \times \mathcal{D}'_{L^\infty} \rightarrow \mathcal{D}'_{L^\infty}$.

7.5.2 Continuous case

Here $G = \mathbb{R}$ with $H = 2\pi\mathbb{Z}$. The space $\mathcal{G}(\mathbb{R}/2\pi\mathbb{Z})$ of test functions on the unit circle is isomorphic to the space of periodic test functions $\mathcal{A}_{2\pi}$ from [133]. Thus $\mathcal{D}'(\mathbb{R}/2\pi\mathbb{Z})$ is isomorphic to $\mathcal{A}'_{2\pi}$. Define the space of 2π -periodic distributions as

$$\mathcal{D}'_{2\pi}(\mathbb{R}) := \{f \in \mathcal{D}'(\mathbb{R}); T_{2\pi z}f = f \text{ for all } z \in \mathbb{Z}\}. \tag{7.40}$$

Distributions from $\mathcal{D}'(\mathbb{R}/2\pi\mathbb{Z})$ are lifted uniquely as

$$\begin{aligned} \mathcal{D}'(\mathbb{R}/2\pi\mathbb{Z}) &\rightarrow \mathcal{D}'_{2\pi}(\mathbb{R}) \\ f &\mapsto f_{2\pi} := P_{2\pi\mathbb{Z}}^*(f) \end{aligned} \tag{7.41}$$

from $\mathcal{D}'(\mathbb{R}/2\pi\mathbb{Z})$ to $\mathcal{D}'_{2\pi}(\mathbb{R})$. For $\alpha \in \mathbb{H}$ one has $Y_{-\alpha} \in \mathcal{D}'_{L^1}(\mathbb{R})$ and

$$P_{2\pi\mathbb{Z}}(Y_{-\alpha}) = \Psi_{-\alpha} \tag{7.42}$$

extending Ψ_α from eq. (2.11) to negative indices, which is then not convergent as a function, but only as a distribution. The series

$$\Psi_\alpha(x) = \sum_{k \in \mathbb{Z}^\times} (ik)^\alpha e^{ikx} \tag{7.43}$$

converges absolutely in $\mathcal{D}'(\mathbb{R}/2\pi\mathbb{Z})$ [5, Ch. VII §1] for all $\alpha \in \mathbb{C}$. Eqs. (7.38) and (7.42) imply that for all $\alpha \in \mathbb{H} \cup \{0\}$ and for $f \in \mathcal{D}'_{2\pi}(\mathbb{R})$ one has

$${}_pD^\alpha f = \Psi_{-\alpha} * f = Y_{-\alpha} * f_{2\pi} = D_+^\alpha f_{2\pi} \tag{7.44}$$

where the fractional derivative ${}_pD^\alpha$ on $\text{dom}({}_pD^\alpha) = \mathcal{D}'(\mathbb{R}/2\pi\mathbb{Z})$ is defined as convolution with $\Psi_{-\alpha}$. This fractional derivative is naturally induced by the generalized Schwartz fractional derivative D_+^α on the space of periodic distributions.

7.5.3 Discrete case

In the discrete case one has $G = h\mathbb{Z}$ with $H = 2\pi\mathbb{Z}$. Now the projected discrete Grünwald-Letnikov kernel is defined as

$$P_{2\pi\mathbb{Z}}(y_{-\alpha,h}) = \psi_{-\alpha,h,\Delta} \tag{7.45}$$

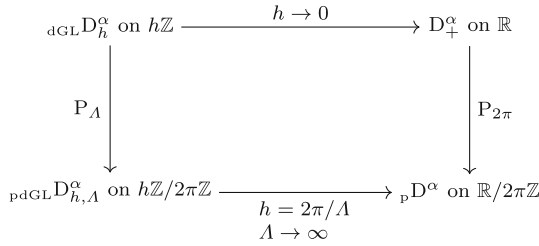


Fig. 3 Diagram illustrating the continuum limit $h \rightarrow 0$ and the projections P_Λ , resp. $P_{2\pi}$, of fractional derivative operators on discrete ${}_{d\text{GL}}D_h^\alpha$, resp. continuous D_+^α , distributions to their periodic counterparts ${}_{pd\text{GL}}D_{h,\Lambda}^\alpha$, resp. ${}_pD^\alpha$. The horizontal arrows hold for all $\alpha \in \mathbb{C}$, the vertical arrows for $\alpha \in \mathbb{H} \cup \{0\}$

with $\Lambda \in \mathbb{N}$ and $h = 2\pi/\Lambda$ and reads as

$$\psi_{-\alpha,h,\Lambda}(h(m + \Lambda\mathbb{Z})) = \frac{1}{h^\alpha} \sum_{k \in m + \Lambda\mathbb{N}_0} (-1)^k \binom{\alpha}{k} \tag{7.46}$$

for $m \in \mathbb{N}_0$ and $0 \leq m < \Lambda$. The *generalized Grünwald-Letnikov fractional derivative of order $\alpha \in \mathbb{C}$ with step $h = 2\pi/\Lambda$ and period Λ* is then defined as

$${}_{\text{GL}}D_{h,\Lambda}^\alpha f := f * \psi_{-\alpha,h,\Lambda} \tag{7.47}$$

with domain

$$d\text{om} \left({}_{\text{GL}}D_{h,\Lambda}^\alpha \right) = \mathcal{a}(h\mathbb{Z}/2\pi\mathbb{Z}), \tag{7.48}$$

where $\mathcal{a}(h\mathbb{Z}/2\pi\mathbb{Z})$ is the space of all sequences on the quotient $h\mathbb{Z}/2\pi\mathbb{Z}$, which can be identified with the space of periodic sequences with period $\Lambda = 2\pi/h$. For $\alpha \in \mathbb{C}$ the limit

$$\lim_{\Lambda \rightarrow \infty} \Lambda^\alpha \sum_{m=0}^{\Lambda-1} \sum_{k \in \mathbb{N}_0} (-1)^k \binom{\alpha}{m + k\Lambda} \delta_{2\pi m/\Lambda} = \Psi_{-\alpha} \tag{7.49}$$

recovers the kernel on $\mathbb{R}/2\pi\mathbb{Z}$, extending the definition of $\Psi_{-\alpha}$ to $\alpha \in i\mathbb{R} \setminus \{0\}$.

Due to the generality of \mathcal{D}' -convolution the continuum limit from discrete to continuous fractional derivatives and the projection from non-periodic to periodic distributions can be freely combined as illustrated in Figure 3.

7.6 Extensions to higher dimension

Let $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{C}^d$, $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, and

$$Y_\alpha^{(d)} = \bigotimes_{i=1}^d Y_{\alpha_i} = Y_{\alpha_1} \otimes \dots \otimes Y_{\alpha_d} \tag{7.50}$$

the *multinomial homogeneous distribution*, which generalizes the one-dimensional monomial distribution from (5.2). The *generalized Schwartz fractional integral* I_+^α is defined as the mapping

$$I_+^\alpha : \mathcal{D}'(\mathbb{R}^d) \rightarrow \mathcal{D}'(\mathbb{R}^d)$$

$$f \mapsto f * Y_\alpha^{(d)} \tag{7.51}$$

for all \mathcal{D}' -convolvable $(d + 1)$ -tuples $(f, Y_{\alpha_1}, \dots, Y_{\alpha_d})$. Its inverse

$$D_+^\alpha f = f * Y_{-\alpha}^{(d)} = I_+^\alpha f \tag{7.52}$$

is the *generalized Schwartz fractional derivative* D_+^α . On the severely restricted domain $\mathcal{D}'(\mathbb{R}_{0+}^d) \subset \mathcal{D}'(\mathbb{R}^d)$ the generalized law of additivity

$$I_+^\alpha I_+^\beta = I_+^{\alpha+\beta} \tag{7.53}$$

holds true for all $\alpha, \beta \in \mathbb{C}^d$ [52]. Note that $\mathcal{D}'(\mathbb{R}_{0+}^d) \subset \text{dom}({}_sD_+^\alpha)$, even for $d = 1$.

Another often studied generalization of fractional derivatives to higher dimension are fractional powers $(-\Delta)^{\alpha/2}$ of the (negative) Laplacian with general $\alpha \in \mathbb{C}$ [32, 45, 150–154]. These operators differ from the operators discussed so far, and in particular from the generalized Schwartz fractional derivative D_+^α , because their kernel is different. The *generalized fractional Riesz derivative* ${}_RD^\alpha$ for distributions is defined as the convolution mapping

$${}_RD^\alpha : (\mathbb{R}_{-\alpha})_{\mathcal{D}'}^* \rightarrow \mathcal{D}'(\mathbb{R}^d)$$

$$f \mapsto {}_RD^\alpha f = (-\Delta)^{\alpha/2} f := R_{-\alpha} * f \tag{7.54}$$

where

$$R_\alpha(x) := \frac{\Gamma((d - \alpha)/2)}{2^\alpha \pi^{d/2} \Gamma(\alpha/2)} |x|^{\alpha-d} \tag{7.55}$$

for all $\alpha \in \mathbb{H} \setminus (d + 2\mathbb{N}_0)$, is the *Riesz kernel* [32], [136, p. 369], [152]. Because the Riesz kernels are convolvable if and only if $\text{Re}(\alpha + \beta) < d$ the fractional Laplacian becomes an endomorphism on the domain

$$\text{dom}({}_RD^\alpha) = \left\{ f \in \mathcal{D}'(\mathbb{R}^d); \forall \varphi \in \mathcal{D}(\mathbb{R}^d) : \int \frac{|\varphi * f(x)|}{(1 + x^2)^{d+\alpha}} dx < \infty \right\} \tag{7.56}$$

for $\alpha > 0, \alpha \notin 2\mathbb{N}$ [55, Sec. 8].

This domain was envisaged in [151, 152] as the dual of the space

$$\mathcal{D}_\alpha(\mathbb{R}^d) = \left\{ \varphi \in \mathcal{E}(\mathbb{R}^d); \forall \beta \in \mathbb{N}_0^d : \sup \|(\partial^\beta \varphi)(1 + x^2)^{d+\alpha}\|_\infty < \infty \right\} \tag{7.57}$$

equipped with the seminorms $\varphi \mapsto \|(\partial^\beta \varphi)(1 + x^2)^{d+\alpha}\|_\infty$ and $\beta \in \mathbb{N}_0^d$. But with this topology the dual $(\mathcal{D}_\alpha(\mathbb{R}^d))'$ is not contained in $\mathcal{D}'(\mathbb{R}^d)$, contrary to statements in [151, Sec. 2.1] and [152], because $\mathcal{D}(\mathbb{R}^d)$ is not dense in $\mathcal{D}_\alpha(\mathbb{R}^d)$. However, using the seminorms $\varphi \mapsto \|(\partial^\beta \varphi)(1 + x^2)^{d+\alpha} \psi\|_\infty$ with $\psi \in \mathcal{C}_v(\mathbb{R}^d)$ and $\beta \in \mathbb{N}^d$ this shortcoming can be repaired similar to [138, Prop. 1.3.1]. For more details see [55] and [155, II.5.d]

We remark also that the test function space \mathcal{T} introduced for fractional powers of the negative Laplace operator in Definition 3.1 [156, p. 261] is nothing but $\mathcal{T} = \mathcal{D}_{L^1}(\mathbb{R}^d)$. Therefore its dual is $\mathcal{T}' = \mathcal{D}'_{L^\infty}(\mathbb{R}^d)$, which is smaller than our domain $\text{dom}({}_R D^\alpha)$ above. This follows from the structural properties of amalgam spaces described in [155, Sec. II.3]

For $d = 1$ the generalized fractional Riesz derivatives enjoy special properties [55]. An example is the factorization into causal and anticausal fractional derivatives

$${}_R D^{2\alpha} f = (-\Delta)^\alpha f = D_+^\alpha (D_-^\alpha f) = D_-^\alpha (D_+^\alpha f) \tag{7.58}$$

for $\text{Re} \alpha > 0$ and all $f \in \text{dom}(D_+^\alpha) \cap \text{dom}(D_-^\alpha)$. Thus, the factorization is valid only on a strict subdomain of the fractional Laplacian. Note, that the power of the Laplacian in [55, eq. (8.5)] should be α , not 2α .

8 Laplace multiplier approach for distributions

Liouville’s formula as used by Weyl [82, Satz 2] defines fractional derivatives as Laplace multipliers. This mathematical interpretation provides yet another approach to define fractional derivatives of distributions. At the first glance, Fourier transform of tempered distributions [5, Ch. VIII] seems to be the right tool for this. However, defining domains is difficult, because the multiplication of a tempered distribution with a continuous function, such as $\mathbb{R} \ni x \mapsto x^\alpha, \alpha > 0$, is not necessarily well defined. This issue will be resolved using pointwise defined \mathcal{D}'_{L^1} -Fourier-Laplace transforms that are based on integration of distributions. This approach was only briefly mentioned in [5, Ch. VIII] and has, up to some exceptions such as [139], not been discussed further in the literature.

Let $\exp_z : \mathbb{R}^d \rightarrow \mathbb{C}$ with $\exp_z(x) = \exp(z \cdot x)$. Because $\exp_z \in \mathcal{E}$ for all $z \in \mathbb{C}^d$ a common approach is to define the Fourier-Laplace transform of a distribution via the dual system $(\mathcal{E}', \mathcal{E})$ as

$$L\{f\}(z) := \langle f, \exp_{-iz} \rangle \tag{8.1}$$

and then to extend it at least in some subsets of \mathbb{C}^d to more general distributions [116, 137, 139, 157]. Using the dual pair $(\mathcal{S}', \mathcal{S})$ Schwartz also studied the Fourier-Laplace transform as the distribution-valued mapping

$$\left\{ z \in \mathbb{C}^d : f \exp_z \in \mathcal{S}' \right\} \ni \zeta \mapsto L\{f \exp_\zeta\} \in \mathcal{S}'. \tag{8.2}$$

Using the dual pair $(\mathcal{D}'_{L^1}, \mathcal{D}_{L^\infty})$ instead, the Fourier-Laplace transform can be understood as a function-valued mapping.

For any $f \in \mathcal{D}'$, $E \subseteq \mathcal{D}'$ define the f -dependent set of E -conjugate (Fourier)-Laplace variables as

$$\mathbb{L}^d(f, E) := \left\{ z \in \mathbb{C}^d : f \exp_{-z} \in E \right\}. \tag{8.3}$$

The \mathcal{D}'_{L^1} -Laplace transform of a distribution $f \in \mathcal{D}'(\mathbb{R}^d)$ is defined via the distributional integral at the points $z \in \mathbb{L}^d(f, \mathcal{D}'_{L^1}) \subseteq \mathbb{C}^d$ as

$$L\{f\}(z) := \int f \exp_{-z}. \tag{8.4}$$

Proofs for the following continuity properties of these Laplace transforms were briefly sketched in [5, Ch. VIII]:

Theorem 5 *Let $f \in \mathcal{D}'$. The sets $\mathbb{L}^d(f, \mathcal{D}'_{L^1})$, $\mathbb{L}^d(f, \mathcal{O}'_C)$ and $\mathbb{L}^d(f, \mathcal{S}')$ are of the form $A + i\mathbb{R}^d$ where $A \subseteq \mathbb{R}^d$ is convex. All three sets have the same interior. The function*

$$\mathbb{L}^d(f, \mathcal{D}'_{L^1}) \ni z \mapsto L\{f\}(z) \tag{8.5}$$

is continuous on $P + i\mathbb{R}^d$ for every convex polytope $P \subseteq \text{Re}(\mathbb{L}^d(f, \mathcal{D}'_{L^1}))$. It is holomorphic on the interior of $\mathbb{L}^d(f, \mathcal{S}')$, where it agrees with the generalized rotated Fourier-Laplace transform $\hat{f}(iz)$ in the sense of [116, Sec. 7.4].

Proof The equation $\mathcal{D}'_{L^1} \cdot \exp_{ix} = \mathcal{D}'_{L^1}$ holds for all $x \in \mathbb{R}^d$, which proves that $\mathbb{L}^d(f, \mathcal{D}'_{L^1}) = \text{Re}[\mathbb{L}^d(f, \mathcal{D}'_{L^1})] + i\mathbb{R}^d$.

Let P be a convex polytope with vertices $\xi_1, \dots, \xi_n \in \text{Re}[\mathbb{L}^d(f, \mathcal{D}'_{L^1})] \subseteq \mathbb{R}^d$, $n \in \mathbb{N}$. Set $h := \exp_{\xi_1} + \dots + \exp_{\xi_n}$. Then $h \in \mathcal{E}$, $f \cdot h \in \mathcal{D}'_{L^1}$ and

$$\left\{ \frac{\exp_z}{h} : z \in P + i[-R, R] \right\} \in \mathfrak{B}(\mathcal{D}_{L^\infty}) \tag{8.6}$$

holds for all $R > 0$, where $\mathfrak{B}(\mathcal{D}_{L^\infty})$ is the set of all bounded subsets of \mathcal{D}_{L^∞} . From this follows $P \subseteq \mathbb{L}^d(f, \mathcal{D}'_{L^1})$, because

$$f \exp_z = (fh) \frac{\exp_z}{h} \in \mathcal{D}'_{L^1} \cdot \mathcal{D}_{L^\infty} \subseteq \mathcal{D}'_{L^1} \tag{8.7}$$

for all $z \in P$. This proves that $\text{Re}[\mathbb{L}^d(f, \mathcal{D}'_{L^1})]$ is convex.

If P is a convex polytope contained in $\text{Re}[\mathbb{L}^d(f, \mathcal{D}'_{L^1})]$ and z a point in its interior, then $\exp_z/h \in \mathcal{S}$ and

$$\mathcal{S}' \cdot \left(\frac{\exp_z}{h} \right) \subseteq \mathcal{O}'_C \tag{8.8}$$

implying $z \in \mathbb{L}^d(f, \mathcal{O}'_C)$. Thus $\mathbb{L}^d(f, \mathcal{O}'_C)$ contains the interior of $\mathbb{L}^d(f, \mathcal{S}')$.

Further, if $z \in P$ converges to $z_0 \in P$, then \exp_z/h converges to \exp_{z_0}/h in \mathcal{G} and thus also within \mathcal{B}_C [138, p. 11]. Because multiplication $\mathcal{D}'_{L^1} \times \mathcal{B}_C \rightarrow \mathcal{D}'_{L^1}$ is hypocontinuous, it follows that

$$L\{f\}(z) = \int fh \frac{\exp_z}{h} \xrightarrow{z \rightarrow z_0} \int fh \frac{\exp_{z_0}}{h} = L\{f\}(z_0) \tag{8.9}$$

and thus $L\{f\}$ is continuous on $P + i\mathbb{R}^d$.

The convexity of $\mathbb{L}^d(f, \mathcal{S}')$ can be proved in the same way (see [117, p. 191]). For $g \in \mathcal{D}'_{L^1}(\mathbb{R}^d)$ the function $z \mapsto \tilde{g}(z) := \mathcal{D}'_{L^1}\langle g, \exp_{-iz} \rangle_{\mathcal{D}_{L^\infty}}$ is continuous with at most polynomial growth by Proposition 1.6.6(2) from [139]. Hence $L\{f\}(z) = \tilde{f}(iz)$ and applying Theorem 7.4.2 from [116] yields analyticity of $L\{f\}(z)$ on the interior of $\mathbb{L}^d(f, \mathcal{S}')$. □

Remark 1 The injectivity of the Fourier transform on \mathcal{D}'_{L^1} entails the injectivity of the \mathcal{D}'_{L^1} -Fourier-Laplace transform from Equation (8.4).

Proposition 7 Let $f, g \in \mathcal{D}'$, $z \in \mathbb{C}^d$ and $i \in \{1, \dots, d\}$.

(1) If (f, g) is \mathcal{D}' -convolvable, then $(f \exp_z, g \exp_z)$ is also \mathcal{D}' -convolvable and

$$(f \exp_z) * (g \exp_z) = (f * g) \exp_z. \tag{8.10}$$

(2) If $z \in \mathbb{L}^d(f, \mathcal{D}'_{L^1})$, then

$$f * \exp_z = L\{f\}(z) \cdot \exp_z. \tag{8.11}$$

(3) If $(\partial_i f, g)$ and $(f, \partial_i g)$ are \mathcal{D}' -convolvable and $(\varphi * f) \cdot (\psi * \tilde{g})$ is vanishing at infinity for all $\varphi, \psi \in \mathcal{D}$, then

$$(\partial_i f) * g = f * (\partial_i g) \tag{8.12}$$

Proof Let $\varphi \in \mathcal{D}$.

(1): Using $(f \exp_z) \otimes (g \exp_z) = (f \otimes g) \exp_z^\Delta$ and $(\varphi \exp_z)^\Delta = \varphi^\Delta \exp_z^\Delta$ gives

$$\begin{aligned} ((f \exp_z) * (g \exp_z), \varphi) &= \int (f \otimes g)(\varphi \exp_z)^\Delta \\ &= \langle f * g, \varphi \exp_z \rangle = \langle (f * g) \exp_z, \varphi \rangle. \end{aligned} \tag{8.13}$$

(2): Equation (8.11) is clear for $f \in \mathcal{S}'$ and can be extended to the general case via approximate units using Lemma 3.5 and Theorem 7.1 from [158].

(3): This follows from Proposition 1 in [159, p. 534] and the remarks subsequent to Equation (1) in [160, p. 202–203].

□

Theorem 6 (Convolution Theorem) *If the pair (f, g) of distributions is \mathcal{D}' -convolvable, then $\mathbb{L}^d(f * g, \mathcal{D}'_{L^1}) \supseteq \mathbb{L}^d(f, \mathcal{D}'_{L^1}) \cap \mathbb{L}^d(g, \mathcal{D}'_{L^1})$ and*

$$L\{f * g\}(z) = L\{f\}(z) \cdot L\{g\}(z) \tag{8.14}$$

for all $z \in \mathbb{L}^d(f, \mathcal{D}'_{L^1}) \cap \mathbb{L}^d(g, \mathcal{D}'_{L^1})$.

Proof Let $z \in \mathbb{L}^d(f, \mathcal{D}'_{L^1}) \cap \mathbb{L}^d(g, \mathcal{D}'_{L^1})$, $f, g \in \mathcal{D}'$ and $\varphi, \psi \in \mathcal{D}$. Then one has $|f * \varphi| \cdot \exp_{-z} \in L^1(\mathbb{R}^d)$, $|g * \psi| \cdot \exp_{-z} \in L^1(\mathbb{R}^d)$ and thus

$$(|f * \varphi| \cdot |g * \psi|) \cdot \exp_{-z} = (|f * \varphi| \cdot \exp_{-z}) * (|g * \psi| \cdot \exp_{-z}) \in L^1(\mathbb{R}^d) \tag{8.15}$$

This entails \mathcal{D}' -convolvability of (f, g, \exp_z) and thus also \mathcal{D}' -convolvability of $(f * g, \exp_z)$. Computing

$$\begin{aligned} L\{f * g\}(z) \cdot \exp_z &= (f * g) * \exp_z \\ &= f * (g * \exp_z) \\ &= (f * \exp_z) \cdot L\{g\}(z) \\ &= L\{f\}(z) \cdot L\{g\}(z) \cdot \exp_z \end{aligned} \tag{8.16}$$

concludes the proof. □

Corollary 1 *Let $f \in \mathcal{D}'(\mathbb{R}^d)$ and $\kappa \in \mathbb{N}_0^d$. Then $\mathbb{L}^d(f, \mathcal{D}'_{L^1}) \subseteq \mathbb{L}^d(\partial^\kappa f, \mathcal{D}'_{L^1})$ and*

$$L\{\partial^\kappa f\}(z) = z^\kappa L\{f\}(z) \tag{8.17}$$

for all $z \in \mathbb{L}^d(f, \mathcal{D}'_{L^1})$.

Proof It is left to the reader to prove that $\mathbb{L}^d(\partial^\kappa \delta, \mathcal{D}'_{L^1}) = \mathbb{C}^d$ and $L\{\partial^\kappa \delta\}(z) = z^d$. Then equation (8.17) follows from the convolution theorem. □

As stated in Theoreme XXV, 1° in [5, p. 201] and the corollary of Proposition 3.5.3 in [117, p. 347] the space \mathcal{D}'_{L^1} can be represented as

$$\begin{aligned} \mathcal{D}'_{L^1}(\mathbb{R}^d) &= \left\langle \partial^\kappa f : f \in L^1(\mathbb{R}^d), \kappa \in \mathbb{N}_0^d \right\rangle_{\mathbb{C},+} \\ &= \left\langle \partial^\kappa \mu : \mu \in \mathcal{M}^1(\mathbb{R}^d), \kappa \in \mathbb{N}_0^d \right\rangle_{\mathbb{C},+} \end{aligned} \tag{8.18}$$

where $\langle \cdot \rangle_{\mathbb{C},+}$ denotes the linear span. Accordingly, represent $f \in \mathcal{D}'_{L^1}$ as the sums $f = \sum_\kappa \partial^\kappa f_\kappa = \sum_\kappa \partial^\kappa \mu_\kappa$ with a finite number of $f_\kappa \in L^1(\mathbb{R}^d)$ and $\mu_\kappa \in \mathcal{M}^1(\mathbb{R}^d)$. Then Corollary 1 above implies that

$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^d} f_0(x) dx = \int_{\mathbb{R}^d} d\mu_0(x) \tag{8.19}$$

holds for the distributional integral. Using this one computes

$$\mathbb{L}\{Y_{-\alpha}\}(z) = z^\alpha \tag{8.20}$$

for all $(\alpha, z) \in \mathbb{H}^d \times \mathbb{H}^d$. This can be extended to all $(\alpha, z) \in (\mathbb{H} \cup \{0\})^d \times \overline{\mathbb{H}}^d$ by virtue of Theorem 5, and to all $(\alpha, z) \in \mathbb{C}^d \times \mathbb{H}^d$ using Corollary 1.

Let $d = 1$ and $\alpha \in \mathbb{C}$. The Laplace multiplier fractional derivative ${}_{\text{LT}}D^\alpha$ is defined implicitly by the relation

$$\mathbb{L}\left\{{}_{\text{LT}}D^\alpha f\right\}(z) = z^\alpha \mathbb{L}\{f\}(z) \tag{8.21}$$

for all $z \in \mathbb{L}(f, \mathcal{D}'_{L^1}) \cap \mathbb{H}$, or $z \in \mathbb{L}(f, \mathcal{D}'_{L^1}) \cap \overline{\mathbb{H}}$ in case $\text{Re}\alpha > 0$ or $\alpha = 0$. As usual, the fractional power $z^\alpha = \exp(\alpha(\log|z| + i \arg(z)))$ with $\arg(z) \in (-\pi, \pi)$ is taken on the principal branch. The distribution ${}_{\text{LT}}D^\alpha f$ is well-defined due to Remark 1. The domain of definition

$$\text{dom}\left({}_{\text{LT}}D^\alpha\right) = \bigcup_{\mu \in M} \mathcal{D}'_{L^1}(\mathbb{R}) \cdot e^{\mu x} \tag{8.22}$$

with the sets

$$M = \begin{cases} \mathbb{R}_{0+} & \alpha \in \mathbb{H} \setminus \mathbb{N}, \\ \mathbb{R}_+ & \alpha \in -\overline{\mathbb{H}} \setminus \{0\}, \\ \mathbb{R} & \alpha \in \mathbb{N}_0, \end{cases} \tag{8.23}$$

is smaller than the domain of the corresponding generalized Schwartz fractional operator from equation (7.16). The Laplace multiplier fractional integral ${}_{\text{LT}}I^\alpha$ is defined as ${}_{\text{LT}}I^\alpha := {}_{\text{LT}}D^{-\alpha}$.

A close relation with Lamb’s operators in Section 6.5 emerges from comparison with eq. (6.44). The spaces $D_{p,\mu}$ with $\mu < 0$ in (6.44) are an inductive system with respect to set-theoretic inclusion. If one takes their inductive limit one obtains the Laplace multiplier domains in (8.22). One finds $\text{dom}({}_{\text{Lb}}I_+^\alpha) = \text{dom}({}_{\text{LT}}I^\alpha)$ for $\text{Re}\alpha \geq 0$ and $\alpha \neq 0$, while $\text{dom}({}_{\text{Lb}}D_+^\alpha) \subset \text{dom}({}_{\text{LT}}D^\alpha)$ for $\text{Re}\alpha > 0$ or $\alpha = 0$. The relation between Lamb’s fractional operators and Laplace multipliers may become understandable, because z^α can be viewed as the fractional power of a multiplication operator, so that both approaches are based on fractional powers.

9 Towards unification of fractional calculus

9.1 Illustrative examples

To illustrate the greater generality of D_+^α as compared to other fractional derivatives, this subsection collects various examples of functions and distributions

$f : \mathbb{R} \rightarrow \mathbb{C}$. They show that D_+^α holds some promise for the unification of translation invariant approaches to fractional calculus. The examples are

- A : $f_A(x) = 1$
- B : $f_B(x) = e^x$
- C : $f_C(x) = e^{ix}$
- D : $f_D(x) = e^{ix^2}$
- E : $f_E = e^{-\sqrt{1+x^2}}$
- F : $f_F(x) = \Theta(x)e^{x^2-x^{-2}}$
- G : $f_G(x) = (\sqrt{1+x^2})^{\alpha-0.1}, \quad \alpha > 0$
- H : $f_H(x) = \Theta(-x)$
- I : $f_I(x) = \delta(x)$
- J : $f_J(x) = \delta'(x)$
- K : $f_K(x) = Y_{1/2}(x)$ $\notin_K: 2\alpha < 1$
- L : $f_L(x) = \sum_{k=1}^{\infty} \varphi[k(x+k)]k^{\alpha+1/2}, \quad \alpha > 0, \varphi \in \mathcal{D}, \varphi > 0$ $\notin_L: 2\alpha \geq 1$
- M : $f_M(x) = \text{sgn}(\sin(x))$ $\notin_M^1: 2\alpha \geq 1, \notin_M^2: \alpha p > 1$

An entry \in in the table below means that $f_i \in \text{dom}(XYD^\alpha)$ **for all** $\alpha > 0$, where XY indicates the type of fractional derivative. An entry \notin means that $f_i \notin \text{dom}(XYD^\alpha)$ **for all** $\alpha > 0$. The table lists \notin_i if inclusion or exclusion depends on α , where the condition is given above together with the definition of f_i .

f_A	f_B	f_C	f_D	f_E	f_F	f_G	f_H	f_I	f_J	f_K	f_L	f_M	$\text{dom}(\cdot)$	equation
\in	\notin	\in	\notin	\notin	\notin	\notin	\notin	\notin	\notin	\notin	\in	\notin_M^1	LwD^α	(2.8)
\in	\notin	\in	\notin	\notin	\notin	\notin	\notin	\notin	\notin	\notin	\notin	\notin_M^2	LGD_p^α	(2.25)
\in	\notin	\in	\notin	\notin	\notin	\notin	\notin	\notin	\notin	\notin	\notin	\notin	LGD^α	(2.26)
\in	\notin	\in	\notin	\notin	\notin	\notin	\notin	\notin	\notin	\notin	\notin	\in	pD_+^α	(6.52)
\notin	\in	\notin	\notin	\in	\in	\notin	\notin	\notin	\notin	\notin_K	\notin	\notin	wD_+^α	(2.16)
\in	\in	\in	\in	\in	\in	\in	\notin	\notin	\notin	\notin	\notin	\notin	MD_+^α	(2.20)
\in	\in	\in	\in	\in	\notin	\in	\notin	\notin	\notin	\notin	\in	\notin	GLD_+^α	(2.22)
\notin	\notin	\notin	\notin	\notin	\in	\notin	\notin	\in	\in	\in	\notin	\notin	SD_+^α	(5.5)
\notin	\in	\notin	\notin	\in	\in	\notin	\notin	\in	\in	\in	\notin	\notin	aWD_+^α	(6.7)
\notin	\in	\notin	\in	\in	\notin	\notin	\in	\in	\in	\in	\notin_L	\notin	LTD_+^α	(8.22)
\in	\in	\in	\in	\in	\in	\in	\in	\in	\in	\in	\in	\in	D_+^α	(7.26)

Riemann-Liouville operators and other non-translation-invariant operators are missing from this table, because they operate on sometimes very different domains. Comparison with translation invariant operators would require extensions, embeddings or projections that influence the comparison. Lizorkin type operators and operational calculus operators are missing, because they operate not on distributions but equivalence classes thereof.

Distributions growing too fast, e.g. exponentially, on the left half axis fall outside the domain of all fractional derivative operators. After they have been excluded, there remains only one domain that contains all examples, namely that of the generalized Schwartz fractional derivative $D_+^\alpha = I_+^{-\alpha}$ in the last row.

9.2 Unification theorem

The following theorem summarizes various examples of translation invariant fractional derivative operators that can be obtained as special cases of the generalized Schwartz fractional derivative D_+^α defined in Section 7.3 with domain (7.26).

Theorem 7

- (a) It holds ${}_S D_+^\alpha \subset D_+^\alpha$ for $\alpha \in \mathbb{C}$.
- (b) It holds ${}_M D_+^\alpha \subset D_+^\alpha$ for $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$.
- (c) It holds ${}_{aw} D_+^\alpha \subset D_+^\alpha$ for $\alpha \in \mathbb{C}$.
- (d) It holds ${}_{GL} D_+^\alpha \subset D_+^\alpha$ for $\alpha \in \mathbb{C}$.
- (e) It holds ${}_{Lb} D_+^\alpha \subset D_+^\alpha$ for $\alpha \in \mathbb{H}$.
- (f) It holds ${}_p D_+^\alpha \subset D_+^\alpha$ for $\alpha \in \mathbb{R}_{0+}$.
- (g) It holds ${}_{LT} D_+^\alpha \subset D_+^\alpha$ for $\alpha \in \mathbb{C}$.

Proof The inclusions of the domains are obvious in each case and it remains to verify the extension part. Part (a) is clear from Figure 2. For (c) and (e), one notes first, that

$${}_M D_+^\alpha \varphi = {}_{aw} D_+^\alpha \varphi = {}_{Lb} D_+^\alpha \varphi = D_+^\alpha \varphi \tag{9.1}$$

for all $\varphi \in \mathcal{D}$. The equalities (9.1) extend to the respective domains of ${}_{aw} D_+^\alpha$ and ${}_{Lb} D_+^\alpha$ by the density of \mathcal{D} and the continuity of transpose operators. For (b), one uses (9.1) and continuity properties of Lebesgue integrals. Part (d) follows from the considerations in Subsection 7.4. Part (f) was treated in Subsection 7.5.2. Part (g) follows from the convolution theorem (Theorem 6). □

9.3 Weak Weyl-, Marchaud- and Grünwald-Letnikov derivatives

The generalized Schwartz fractional derivative D_+^α from eq. (7.15) can be interpreted as a weak formulation of fractional derivatives in the following sense:

Theorem 8 (a) Let $\alpha \in \mathbb{C}$ and $f \in \mathcal{O}'_C + \mathcal{D}'_+$. Then

$$(D_+^\alpha f) * \varphi = {}_w D_+^\alpha (f * \varphi) \tag{9.2a}$$

for all $\varphi \in \mathcal{D}$ with the right hand side defined point wise by eq. (2.15).

(b) Let $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$ and $f \in (Y_{-\alpha})^*_{\mathcal{D}'}$. Then

$$(D_+^\alpha f) * \varphi = {}_M D_+^\alpha (f * \varphi) \tag{9.2b}$$

for all $\varphi \in \mathcal{D}$ with the right hand side defined point wise by eq. (2.17).

(c) Let $\alpha \in \mathbb{C}$ and $f \in (Y_{-\alpha})^*_{\mathcal{D}'}$. Then

$$(D_+^\alpha f) * \varphi = {}_{GL} D_+^\alpha (f * \varphi) \tag{9.2c}$$

for all $\varphi \in \mathcal{D}$ with the right hand side defined point wise by eq. (2.21).

Proof Associativity yields $(D_+^\alpha f) * \varphi = D_+^\alpha (f * \varphi)$ for $f \in (Y_{-\alpha})_{\mathcal{D}}$. Thus, it is clear that the equations hold for $f \in \mathcal{E}'$. The general case is obtained by considering approximations $f\theta_n \rightarrow f$ via an approximate unit (θ_n) , that is, a sequence $\theta_n \in \mathcal{D}$ that is bounded in \mathcal{D}_{L^∞} and satisfies $\theta_n \rightarrow 1$ in \mathcal{E} . It is readily seen, that the limits $\lim_{n \rightarrow \infty} {}_x D_+^\alpha ((f\theta_n) * \varphi) = {}_x D_+^\alpha (f * \varphi)$ hold for $X = W, M, GL$. \square

The theorem implies that it is possible to employ the Grünwald-Letnikov fractional derivative of functions to define the generalized Schwartz fractional derivative of distributions by the formula

$$({}_x D_+^\alpha f, \varphi) := {}_{GL} D_+^\alpha (f * \check{\varphi})(0) \tag{9.3}$$

for all $\alpha \in \mathbb{C}$, $f \in (Y_{-\alpha})_{\mathcal{D}}$, and $\varphi \in \mathcal{D}$. The same is possible for the Marchaud fractional derivative if $\alpha \in \mathbb{R}_+ \setminus \mathbb{N}$.

10 Appendix

10.1 General notation

Natural, integer, real and complex numbers are denoted $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$, and \mathbb{N}_0 is the set of natural numbers with zero. Positive real numbers are denoted $\mathbb{R}_+ :=]0, \infty[$, non-negative real numbers as $\mathbb{R}_{0+} := [0, \infty[$. The set $\overline{\mathbb{R}}_+ := [0, \infty]$ consists of the non-negative real numbers including infinity with the convention $0 \cdot \infty = 0$. The open resp. closed complex half plane is written \mathbb{H} resp. $\overline{\mathbb{H}}$. The shorthand notations $X^d := X \times \dots \times X$ for d -fold cartesian products and $X^\times := X \setminus \{0\}$ for excluding zero are used.

10.2 Locally convex spaces

A few basic concepts of locally convex spaces are collected here for convenient reference. A *seminorm* on a vector space X is a function $q : X \rightarrow \mathbb{R}_{0+}$ such that $q(\lambda x) = |\lambda|q(x)$ and $q(x + y) \leq q(x) + q(y)$. Let A be an arbitrary set of indices. The topology of a locally convex space can be defined by a family $(q_a)_{a \in A}$ of seminorms by choosing the family of sets $(\{x \in X; q_a(x) < \lambda\})_{a \in A, \lambda > 0}$ and their finite intersections as neighbourhoods of zero. This topology is Hausdorff if and only if for every $x \neq 0$ there is an $a \in A$ such that $q_a(x) \neq 0$.

A set $B \subseteq X$ is *bounded* if every seminorm $q_a, a \in A$ is bounded on B . The concept of bounded set is not very important in normed spaces $(Y, \|\cdot\|)$, because it is then the same as the concept of “subset of a ball”, where a ball of radius R is defined by $\{y \in Y : \|y\| \leq R\}$. In locally convex spaces there does not in general exist a fundamental system of bounded neighbourhoods of zero. Indeed, a Hausdorff locally convex space is *normable* if and only if it has at least one bounded neighbourhood of zero [117, p. 108]. A locally convex space is *metrizable* if and only if it is Hausdorff and its topology can be generated by a countable family of seminorms.

The *topological dual space* X' of a locally convex space X is the space of continuous linear functionals $x' : X \rightarrow \mathbb{C}$, often written as $x'(x) = \langle x', x \rangle$ a bracket or product. For any subset $A \subset X$ its *polar* A^o in X' is the set of all $x' \in X'$ such that $|\langle x', x \rangle| \leq 1$ for all $x \in A$.

Let \mathfrak{S} denote a family of subsets of X . The so called *\mathfrak{S} -topology on the dual space X'* is the topology which has a fundamental system of neighbourhoods of zero consisting of the finite intersections of the polars of the sets $A \in \mathfrak{S}$. The most important \mathfrak{S} -topologies on X' arise from choosing $\mathfrak{S} = \{\{x\} \subset X; x \in X\}$ or $\mathfrak{S} = \{A \subset X; A \text{ bounded}\}$. The former is called *topology of pointwise convergence* or *weak* topology* and denoted as $\sigma(X', X)$, while the latter is called *topology of bounded convergence* or *strong topology* and denoted as $\beta(X', X)$. When X is a normed space, then $\beta(X', X)$ is the usual dual norm topology.

10.3 Function spaces

Let $\Omega \subseteq \mathbb{R}^d$ be an open set, \mathfrak{M} a σ -algebra of subsets of Ω , and μ a measure on \mathfrak{M} . The *Lebesgue spaces* of equivalence classes of integrable functions over the measure space $(\Omega, \mathfrak{M}, \mu)$ are defined as

$$L^p(\Omega, \mu) := \{f : \Omega \rightarrow \mathbb{R} ; f^p \text{ is integrable}\} \tag{10.1}$$

with norm

$$\|f\|_p := \left(\int_{\Omega} |f(s)|^p d\mu(s) \right)^{1/p} \tag{10.2}$$

and their locally integrable variants are

$$L^p_{\text{loc}}(\Omega, \mu) := \{f : \Omega \rightarrow \mathbb{R} ; f^p \text{ is integrable on every compact } K \subset \Omega\}. \tag{10.3}$$

For $p = \infty$

$$L^\infty(\Omega, \mu) := \{f : \Omega \rightarrow \mathbb{R} \mid f \text{ is measurable and } \|f\|_\infty < \infty\} \tag{10.4}$$

where

$$\|f\|_\infty := \sup \{|z| : z \in f_{\text{ess}}(\Omega)\} \tag{10.5}$$

and

$$f_{\text{ess}}(\Omega) := \{z \in \mathbb{C} : \mu(\{x \in \Omega : |f(x) - z| < \varepsilon\}) \neq 0 \text{ for all } \varepsilon > 0\} \tag{10.6}$$

is the *essential range* of f .

Let $\Omega \subseteq \mathbb{R}^d$ be open with boundary $\partial\Omega$. For $\Omega = \mathbb{R}^d$ the phrase “for all $x \in \partial\Omega$ ” means “for all $|x| \rightarrow \infty$ ”. Denote with \mathbb{C}^Ω the set of all complex valued functions

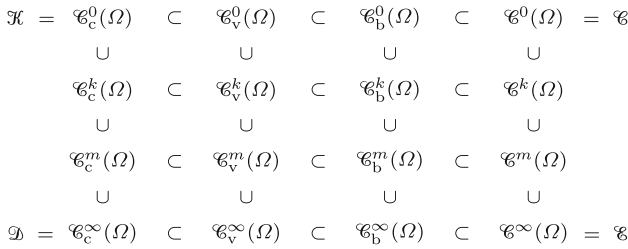


Fig. 4 Inclusion relations between some function spaces ($k \leq m$)

$f : \Omega \rightarrow \mathbb{C}$. The notation for spaces of functions is ($\kappa \in \mathbb{N}^d, |\kappa| = \kappa_1 + \dots + \kappa_d, k \in \mathbb{N}$)

$$\mathcal{C}(\mathbb{C}, \Omega) := \mathcal{C}(\Omega) := \mathcal{C}^0(\Omega) := \{f \in \mathbb{C}^\Omega; f \text{ is continuous}\} \tag{10.7}$$

$$\mathcal{C}^k(\Omega) := \{f \in \mathbb{C}^\Omega; \partial^\kappa f \text{ is continuous, } |\kappa| \leq k\} \tag{10.8}$$

$$\mathcal{C}_b^k(\Omega) := \{f \in \mathcal{C}^k(\Omega); \partial^\kappa f \text{ is bounded, } |\kappa| \leq k\} \tag{10.9}$$

$$\mathcal{C}_v^k(\Omega) := \{f \in \mathcal{C}^k(\Omega); (\partial^\kappa f)(x) = 0 \text{ for all } x \in \partial\Omega, |\kappa| \leq k\} \tag{10.10}$$

$$\mathcal{C}_c^k(\Omega) := \{f \in \mathcal{C}^k(\Omega); \partial^\kappa f \text{ has compact support, } |\kappa| \leq k\} \tag{10.11}$$

$$\mathcal{E}(\Omega) := \mathcal{C}^\infty(\Omega) \tag{10.12}$$

$$\mathcal{D}(\Omega) := \mathcal{C}_c^\infty(\Omega) \tag{10.13}$$

$$\mathcal{K}(\Omega) := \mathcal{C}_c^0(\Omega) \tag{10.14}$$

$$\mathcal{D}_{0+}(\Omega) := \{f \in \mathcal{E}(\Omega); \text{supp } f \subseteq \mathbb{R}_{0+}^d\} \tag{10.15}$$

$$\mathcal{D}_{L^p}^k(\Omega) := \{f \in \mathcal{C}^k(\Omega); \partial^\kappa f \in L^p(\Omega), |\kappa| \leq k\}, 1 \leq p \leq \infty \tag{10.16}$$

$$\mathcal{D}_{L^p}^\infty(\Omega) := \mathcal{D}_{L^p}^\infty(\Omega) = \{f \in \mathcal{E}(\Omega) : \partial^\kappa f \in L^p(\Omega), |\kappa| < \infty\}, 1 \leq p \leq \infty \tag{10.17}$$

$$\mathcal{B}(\Omega) := \mathcal{D}_{L^\infty}(\Omega) = \mathcal{D}_{L^\infty}^\infty(\Omega) = \mathcal{C}_b^\infty(\Omega) \tag{10.18}$$

$$\dot{\mathcal{B}}(\Omega) := \{f \in \mathcal{E}(\Omega) : \partial^\kappa f \in \mathcal{C}_v^0(\Omega), |\kappa| \leq k\} \tag{10.19}$$

$$\begin{aligned}
 \mathcal{S}(\Omega) &:= \{f \in \mathcal{E}(\Omega); \forall k \in \mathbb{Z}, \kappa \in \mathbb{N}_0^d, \varepsilon > 0 \exists \rho > 0 : \\
 &\quad |(1 + |x|^2)^k \partial^\kappa f| \leq \varepsilon \text{ if } |x| > \rho\} \tag{10.20}
 \end{aligned}$$

$$\mathcal{O}_C(\Omega) := \left\{ f \in \mathcal{E}(\Omega); \exists k \in \mathbb{N}_0, \forall \kappa \in \mathbb{N}_0^d : (1 + |x|^2)^{-k} \partial^\kappa f \in \mathcal{E}(\Omega) \right\} \tag{10.21}$$

$$\mathcal{O}_M(\Omega) := \left\{ f \in \mathcal{E}(\Omega); \forall \kappa \in \mathbb{N}_0^d, \exists k \in \mathbb{N}_0 : (1 + |x|^2)^{-k} \partial^\kappa f \in \mathcal{E}(\Omega) \right\} \tag{10.22}$$

$$\begin{aligned}
 \mathcal{C}^{k,\alpha}(\Omega) &:= \{f \in \mathcal{C}^k(\Omega); \exists C > 0, \forall x, y \in \Omega : \\
 &\quad |\partial^\kappa f(x) - \partial^\kappa f(y)| \leq C|x - y|^\alpha, |\kappa| \leq k\}, 0 < \alpha \leq 1 \tag{10.23}
 \end{aligned}$$

If $K \subset \Omega$ is compact, the spaces of functions $f : K \rightarrow \mathbb{C}$ are denoted

$$\mathcal{X}(K) := \{f \in \mathcal{X}; \text{supp } f \subseteq K\} \tag{10.24}$$

Table 1 Comparison of notations for function spaces, $\Omega \subseteq \mathbb{R}^d$ open

here	[5], page#	[117], page#	[161], page#
$\mathcal{C}(\Omega)$	\mathcal{C}_Ω , 20	$\mathcal{C}(\Omega)$, 83	$C(\Omega)$, 9
$\mathcal{E}^k(\Omega)$		$\mathcal{E}^k(\Omega)$, 89	$C^k(\Omega)$, 9
$\mathcal{E}_b^k(\Omega)$		$\mathcal{B}^k(\Omega)$, 91	$C_B^k(\Omega)$, 95
$\mathcal{E}_v^k(\Omega)$		$\mathcal{B}_0^k(\Omega)$, 91	
$\mathcal{E}_c^k(\Omega)$	\mathcal{D}^k , 21	$\mathcal{D}^k(\Omega)$, 171	
$\mathcal{E}(\Omega)$	\mathcal{E} , 88	$\mathcal{E}(\Omega)$, 165	$C^\infty(\Omega)$, 9
$\mathcal{D}(\Omega)$	\mathcal{D}_Ω , 26	$\mathcal{D}(\Omega)$, 165	$C_0^\infty(\Omega)$, 9
$\mathcal{H}(\Omega)$		$\mathcal{K}(\Omega)$, 164	$C_0(\Omega)$, 9
$\mathcal{D}_{L^p}^k(\Omega)$			$W^{k,p}(\Omega)$, 45
$\mathcal{D}_{L^p}(\Omega)$	\mathcal{D}_{L^p} , 199		$W^{\infty,p}(\Omega)$, 45
$\mathcal{B}(\Omega)$	\mathcal{B} , 199	$\mathcal{B}(\Omega)$, 92	$W^{\infty,\infty}(\Omega)$, 45
$\dot{\mathcal{B}}(\Omega)$	$\dot{\mathcal{B}}$, 199	$\mathcal{B}_0(\Omega)$, 91	
$\mathcal{S}(\mathbb{R}^d)$	\mathcal{S} , 233	\mathcal{S} , 91	

where $\mathcal{X} \in \{\mathcal{C}, \mathcal{E}^k, \mathcal{E}_v^k, \mathcal{E}_b^k, \mathcal{E}, \mathcal{D}, \dot{\mathcal{B}}\}$. In these cases the topology of $\mathcal{X}(K)$ is the subspace topology inherited from $\mathcal{X}(\Omega)$. As an example, for $d = 1$ and $K = [a, b] \subset \mathbb{R}$

$$\mathcal{A}\mathcal{E}^k([a, b]) := \{f \in C^k([a, b]); f^{(k)} \text{ is absolutely continuous}\} \tag{10.25}$$

are the absolutely continuous functions on $[a, b]$.

The spaces $\mathcal{E}^{k,\alpha}(\Omega)$ are known as Hölder spaces. For $0 < \alpha < \beta \leq 1$ they obey $\mathcal{E}^{k,\beta}(K) \subset \mathcal{E}^{k,\alpha}(K) \subset \mathcal{E}^k(K)$ if $K \subset \Omega$ [161, p. 10]. The space $\mathcal{O}_M(\Omega)$, consists of slowly increasing smooth functions, while $\mathcal{O}_C(\Omega)$ is the space of very slowly increasing smooth functions. The spaces $\mathcal{D}_{L^p}(\Omega)$ of Lebesgue integrable test functions are known as Sobolev spaces, and often denoted as $W^{\infty,p}(\Omega)$. Table 1 compares our notation for some function spaces with other widespread notations in the literature.

$\mathcal{B}_c(\Omega)$ is defined as the space $\mathcal{B}(\Omega)$ equipped with the finest locally convex topology that induces the topology of $\mathcal{E}(\Omega)$ on the subsets of $\mathcal{B}(\Omega)$ which are bounded w.r.t its Frechet topology generated by the seminorms $\varphi \mapsto \|\partial^\kappa \varphi\|_\infty, \kappa \in \mathbb{N}_0^d$. This topology is weaker than the Frechet topology on $\mathcal{B}(\Omega) = \mathcal{D}_{L^\infty}(\Omega)$. However, $\mathcal{B}(\Omega)$ and $\mathcal{B}_c(\Omega)$ have the same bounded subsets.

10.4 Spaces of Radon measures

Spaces of Radon measures are defined as topological duals of spaces of continuous functions,

$$\mathcal{M}(\Omega) := (\mathcal{H}(\Omega))' \tag{10.26}$$

$$\mathcal{M}_c(\Omega) := (\mathcal{E}(\Omega))' \tag{10.27}$$

$$\mathcal{M}^p(\Omega) := \left(\mathcal{D}_{L^q}^0(\Omega)\right)' \quad 1/p + 1/q = 1, p, q \neq \infty, \tag{10.28}$$

Table 2 Notation of common distribution spaces using eqs. (10.35) and (10.36)

X	\mathcal{D}_X	Y	Ref
\mathcal{E}_c	\mathcal{D}	\mathcal{D}'	
\mathcal{E}	\mathcal{E}	\mathcal{E}'	[5, p. 89]
\mathcal{E}_P	\mathcal{S}	\mathcal{S}'	[5, p. 237]
L^q	\mathcal{D}_{L^q}	\mathcal{D}'_{L^p}	[5, p. 200]
\mathcal{E}_v	\mathcal{B}	\mathcal{D}'_{L^1}	[5, p. 200]

Given a locally convex space X of distributions, Schwartz defined its pre-image under the maps $\mathcal{E} \ni f \mapsto f^{(m)}, m \in \mathbb{N}_0$ as the following set of smooth functions

$$\mathcal{D}_X = \left\{ f \in \mathcal{E} : f^{(m)} \in X, \forall m \in \mathbb{N}_0 \right\} \tag{10.35}$$

equipped with the coarsest topology such that all the maps $\mathcal{D}_X \ni f \mapsto f^{(m)}, m \in \mathbb{N}_0$ are continuous (projective topology). He then introduced the distribution spaces

$$Y = (\mathcal{D}_X)' \tag{10.36}$$

the space of continuous linear functionals on \mathcal{D}_X equipped with the topology of uniform convergence on bounded subsets of \mathcal{D}_X (Table 2).

The reflection of a locally integrable function $f(x)$ with respect to the origin, denoted $\check{f}(x)$, is defined as [5, p. 167]

$$\check{f}(x) = f(-x) \tag{10.37}$$

so that with

$$\langle \check{f}, \varphi \rangle = \int_{-\infty}^{\infty} f(-x)\varphi(x)dx = \int_{-\infty}^{\infty} f(x)\varphi(-x)dx = \langle f, \check{\varphi} \rangle \tag{10.38}$$

the reflection of a distribution is defined by

$$\langle \check{f}, \varphi \rangle = \langle f, \check{\varphi} \rangle \tag{10.39}$$

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Declarations

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