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LECTURE 9

Fractional Derivatives in Static and Dynamic Scaling

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1. INTRODUCTION

A derivative or integral of fractional order is usually defined by analytically continuing a suitable definition of the derivative or integral of integer order $n \in \mathbb{N}$ to real or complex values of n.

My purpose here is to give some of the definitions and then to review briefly recent applications of fractional derivatives and integrals in physics with emphasis on static and dynamic scaling.

Derivatives of fractional order have recently emerged in physics as generators of time evolutions in ergodic theory [1, 2, 3, 4, 5, 6], and as a tool for classifying phase transitions in thermodynamics by generalizing the classification scheme of Ehrenfest [1, 7, 8, 9, 10].

Given the fact that a fractional integral is, loosely speaking, a convolution operator with a power law kernel it is perhaps not too surprising that fractional integrals and derivatives are useful tools in scaling theory.

2. FRACTIONAL DERIVATIVES

Let $f : [a, b] \to \mathbb{R}$ be a real function and recall that its n-th order integral is given by

$$\begin{aligned}(I^n_{a+}f)(x) &= \int_a^x \int_a^{y1} ... \int_a^{y_{n-1}} f(y_n)\, dy_n ... dy_1 \qquad (x > a) \\ &= \frac{1}{(n-1)!}\int_a^x (x-y)^{n-1} f(y)\, dy \end{aligned} \tag{1}$$

as is readily proven by induction. Generalizing equation (1) to noninteger n defines the *Riemann-Liouville fractional integral of order* $\alpha > 0$ as [11, 12]

$$(I^\alpha_{a+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-y)^{\alpha-1} f(y)\, dy \tag{2}$$

for $x > a$ and $f \in \mathrm{L}_1([a, b])$ to ensure that the integral will be finite. A dual form is the right-sided integral

$$(I^\alpha_{b-}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (x-y)^{\alpha-1} f(y)\, dy \tag{3}$$

for $x < b$. The Riemann-Liouville integral (2) is perhaps the most commonly employed definition [13, 14]. For $0 < a < 1$ these definitions are extended to the whole real axis $\mathbb{R}$ as

$$(I^\alpha_{+}f)(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^x (x-y)^{\alpha-1} f(y)\, dy \tag{4}$$

$$(I^\alpha_{-}f)(x) = \frac{1}{\Gamma(\alpha)}\int_x^{\infty} (x-y)^{\alpha-1} f(y)\, dy \tag{5}$$

for $f \in \mathrm{L}_p(\mathbb{R})$ with $1 \leq p < 1/\alpha$. Formula (5) is often referred to as the *Weyl fractional integral of order* α [15].

The definition of fractional derivatives is frequently based on that of fractional integrals through

$$(\partial^\alpha_i f)(x) = \frac{d^{n+1}}{dx^{n+1}}(I^{n+1-\alpha}_i f)(x) \tag{6}$$

where $i \in \{a+, b-, +, -\}$ and $n = [\alpha]$ is the largest integer less than or equal to α. An alternative would be to interchange the orders of differentiation and fractional integration in (6).

Another approach to define a fractional derivative tries to replace α with $-\alpha$ directly in the definitions (2)–(5). However the resulting integral is divergent and needs to be regularized. The traditional regularization uses Hadamards finite part. Starting from (6) and (4) and proceeding formally one arrives at [15, 16]

$$\begin{aligned}(\partial_+^\alpha f)(x) &= \frac{d}{dx}(I_+^{1-\alpha}f)(x)\\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{-\infty}^{x}(x-y)^{-\alpha}f(y)\,dy\\ &= \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_0^\infty t^{-\alpha}f(x-t)\,dt\\ &= \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty f'(x-t)\int_t^\infty \frac{1}{z^{1+\alpha}}\,dzdt\\ &= \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{f(x)-f(x-t)}{t^{1+\alpha}}\,dt \end{aligned} \tag{7}$$

where the last equality serves to define $(\partial_+^\alpha f)$ for $0 < \alpha < 1$. Here f' denotes the first derivative of f with respect to its argument. Similar definitions can be given for $\alpha > 1$. The main difference between the definitions (7) and (6) is that (7) allows more freedom for the behaviour of f at infinity. Thus (7) is defined for $f(x) = const$ while (6) is not.

The preceding definitions define fractional differentiation as an inverse operation to fractional integration, i.e. as integration of order $-\alpha$. A more direct approach arises from the fact that derivatives are limits of difference quotients. Let T^h denote the translation

$$(T^h f)(x) = f(x-h) \tag{8}$$

by h. The finite difference of order α is defined as

$$(\Delta_h^\alpha f)(x) = (\mathbf{1}-T^h)^\alpha f(x) = \sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}f(x-kh) \tag{9}$$

with $\mathbf{1} = T^0$ the identity, and

$$\binom{\alpha}{k} = \frac{(-1)^{k-1}\alpha\Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k+1)} \tag{10}$$

It reduces to the familiar finite difference of integer order for $\alpha \in \mathbb{N}$. The *Grünwald fractional derivative of order* α is then defined as [17, 18]

$$(\partial_\pm^\alpha f)(x) = \lim_{h\to 0^+} h^{-\alpha}(\Delta_{\pm h}^\alpha f)(x) \tag{11}$$

as the limit of a fractional finite difference quotient. There are several possibilities to define this limit, e.g. pointwise, almost everywhere, or in the norm of a Banach space. The choice depends upon the question at hand. It is possible to show that, in a suitable sense, the definitions (11) and (7) coincide.

As equation (9) suggests it is also possible to define fractional derivatives as fractional powers of the differentiation operator, i.e. $\partial_+^\alpha = (d/dx)^\alpha$ [19, 18]. More generally one considers fractional powers of the infinitesimal generators of strongly continuous semigroups. A strongly continuous semigroup on a Banach space X is a one parameter family of maps $T(t) : X \to X$ fulfilling

$$1. \quad T(t)T(s) = T(t+s) \tag{12}$$

$$2. \quad T(0) = \mathbf{1} \tag{13}$$

$$3. \quad \lim_{t\to s} ||T(t)f - T(s)f|| = 0 \qquad \text{for all} \quad f \in X \tag{14}$$

whose infinitesimal generator A is defined by

$$Af = \lim_{t\to 0^+} \frac{1}{t}(T(t) - \mathbf{1})f \tag{15}$$

The fractional power $(-A)^\alpha$ can be defined analogous to eq. (11) or analogous to eq. (7) through

$$(-A)^\alpha f = \frac{1}{\Gamma(-\alpha)} \int_0^\infty t^{-\alpha-1}(T(t) - \mathbf{1})f \; dt \tag{16}$$

for $0 < \alpha < 1$ and f within the domain of A. Equation (16) is known as *Balakrishnan's formula*. Obviously, (16) generalizes (7) and (11).

Finally, fractional derivatives may also be defined through analytic continuation in Fourier space. If

$$\widehat{f}(k) = \int_{-\infty}^{\infty} e^{ikx} f(x) \; dx \tag{17}$$

denotes the Fourier transform of f then a fractional derivative of order α can be defined as the operator

$$(\partial_\pm^\alpha f)(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mp ik)^\alpha \widehat{f}(k) e^{-ikx} \; dk \tag{18}$$

for functions f such that the integrals exist (e.g. smooth with compact support).

3. SCALING IN THERMODYNAMICS

Fractional derivatives have been employed in thermodynamics to generalize the Ehrenfest classification scheme for phase transitions [1, 7, 8, 9]. It is found that this generalization gives rise to a generalized form of static scaling at the transition point.

To be concrete consider the pressure $p(T, \mu)$ as a function of temperature T and chemical potential μ. Let $p(T, \mu)$ have a phase transition (i.e. a singular point) at (T_c, μ_c). Let furthermore $\mathcal{C} : \sigma \to (T(\sigma), \mu(\sigma))$ be a thermodynamic process parametrized by σ such that $\sigma = 0$ corresponds to the critical point, i.e. $(T(0), \mu(0)) = (T_c, \mu_c)$. Let p_s denote the singular part of p, i.e. $p = p_r + p_s$ where p_r is the regular part. Let

$$F(\mathcal{C}, q; \sigma) = \begin{cases} (\partial^q_{0+} p_s)(\sigma) & : \quad \text{for} \quad \sigma > 0 \\ (\partial^q_{0-} p_s)(\sigma) & : \quad \text{for} \quad \sigma < 0 \end{cases} \tag{19}$$

denote the q-th order derivative of the function $p_s(\sigma)$.

Ehrenfest [20] suggested to classify phase transitions according to their order. The phase transition at $\sigma = 0$ is defined to have Ehrenfest order $n \in \mathbb{N}$ if and only if $F(\mathcal{C}, n; \sigma)$ has a jump discontinuity at $\sigma = 0$, i.e. iff

$$\lim_{\sigma \to 0^+} F(\mathcal{C}, n; \sigma) \neq \lim_{\sigma \to 0^-} F(\mathcal{C}, n; \sigma) \tag{20}$$

In [1, 7, 8, 9] Ehrenfests classification was generalized to allow noninteger orders and slowly varying confluent behaviour. The phase transition at $\sigma = 0$ is defined to be of order $\lambda^\pm > 0$ if and only if [1, 7, 8, 9]

$$\lim_{\sigma \to 0^\pm} \frac{F(\mathcal{C}, \lambda^\pm; b\sigma)}{F(\mathcal{C}, \lambda^\pm; \sigma)} = 1 \tag{21}$$

for all $b > 0$, i.e. iff $F(\mathcal{C}, \lambda^\pm; \sigma)$ is a slowly varying function in the sense of Karamata [21].

It follows readily from this definition that $p_s(\sigma)$ is regularly varying of order $\lambda^\pm(\mathcal{C})$. Note that the order depends on the choice of the curve $\mathcal{C}$. Hence

$$p_s(T(\sigma), \mu(\sigma)) = \begin{cases} \Lambda^+(\mathcal{C}; \sigma)\sigma^{\lambda^+(\mathcal{C})} & : \quad \text{for} \quad \sigma > 0 \\ \Lambda^-(\mathcal{C}; \sigma)\sigma^{\lambda^-(\mathcal{C})} & : \quad \text{for} \quad \sigma < 0 \end{cases} \tag{22}$$

where the functions $\Lambda^\pm(\mathcal{C}; \sigma)$ are slowly varying functions at $\sigma = 0$.

These results show that phase transitions with noninteger Ehrenfest order exhibit algebraic scaling behaviour with an exponent that is in general path dependent.

What is the interest of this new classification ? First of all classifying phenomena is an important task of science. An exhaustive classification of fixed points in renormalization group theory of phase transitions does not exist [22]. An exhaustive classification allows to predict that certain cases cannot occur or that new cases beyond known ones must be expected. Secondly the generalized Ehrenfest scheme has led to a wealth of new predictions ranging from new classes of phase transitions [1] to the prediction of scaling functions and universal amplitude ratios for finite size scaling [23, 24]. Finally the same ideas giving rise to a classification of static macroscopic phenomena are also useful to classify macroscopic dynamics as discussed next.

4. SCALING IN MACROSCOPIC DYNAMICS

First order time derivatives play a fundamental role as the infinitesimal generators of time evolution for a dynamical system. A growing number of publications uses fractional time derivatives to describe the time evolution of physical systems [6, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38]. This raises the question whether fractional time derivatives are also infinitesimal generators of time evolution. Recent work has given a positive answer to this question [3, 4, 5]. The purpose of this section is to present this result.

Let Γ be the microscopic phase or state space of a dynamical system in the sense of ergodic theory [39, 40], with σ-algebra $\mathcal{G}$, measure μ, and time evolution $T(t) = T^t$ with either $t \in \mathbb{R}$ or $t \geq 0$, i.e.

$$T(t)\mu(G, t_0) = \mu(G, t_0 - t) \tag{23}$$

for all $G \in \mathcal{G}$. Let $\Gamma^* \subset \Gamma$ be the microscopic phase or state space associated with a subsystem. If $\mu(\Gamma^*) > 0$ then the time evolution $T(t)$ on Γ canonically induces also a time evolution on Γ^* [39, 40]. In situations of physical interest, however, one often has $\mu(\Gamma^*) = 0$. An example would be the solid phase in a system of interacting classical particles exhibiting macroscopic solid-fluid coexistence at low temperatures.

The physically interesting case with $\mu(\Gamma^*) = 0$ was first discussed in [3, 4, 5] in the context of ergodic theory with the following result: If the subset Γ^* is suitably chosen, then the limit of macroscopic times (ultralong time limit) of the induced time evolution $T^*(t^*)$ on Γ^* is given by a one-parameter family of semigroups parametrized by $0 < \alpha \leq 1$

$$\begin{aligned} T^*_\alpha(t^*)\mu^*(B, t^*_0) &= \int_0^\infty \mu^*(B, t^*_0 - t) h_\alpha\left(\frac{t}{t^*}\right)\frac{dt}{t^*} \\ &= \frac{1}{t^*}\int_0^\infty h_\alpha(t/t^*)T(t)\mu^*(B, t^*_0)dt \end{aligned} \tag{24}$$

with $t^* \geq 0$. Here $B \subset \Gamma^*$, μ^* is a measure on Γ^*, and

$$h_\alpha(x) = \frac{1}{x\alpha} H^{10}_{11}\left(\frac{1}{x}\,\middle|\, \begin{matrix}(0,1)\\(0,1/\alpha)\end{matrix}\right) \tag{25}$$

is a one-parameter familiy of functions with H^{10}_{11} defined through its Mellin transform [41]

$$\int_0^\infty H^{10}_{11}\left(x\,\middle|\, \begin{matrix}(0,1)\\(0,1/\alpha)\end{matrix}\right) x^{s-1}\, dx = \frac{\Gamma(s/\alpha)}{\Gamma(s)} \tag{26}$$

or through the Laplace transform

$$\int_0^\infty \frac{e^{-ux}}{x\alpha} H^{10}_{11}\left(\frac{1}{x}\,\middle|\, \begin{matrix}(0,1)\\(0,1/\alpha)\end{matrix}\right) dx = e^{-u^\alpha}. \tag{27}$$

Obviously

$$h_1(x) = \lim_{\alpha\to 1^-} h_\alpha(x) = \delta(x-1) \tag{28}$$

and hence from (24)

$$T_1^*(t^*)\mu^*(B,t_0^*) = \mu^*(B,t_0^*-t^*) \tag{29}$$

is again a translation as in the microscopic evolution (23) except that here the macroscopic time parameter obeys always $t^* \geq 0$ while the microscopic time parameter can also obey $t \in \mathbb{R}$. This means that the macroscopic time evolution is always irreversible while the microscopic evolution may be either reversible or irreversible.

Fractional derivatives appear when the infinitesimal generator, (15), of the induced time evolution $T^*(t^*)$, given in (24), is determined as

$$A_\alpha^* = \lim_{t^*\to 0^+} \frac{1}{t^*}(T_\alpha^*(t^*) - \mathbf{1}) = c^+ \int_0^\infty t^{-\alpha-1}(T(t)-T(0))\,dt \tag{30}$$

where $c^+ > 0$ is a constant. Comparison with Balakrishnans formula (16) shows that

$$A_\alpha^* = c\partial_+^\alpha \tag{31}$$

for $0 < \alpha < 1$ with c a constant, and

$$A_1^* = \frac{d}{dt^*} \tag{32}$$

for $\alpha = 1$ are the infinitesimal generators of the induced time evolution $T_\alpha^*(t^*)$.

An immediate consequence of (31) was proposed in [1]. It identifies A_α^* with a generalized Liouvillian, and suggests to generalize the Liouville equation for the time evolution of probability measures in phase space into a *fractional Liouville equation*

$$\partial_+^\alpha \mu^*(B,t^*) = A_\alpha^* \mu^*(B,t^*) \tag{33}$$

where A_α^* is interpreted as a Liouville operator. Explicit examples are given by anomalous fractional diffusion where the right hans side is the Laplacian [33] or fractional relaxation [1, 31, 32].

A second consequence emerges from the condition of invariance (or stationarity) of a measure under the induced macrotime evolution $T^*(t^*)$ which reads

$$T_\alpha^*(t^*)\mu^*(B,t_0^*) = \mu^*(t_0^*). \tag{34}$$

This becomes infinitesimally a fractional differential equation

$$\partial_+^\alpha \mu^*(B,t^*) = 0 \tag{35}$$

whose solution is

$$\mu^*(B,t^*) = C_0 {t^*}^{\alpha-1}. \tag{36}$$

Thus algebraic decay in time can be a sign of stationarity for induced dynamics on subsets of measure zero. The algebraic time decay shows that order α of the derivative plays the role of a dynamic scaling exponent.

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