D. Helbing H. J. Herrmann
M. Schreckenberg D. E. Wolf
Editors

TRAFFIC AND GRANULAR FLOW '99

Social, Traffic, and Granular Dynamics

With 213 Figures, 20 in Colour

Springer

Editors

Dirk Helbing
Institut für Wirtschaft und Verkehr
Technische Universität Dresden
01062 Dresden, Germany
e-mail: helbing@trafficforum.de

Hans J. Herrmann
ICA 1
Universität Stuttgart
Pfaffenwaldring 57/III
70550 Stuttgart, Germany
e-mail: hans@ica1.uni-stuttgart.de

Michael Schreckenberg
Physik von Transport und Verkehr
Gerhard-Mercator-Universität Duisburg
Lotharstraße 1
47048 Duisburg, Germany
e-mail: schreckenberg@uni-duisburg.de

Dietrich E. Wolf
Theoretische Physik
Gerhard-Mercator-Universität Duisburg
Lotharstraße 1
47048 Duisburg, Germany
e-mail: wolf@rs1.comphys.uni-duisburg.de

Mathematics Subject Classification (1991):
92Hxx, 92Bxx, 90Axx, 93Exx, 82-XX, 82D99

Cataloging-in-Publication Data applied for

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Traffic and granular flow '99 : social, traffic, and granular dynamics
/ D. Helbing ... ed.. - Berlin ; Heidelberg ; New York ; Barcelona ;
Hong Kong ; London ; Milan ; Paris ; Singapore ; Tokyo : Springer,
2000
ISBN 3-540-67091-2

ISBN 3-540-67091-2 Springer-Verlag Berlin Heidelberg New York

Cover design: *design & production* GmbH, Heidelberg
Typeset by the authors using a Springer LaTeX macro package

SPIN 10750893 46/3143LK – 5 4 3 2 1 0 Printed on acid-free paper

Fractional Evolution Equations and Irreversibility

R. Hilfer[1,2]

[1] ICA-1, Universität Stuttgart, Pfaffenwaldring 27, 70569 Stuttgart, Germany
[2] Institut für Physik, Universität Mainz, 55099 Mainz, Germany

Abstract. The paper reviews a general theory predicting the general importance of fractional evolution equations. Fractional time evolutions are shown to arise from a microscopic time evolution in a certain long time scaling limit. Fractional time evolutions are generally irreversible. The infinitesimal generators of fractional time evolutions are fractional time derivatives. Evolution equations containing fractional time derivatives are proposed for physical, economical and traffic applications. Regular non-fractional time evolutions emerge as special cases from the results. Also for these regular time evolutions it is found that macroscopic irreversibility arises in the scaling limit.

1 Introduction

An evolution equation for the time evolution of physical and other systems is typically of the form [1]

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = \mathrm{B} f(t) \tag{1}$$

where $t \in \mathbb{R}$ denotes time and B is an operator on a Banach space $(B, \|\cdot\|)$. Depending on the initial data $f(0) = f_0$ for the state or observable f of the system at time $t = 0$ the problem is to find $f(t)$ at later times $t > 0$.

Many examples of (1) arise not only in physics but also in the social or economical sciences (see other articles in this book and also [2]). All evolution equations pertain to phenomena on a characteristic microscopic or macroscopic time scale. If the time scale is changed then the form of an evolution equation will usually also change. Of course this micro-macro transition affects the variables f, or the operator B, or both. Recently, it was found that not only f and B, but also the generator $\mathrm{d}/\mathrm{d}t$ of the time evolution in (1) may change in a transition from microscopic to macroscopic of time scales. Expressed somewhat imprecisely the result states that the infinitesimal generator $\mathrm{d}/\mathrm{d}t$ of time evolution may be changed into a fractional derivative "$\mathrm{d}^\alpha/\mathrm{d}t^\alpha$" of order α with $0 < \alpha \leq 1$. My objective in this paper is to discuss the origin of this result.

Derivates of fractional order exhibit algebraic scaling properties with non-integer exponents. Extending classical evolution equations to fractional evolution equations therefore introduces dynamic scaling in a natural way. It provides also a more flexible class of solutions for the comparison with empirical data.

Given the generality and mathematical universality of the appearance of fractional derivatives it is tempting to propose fractional evolution equations also for economical, social or traffic dynamics. Let me point out that this was done implicitly already in [2] where continuous time random walk (CTRW) models were introduced for modeling stochastic processes in the economical sciences. One of the input functions for a CTRW model is the so called waiting time density $\psi(t)$ that describes the distribution of time intervals between random events (jumps). Random events may correspond to hopping (jumps) of particles in physics, to purchases or business transactions in economical applications, or to delays due to a traffic jam in a traffic model. Exploiting the relationship between continuous time random walks and generalized master equations it was shown in [3] that CTRW-models to a fractional master equation whenever $\psi(t)$ is a generalized Mittag-Leffler function. Many applications of fractional derivatives in the economical sciences are therefore obtained simply by combining the results from [2] and [3].

A concrete example from the economical sciences is the model for the probability distribution $f(z,t)$ of cumulative sales z after time t introduced in [2, p. 138]. Combined with the results from [3,4] the model from [2] is equivalent to a fractional diffusion equation of the form

$$\mathrm{D}_t^\alpha f(z,t) = B\frac{\partial^2}{\partial z^2}f(z,t) \tag{2}$$

where D_t^α denotes a suitably defined fractional derivative operator of order α and B is a constant. [1] A precise formulation of such a fractional diffusion equation and its exact solution are given in [6]. Exact solutions for a whole class of fractional derivative operators are also given in [7]. Fractional equations could also be postulated for traffic flow. An example would be the fractional differential equation

$$\mathrm{D}_t^\alpha f(z,t) = A\frac{\partial}{\partial z}f(z,t) + B\frac{\partial^2}{\partial z^2}f(z,t) \tag{3}$$

where now $f(z,t)$ denotes the probability distribution for the number z of cars in a traffic jam at time t, and where A, B are phenomenological parameters. For $\alpha = 1$ this equation reduces to an equation proposed in [8]. Another example would be a fractional Lighthill-Whitham model defined by the equations [9]

$$\frac{\partial}{\partial t}f_1(z,t) = -\frac{\partial(f_1 f_2)}{\partial z} \tag{4a}$$

$$\mathrm{D}_t^\alpha f_2(z,t) = A - Bf_2 \tag{4b}$$

where A and B are phenomenological parameters. Here $f_1(z,t)$ is a macroscopic density of vehicles at position z at time t, and $f_2(z,t)$ is their average velocity. An exact solution for the fractional relaxation equation (4b) is given in [7].

My purpose in this article is not to discuss specific applications of fractional calculus (see [5] for other examples), but rather to investigate the reasons for

[1] Not all definitions of fractional derivatives are suitable. For information about "suitable" definitions of fractional derivatives see [5].

the importance of fractional time derivatives from a more fundamental point of view. The results presented here provide a general justification for the usage of fractional time derivatives in model equations pertaining to macroscopic time scales [7]. Fractional time derivatives appear universally because they arise as attractive fixed points in a scaling limit. This is similar to the thermodynamic limit in the theory of phase transitions, and to the concept of universality in that context. The appropriate scaling limit is introduced as a long time limit describing a change between microscopic and macroscopic time scales. The results also contain a mathematical mechanism for the emergence of macroscopic irreversibility from microscopic reversibility that does not depend on traditional [10,11] phase space arguments.

2 Semi-groups and Long Time Limit

Consider a general time evolution $\mathrm{T}(t)$ in physics, economics, or other sciences. An example of what is meant by $\mathrm{T}(t)$ is given by the left hand side of (1). In that case it is identified as a simple translation, defined as

$$\mathcal{T}(t)f(s) = f(s-t), \tag{5}$$

because the infinitesimal generator $-\mathrm{d}/\mathrm{d}t$ of $\mathcal{T}(t)$ appears on the left hand side of eq. (1). The subsequent considerations concern the possible operators that may appear on the left hand side of an evolution equation.

A time evolution may be characterized generally as a pair $(\{\mathrm{T}_\tau(t) : 0 \leq t < \infty\}, (B_\tau, \|\cdot\|))$. Here $\mathrm{T}_\tau(t) = \mathrm{T}(t\tau)$ is a semi-group of operators $\{\mathrm{T}(t) : 0 \leq t < \infty\}$ mapping a Banach space $(B_\tau(\mathbb{R}), \|\cdot\|)$ of functions $f_\tau(s) = f(s\tau)$ on $\mathbb{R}$ to itself. The elements of the Banach space represent the observables or states of the system. The argument $t \geq 0$ of $\mathrm{T}_\tau(t)$ represents a time duration, the argument $s \in \mathbb{R}$ of $f_\tau(s)$ a time instant. The index $\tau > 0$ indicates the units (or scale) of time. Below, τ will again be frequently suppressed to simplify the notation. The elements $f_\tau(s) = f(s\tau) \in B_\tau$ represent observables or the state of a physical system as function of the time coordinate $s \in \mathbb{R}$. The semi-group conditions require

$$\mathrm{T}_\tau(t_1)\mathrm{T}_\tau(t_2)f_\tau(t_0) = \mathrm{T}_\tau(t_1+t_2)f_\tau(t_0) \tag{6a}$$

$$\mathrm{T}_\tau(0)f_\tau(t_0) = f_\tau(t_0) \tag{6b}$$

for $t_1, t_2 > 0$, $t_0 \in \mathbb{R}$ and $f_\tau \in B_\tau$. The first condition defining the composition law of the semi-group may be viewed as representing the unlimited divisibility of time. The second condition is the unit element of the semi-group.

Reproducibility of experiments requires homogeneity with respect to time translations. The postulate of homogeneity assumes that $\mathrm{T}(t)$ commutes with translations, i.e.

$$\mathcal{T}(t_1)\mathrm{T}(t_2)f(t_0) = \mathrm{T}(t_2)\mathcal{T}(t_1)f(t_0) \tag{7}$$

for all $t_2 > 0$ and $t_0, t_1 \in \mathbb{R}$. This postulate allows to shift the origin of time and it reflects the basic symmetry of time translation invariance for scientific laws.

A time evolution $\mathrm{T}(t)$ should also be causal in the sense that the function $g(t_0) = (\mathrm{T}(t)f)(t_0)$ should depend only on values of $f(s)$ for $s < t_0$.

Finally, the time evolution $\mathrm{T}(t)$ is assumed to be strongly continuous in t by demanding

$$\lim_{t\to 0} ||\mathrm{T}(t)f - f|| = 0 \tag{8}$$

for all $f \in B$. It is also assumed to preserve boundedness in the sense that for all t and for $f \in L^p(\mathbb{R})$ the assumption $0 \leq f \leq 1$ almost everywhere implies $0 \leq \mathrm{T}(t)f \leq 1$ almost everywhere. For simplicity it will also be assumed that the operators $\mathrm{T}(t)$ are linear. For the general case see [7].

Let $L^p(\mathbb{R}^n)$ denote the Lebesgue spaces of p-th power integrable functions, and let $\mathcal{S}$ denote the Schwartz space of test functions for tempered distributions [12]. It is well known that all bounded linear operators on $L^p(\mathbb{R}^n)$ commuting with translations (i.e. fulfilling (7)) are of convolution type [12]. More precisely, suppose the operator $\mathrm{T} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$, $1 \leq p, q, \leq \infty$ is linear, bounded and commutes with translations. Then there exists a unique tempered distribution μ such that $\mathrm{T}f = \mu * f$ for all $f \in \mathcal{S}$. For $p = q = 1$ the tempered distributions in this theorem are finite Borel measures. If the measure is bounded and positive this means that the operator T can be viewed as a weighted averaging operator. In the following the case $n = 1$ will be of interest. A positive bounded measure μ on $\mathbb{R}$ is uniquely determined by its distribution function $\widetilde{\mu} : \mathbb{R} \to [0, 1]$ defined by

$$\widetilde{\mu}(x) = \frac{\mu(]-\infty, x[)}{\mu(\mathbb{R})}. \tag{9}$$

The tilde will again be omitted to simplify the notation.

Now, let $\mathrm{T}(t)$ be a strongly continuous time evolution as defined above fulfilling the conditions of homogeneity and causality, and being such that $f \in L^p(\mathbb{R})$ and $0 \leq f \leq 1$ almost everywhere implies $0 \leq \mathrm{T}f \leq 1$ almost everywhere. Then $\mathrm{T}(t)$ corresponds uniquely to a convolution semi-group of measures μ_t through the formula

$$\mathrm{T}(t)f(s) = (\mu_t * f)(s) = \int_{-\infty}^{\infty} f(s - s')\mathrm{d}\mu_t(s') \tag{10}$$

with $\operatorname{supp} \mu_t \subset \mathbb{R}_+$ for all $t \geq 0$. Here a convolution semi-group of measures on $\mathbb{R}$ is a family $\{\mu_t : t > 0\}$ of positive bounded measures on $\mathbb{R}$ with the properties that

$$\mu_t(\mathbb{R}) \leq 1 \quad \text{for } t > 0, \tag{11a}$$

$$\mu_{t+s} = \mu_t * \mu_s \quad \text{for } t, s > 0, \tag{11b}$$

$$\delta = \lim_{t\to 0} \mu_t \tag{11c}$$

where δ is the Dirac measure at 0 and the limit is the weak limit.

The problem of interest is to take a scaling limit of $\mathrm{T}(t)f(s)$ in which both $t \to \infty$ and $s \to \infty$ simultaneously. [2] This is conveniently done by discretizing $\mathrm{T}(t)$ using $t \in \mathbb{N}$. Then reintroducing the time scale τ one has $\mathrm{T}_\tau(n) = \mathrm{T}(n\tau) = \mathrm{T}(\tau)^n = \mathrm{T}_\tau(1)^n$. Suppressing again the time scale τ the scaling limit of interest may be defined through iteration of $\mathrm{T}(1) = \mathrm{T}$ as

$$\lim_{\substack{n,s\to\infty \\ s=\sigma_n \bar{s}}} (\mathrm{T}^n f)(s) \tag{12}$$

whenever it exists. Here σ_n denotes a sequence of rescaling factors such that $\lim_{n\to\infty} \sigma_n = \infty$. The scaling limit is called causal if $\operatorname{supp}\mu \subset \mathbb{R}_+$ where $\mu = \mu_1$ is the measure corresponding to $\mathrm{T} = \mathrm{T}(1)$ by virtue of (10)

3 Result

If a causal (ultra-)long time limit, as defined in (12), exists and is non-degenerate then the sequence σ_n of rescaling factors must have the form

$$\sigma_n = n^{1/\alpha} \Lambda(n) \tag{13}$$

where $0 < \alpha \le 1$ and $\Lambda(n)$ fulfills $\lim_{n\to\infty} \Lambda(bn)/\Lambda(n) = 1$ for all $b > 0$. Such a function $\Lambda(n)$ is called slowly varying [14,15]. The universal exponent α and the slowly varying function $\Lambda(n)$ are determined by the asymptotic behavior of the measure μ_1 corresponding to $\mathrm{T} = \mathrm{T}(1)$ by (10), i.e. they are determined by the microscopic evolution.

The basic result was obtained in [6,13,16–19]. Define the Fourier transform of a function f as usual through

$$\widehat{f}(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} f(t)\, \mathrm{d}t. \tag{14}$$

Let $f(s)$ be such that the limit $\lim_{a\to 0} a\widehat{f}(a\omega) = \widehat{\overline{f}}(\omega)$ defines the Fourier transform of a function $\overline{f}(s)$. Then the (ultra-)long time scaling limit exists if and only if it is of the form

$$\begin{aligned}
\lim_{\substack{n,s\to\infty \\ s=\sigma_n \bar{s}}} (\mathrm{T}^n f)(s) &= \int_0^\infty \overline{f}(\bar{s}-y) h_\alpha\left(\frac{y}{\bar{t}}\right) \frac{\mathrm{d}y}{\bar{t}} = \int_0^\infty \overline{\mathrm{T}}_y \overline{f}(\bar{s}) h_\alpha\left(\frac{y}{\bar{t}}\right) \frac{\mathrm{d}y}{\bar{t}} \\
&= \overline{\mathrm{T}}_\alpha(\bar{t})\overline{f}(\bar{s})
\end{aligned} \tag{15}$$

[2] This scaling limit was called "ultra-long time limit" in [13] because it goes over to time evolutions on time scales longer than infinite.

where $\bar{t} \geq 0$ and $0 < \alpha \leq 1$ are constants, the functions $h_\alpha(x)$ are given by

$$h_\alpha(x) = \frac{1}{\alpha x} H_{11}^{10}\left(\frac{1}{x}\middle|\begin{matrix}(0,1)\\(0,1/\alpha)\end{matrix}\right), \tag{16}$$

and $\overline{\mathrm{T}}_{\bar{t}}\bar{f}(\bar{s}) = \bar{f}(\bar{s} - \bar{t})$ denotes the translation semi-group in the rescaled variables $\bar{t}$ and $\bar{s}$. The definition of the H-function $H_{p,q}^{m,n}(x)$ in (16) is given in the Appendix. The constant α is determined by the measure μ_1 corresponding to T, and agrees with that appearing in (13) for the normalizing constants.

The constant $\bar{t}$ is nonnegative. This result receives its significance from the fact that the operator $\overline{\mathrm{T}}_\alpha(\bar{t})\bar{f}(\bar{s})$ defined by (15) is again a semi-group in the variable $\bar{t}$. Therefore $\overline{\mathrm{T}}_\alpha(\bar{t})$ may be identified as the macroscopic time evolution arising in the long time scaling limit from the microscopic time evolution defined by T. The result $\bar{t} \geq 0$ then states that a macroscopic time evolution is always a semi-group never a group. The result remains valid also if T is invertible, i.e. if the microscopic time evolution is reversible. This finding seems related to the questions surrounding the origin of macroscopic irreversibility versus microscopic reversibility that have received renewed attention [10,11].

4 Infinitesimal Generators

The fundamental importance of the semi-groups $\overline{\mathrm{T}}_\alpha(\bar{t})$ for time evolutions in physics and other sciences as universal attractors for macroscopic time evolutions seems to have been noticed for the first time in [6,7,13,16–19]. This is surprising because their mathematical importance has long been recognized [14,20–22]. In particular the infinitesimal generators are known to be fractional derivatives [23,24,21,14]. The infinitesimal generators are defined as

$$\mathrm{A}_\alpha \bar{f}(\bar{s}) = \lim_{\bar{t}\to 0} \frac{\overline{\mathrm{T}}_\alpha(\bar{t})\bar{f}(\bar{s}) - \bar{f}(\bar{s})}{\bar{t}}. \tag{17}$$

Formally A_α may be calculated by Laplace transformation in (15).

To obtain the Laplace transformation of h_α note first that

$$h_\alpha(x) = \frac{1}{\alpha} H_{11}^{01}\left(x\middle|\begin{matrix}(1-1/\alpha,1/\alpha)\\(0,1)\end{matrix}\right) \tag{18}$$

by virtue of relations (39) and (37) given in the Appendix. Using the Laplace transform of a general H-function from (40), the order reduction formula (35) and the reciprocal relation (37) one finds

$$\begin{aligned} h_\alpha(u) &= \int_0^\infty e^{-ux} h_\alpha(x)\,\mathrm{d}x \\ &= \frac{1}{\alpha} H_{01}^{10}\left(x\middle|\begin{matrix}(-,-)\\(0,1/\alpha)\end{matrix}\right) \\ &= e^{-u^\alpha} \end{aligned} \tag{19}$$

where the last equality follows from the Mellin transform

$$\int_0^\infty x^{s-1} H_{01}^{10}\left(x \,\middle|\, \begin{matrix}(-,-)\\(0,B_1)\end{matrix}\right) \mathrm{d}x = \Gamma(B_1 s) \tag{20}$$

from the definition (23). With (19), (15), (17) and the convolution theorem one finds by Laplace inversion

$$\begin{aligned}
\mathrm{A}_\alpha \overline{f}(\overline{s}) &= \lim_{\overline{t}\to 0} \frac{1}{2\pi i} \int_{\eta-i\infty}^{\eta+i\infty} e^{\overline{su}} \left(\frac{e^{-\overline{t}\overline{u}^\alpha} - 1}{\overline{t}}\right) \overline{f}(\overline{u})\, \mathrm{d}\overline{u} \\
&= \frac{1}{2\pi i} \int_{\eta-i\infty}^{\eta+i\infty} e^{\overline{su}} \lim_{\overline{t}\to 0} \left(\frac{e^{-\overline{t}\overline{u}^\alpha} - 1}{\overline{t}}\right) \overline{f}(\overline{u})\, \mathrm{d}\overline{u} \\
&= -\frac{1}{2\pi i} \int_{\eta-i\infty}^{\eta+i\infty} e^{\overline{su}}\, \overline{u}^\alpha \overline{f}(\overline{u})\, \mathrm{d}\overline{u}.
\end{aligned} \tag{21}$$

This formal result can be made rigorous [25]. The infinitesimal generator A_α of the macroscopic time evolutions $\overline{\mathrm{T}}_\alpha(\overline{t})$ is therefore related to the infinitesimal generator $\mathrm{A} = -\mathrm{d}/\mathrm{d}\overline{t}$ of $\overline{\mathfrak{T}}_{\overline{t}}$ through

$$\begin{aligned}
\mathrm{A}_\alpha \overline{f}(\overline{s}) = -(-\mathrm{A})^\alpha \overline{f}(\overline{s}) &= -\frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{\overline{f}(\overline{s}-y) - \overline{f}(\overline{s})}{y^{\alpha+1}}\, \mathrm{d}y \\
&= -\frac{1}{\Gamma(-\alpha)} \int_0^\infty y^{-\alpha-1} (\overline{\mathfrak{T}}_y - \mathbf{1}) \overline{f}(\overline{s})\, \mathrm{d}y \\
&= -\mathrm{D}^\alpha \overline{f}(\overline{s})
\end{aligned} \tag{22}$$

showing that A_α is the fractional power of the derivative $\mathrm{d}/\mathrm{d}\overline{t}$. The last equality defines the fractional derivative of order α, denoted as D^α, through the Marchaud-Hadamard-Balakrishnan algorithm [23,25].

5 Conclusion

The preceding results demonstrate that fractional time derivatives may arise from a suitable scaling limit as the infinitesimal generators of time evolutions on macroscopic time scales. The order α of the derivative is restricted to the unit interval, and its value is determined by the microscopic time evolution. Physically, the order α is a quantitative measure for the decay of the temporal correlations or history dependence in the microscopic time evolution. For the most frequent case $\alpha = 1$ the results show that macroscopic irreversibility of regular evolution equations such as (1) may be the viewed as a general consequence of a long time scaling limit.

Appendix: Definition and Properties of H-Functions

The H-function of order $(m,n,p,q) \in \mathbb{N}^4$ and parameters $A_i \in \mathbb{R}_+ (i=1,\dots,p)$, $B_i \in \mathbb{R}_+ (i=1,\dots,q)$, $a_i \in \mathbb{C} (i=1,\dots,p)$, and $b_i \in \mathbb{C} (i=1,\dots,q)$ is defined for $z \in \mathbb{C}, z \neq 0$ by the contour integral [26–30]

$$H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_q,B_q)\end{matrix}\right) = \frac{1}{2\pi i}\int_{\mathcal{L}} \eta(s) z^{-s}\,\mathrm{d}s \tag{23}$$

where the integrand is

$$\eta(s) = \frac{\prod\limits_{i=1}^{m} \Gamma(b_i + B_i s) \prod\limits_{i=1}^{n} \Gamma(1 - a_i - A_i s)}{\prod\limits_{i=n+1}^{p} \Gamma(a_i + A_i s) \prod\limits_{i=m+1}^{q} \Gamma(1 - b_i - B_i s)}. \tag{24}$$

In (23) $z^{-s} = \exp\{-s \log|z| - i \arg z\}$ and $\arg z$ is not necessarily the principal value. The integers m,n,p,q must satisfy

$$0 \le m \le q, \qquad 0 \le n \le p \tag{25}$$

and empty products are interpreted as being unity. The parameters are restricted by the condition

$$\mathbb{P}_a \cap \mathbb{P}_b = \emptyset \tag{26}$$

where

$$\begin{aligned}
\mathbb{P}_a &= \{\text{poles of } \Gamma(1-a_i-A_i s)\} = \left\{\frac{1-a_i+k}{A_i} \in \mathbb{C} : i=1,\dots,n; k \in \mathbb{N}_0\right\}\\
\mathbb{P}_b &= \{\text{poles of } \Gamma(b_i+B_i s)\} = \left\{\frac{-b_i-k}{B_i} \in \mathbb{C} : i=1,\dots,m; k \in \mathbb{N}_0\right\}
\end{aligned} \tag{27}$$

are the poles of the numerator in (24). The integral converges if one of the following conditions holds [30]

$$\mathcal{L} = \mathcal{L}(c-i\infty, c+i\infty; \mathbb{P}_a, \mathbb{P}_b); \quad |\arg z| < C\pi/2; \quad C > 0 \tag{28a}$$

$$\mathcal{L} = \mathcal{L}(c-i\infty, c+i\infty; \mathbb{P}_a, \mathbb{P}_b); \quad |\arg z| = C\pi/2; \quad C \ge 0; \quad cD < -\operatorname{Re} F \tag{28b}$$

$$\mathcal{L} = \mathcal{L}(-\infty+i\gamma_1, -\infty+i\gamma_2; \mathbb{P}_a, \mathbb{P}_b); \quad D > 0; \quad 0 < |z| < \infty \tag{29a}$$

$$\mathcal{L} = \mathcal{L}(-\infty+i\gamma_1, -\infty+i\gamma_2; \mathbb{P}_a, \mathbb{P}_b); \quad D = 0; \quad 0 < |z| < E^{-1} \tag{29b}$$

$$\mathcal{L} = \mathcal{L}(-\infty+i\gamma_1, -\infty+i\gamma_2; \mathbb{P}_a, \mathbb{P}_b); \quad D = 0; \quad |z| = E^{-1}; C \ge 0; \operatorname{Re} F < 0 \tag{29c}$$

$$\mathfrak{L} = \mathfrak{L}(\infty + i\gamma_1, \infty + i\gamma_2; \mathbb{P}_a, \mathbb{P}_b); \quad D < 0; \quad 0 < |z| < \infty \tag{30a}$$

$$\mathfrak{L} = \mathfrak{L}(\infty + i\gamma_1, \infty + i\gamma_2; \mathbb{P}_a, \mathbb{P}_b); \quad D = 0; \quad |z| > E^{-1} \tag{30b}$$

$$\mathfrak{L} = \mathfrak{L}(\infty + i\gamma_1, \infty + i\gamma_2; \mathbb{P}_a, \mathbb{P}_b); \quad D = 0; \quad |z| = E^{-1}; C \geq 0; \operatorname{Re} F < 0 \tag{30c}$$

where $\gamma_1 < \gamma_2$. Here $\mathfrak{L}(z_1, z_2; \mathbb{G}_1, \mathbb{G}_2)$ denotes a contour in the complex plane starting at z_1 and ending at z_2 and separating the points in $\mathbb{G}_1$ from those in $\mathbb{G}_2$, and the notation

$$C = \sum_{i=1}^{n} A_i - \sum_{i=n+1}^{p} A_i + \sum_{i=1}^{m} B_i - \sum_{i=m+1}^{q} B_i \tag{31}$$

$$D = \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i \tag{32}$$

$$E = \prod_{i=1}^{p} A_i^{A_i} \prod_{i=1}^{q} B_i^{-B_i} \tag{33}$$

$$F = \sum_{i=1}^{q} b_i - \sum_{i=1}^{p} a_j + (p-q)/2 + 1 \tag{34}$$

was employed. The H-functions are analytic for $z \neq 0$ and multi-valued (single valued on the Riemann surface of $\log z$).

The following properties of a general H-function are used in the text. First, the order reduction formula

$$H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (a_1, A_1), (a_2, A_2) \ldots, (a_p, A_p) \\ (b_1, B_1), (b_2, B_2) \ldots, (b_{q-1}, B_{q-1})(a_1, A_1) \end{matrix}\right)$$
$$= H^{m,n-1}_{p-1,q-1}\left(z \,\middle|\, \begin{matrix} (a_2, A_2), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_{q-1}, B_{q-1}) \end{matrix}\right) \tag{35}$$

holds for $n \geq 1$ and $q > m$, and similarly

$$H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (a_1, A_1), (a_2, A_2) \ldots, (a_{p-1}, A_{p-1})(b_1, B_1) \\ (b_1, B_1), (b_2, B_2) \ldots, (b_q, B_q) \end{matrix}\right)$$
$$= H^{m-1,n}_{p-1,q-1}\left(z \,\middle|\, \begin{matrix} (a_1, A_1), \ldots, (a_{p-1}, A_{p-1}) \\ (b_2, B_2), \ldots, (b_q, B_q) \end{matrix}\right) \tag{36}$$

for $m \geq 1$ and $p > n$. A change of variables in (23) shows

$$H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (a_1, A_1), \ldots, (a_p, A_p) \\ (b_1, B_1), \ldots, (b_q, B_q) \end{matrix}\right)$$
$$= H^{n,m}_{q,p}\left(\frac{1}{z} \,\middle|\, \begin{matrix} (1-b_1, B_1), \ldots, (1-b_q, B_q) \\ (1-a_1, A_1), \ldots, (1-a_p, A_p) \end{matrix}\right) \tag{37}$$

which allows to transform an H-function with $D > 0$ and $\arg z$ to one with $D < 0$ and $\arg(1/z)$. For $\gamma > 0$

$$\begin{aligned} &\frac{1}{\gamma} H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} \right) \\ &= H^{m,n}_{p,q}\left(z^{\gamma} \,\middle|\, \begin{matrix} (a_1, \gamma A_1), \dots, (a_p, \gamma A_p) \\ (b_1, \gamma B_1), \dots, (b_q, \gamma B_q) \end{matrix} \right) \end{aligned} \tag{38}$$

while for $\gamma \in \mathbb{R}$

$$\begin{aligned} &z^{\gamma} H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} \right) \\ &= H^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} (a_1 + \gamma A_1, A_1), \dots, (a_p + \gamma A_p, A_p) \\ (b_1 + \gamma B_1, B_1), \dots, (b_q + \gamma B_q, B_q) \end{matrix} \right) \end{aligned} \tag{39}$$

holds. Finally, the Laplace transform of an H-function is

$$\begin{aligned} \mathcal{L}\left\{ H^{m,n}_{p,q}(z) \right\}(u) &= \int_0^{\infty} e^{-ux} H^{m,n}_{p,q}\left(x \,\middle|\, \begin{matrix} (a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} \right) \mathrm{d}x \\ &= H^{n+1,m}_{q,p+1}\left(u \,\middle|\, \begin{matrix} (1 - b_1 - B_1, B_1), \dots, (1 - b_q - B_q, B_q) \\ (0,1)(1 - a_1 - A_1, A_1), \dots, (1 - a_p - A_p, A_p) \end{matrix} \right) \\ &= \frac{1}{u} H^{m,n+1}_{p+1,q}\left(\frac{1}{u} \,\middle|\, \begin{matrix} (0,1)(a_1, A_1), \dots, (a_p, A_p) \\ (b_1, B_1), \dots, (b_q, B_q) \end{matrix} \right) \end{aligned} \tag{40}$$

for $\operatorname{Re} s > 0$, $C > 0$, $|\arg z| < \frac{1}{2} C\pi$ and $\min_{1 \le j \le m} \operatorname{Re}(b_j / B_j) > -1$.

References

1. In classical mechanics the states are points in phase space, the observables are functions on phase space, and the operator B is specified by a vector field and Poisson brackets. In quantum mechanics (with finitely many degrees of freedom) the states correspond to rays in a Hilbert space, the observables to operators on this space, and the operator B to the Hamiltonian. In field theories the states are normalized positive functionals on an operator algebra of observables, and then B becomes a derivation on the algebra of observables. The equations (1) need not be first order in time. An example is the initial-value problem for the wave equation for $g(t, x)$

$$\frac{\partial^2 g}{\partial t^2} = c^2 \frac{\partial^2 g}{\partial x^2}$$

in one dimension. It can be recast into the form of (1) by introducing a second variable h and defining

$$f = \begin{pmatrix} g \\ h \end{pmatrix} \quad \text{and} \quad \mathrm{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} c \frac{\partial}{\partial x}$$

2. R. Hilfer, *Stochastische Modelle für die betriebliche Planung*, (GBI – Verlag, München, 1985).
3. R. Hilfer and L. Anton, *Fractional master equations and fractal time random walks,* Phys. Rev. E, Rapid Commun. **51**, 848 (1995).
4. R. Hilfer, *On fractional diffusion and its relation with continuous time random walks,* in: *Anomalous Diffusion: From Basis to Applications*, A. Pekalski, R. Kutner, and K. Sznajd-Weron, (Eds.), p. 77 (Springer, 1999).
5. R. Hilfer, *Applications of Fractional Calculus in Physics,* (World Scientific Publ. Co., Singapore, 2000).
6. R. Hilfer, *Exact solutions for a class of fractal time random walks,* Fractals **3**, 211 (1995).
7. R. Hilfer, *Fractional time evolution,* in: *Applications of Fractional Calculus in Physics*, R. Hilfer, (Ed.), p. 87 (World Scientific, Singapore, 2000).
8. M. Paczuski and K. Nagel, *Self-organized criticality and $1/f$ noise in traffic,* in: *Traffic and Granular Flow*, D.E. Wolf, M. Schreckenberg, and A. Bachem, (Eds.), p. 73 (World Scientific, Singapore, 1996).
9. D. Helbing, *Traffic modeling by means of physical concepts,* in: *Traffic and Granular Flow*, D.E. Wolf, M. Schreckenberg, and A. Bachem, (Eds.), p. 87 (World Scientific, Singapore, 1996).
10. J.L. Lebowitz, *Macroscopic laws, microscopic dynamics, time's arrow and Boltzmann's entropy,* Physica A **194**, 1 (1993).
11. J. Lebowitz, *Statistical mechanics: A selective review of two central issues,* Rev. Mod. Phys. **71**, S346 (1999).
12. E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces,* (Princeton University Press, Princeton, 1971).
13. R. Hilfer, *Foundations of fractional dynamics,* Fractals **3**, 549 (1995).
14. W. Feller, *An Introduction to Probability Theory and Its Applications*, Vol. II, (Wiley, New York, 1971).
15. E. Seneta, *Regularly Varying Functions*, (Springer Verlag, Berlin, 1976).
16. R. Hilfer, *Classification theory for an equilibrium phase transitions,* Phys. Rev. E **48**, 2466 (1993).
17. R. Hilfer, *On a new class of phase transitions,* in: *Random Magnetism and High-Temperature Superconductivity*, W.P. Beyermann, N.L. Huang-Liu, and D.E. MacLaughlin, (Eds.), p. 85 (World Scientific Publ. Co, Singapore, 1994).
18. R. Hilfer, *Fractional dynamics, irreversibility and ergodicity breaking,* Chaos, Solitons & Fractals **5**, 1475 (1995).
19. R. Hilfer, *An extension of the dynamical foundation for the statistical equilibrium concept,* Physica A **221**, 89 (1995).
20. S. Bochner, *Harmonic Analysis and the Theory of Probability,* (University of California Press, Berkeley, 1955).
21. K. Yosida, *Functional Analysis,* (Springer, Berlin, 1965).
22. C. Berg and G. Forst, *Potential Theory on Locally Compact Abelian Groups,* (Springer, Berlin, 1975).
23. A.V. Balakrishnan, *Fractional powers of closed operators and the semigroups generated by them,* Pacific J. Math. **10**, 419 (1960).
24. U. Westphal, *Ein Kalkül für gebrochene Potenzen infinitesimaler Erzeuger von Halbgruppen und Gruppen von Operatoren,* Compos. Math. **22**, 67 (1970).
25. U. Westphal, *Fractional powers of infinitesimal generators of semigroups,* in: *Applications of Fractional Calculus in Physics*, R. Hilfer, (Ed.), (World Scientific, Singapore, 2000)

26. C. Fox, *The G and H functions as symmetrical Fourier kernels,* Trans. Am. Math. Soc. **98**, 395 (1961).
27. B.L.J. Braaksma, *Asymptotic expansions and anlytic continuations for a class of Barnes-integrals,* Compos. Math. **15**, 239 (1964).
28. A.M. Mathai and R.K. Saxena, *The H-function with Applications in Statistics and Other Disciplines,* (Wiley, New Delhi, 1978).
29. H.M. Srivastava, K.C. Gupta, and S.P. Goyal, *The H-functions of One and Two Variables with Applications*, (South Asian Publishers, New Delhi, 1982).
30. A.P. Prudnikov, Yu.A. Brychkov, and O.I. Marichev, *Integrals and Series*, Vol. 3, (Gordon and Breach, New York, 1990).