

Geometry, Dielectric Response and Scaling in Porous Media

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Abstract

Local porosity distributions and local percolation probabilities have been proposed as well defined and experimentally observable geometric characteristics of general porous media. Based on these concepts the dielectric response is analysed using the effective medium approximation and percolation scaling theory. The theoretical origin of static and dynamic scaling laws for the dielectric dispersion and enhancement including Archie's law in the low porosity limit are elucidated. These well known experimental facts are unified within a theoretical framework based on a quantitative characterization of the pore space geometry. In the high porosity limit the zero frequency real dielectric constant is predicted to diverge as $\epsilon'(0) \propto (1 - \phi)^{-m'}$ where ϕ denotes porosity and m' is analogous to the cementation exponent.

An improved theoretical understanding of the physics of porous media is important scientifically and technologically. A much investigated physical question is the dielectric response arising in the study of propagation of electromagnetic waves in heterogeneous media [1–25]. This paper is concerned with the dielectric response of water saturated rocks. The effective, complex and frequency-dependent dielectric function $\epsilon(\omega)$ of water wet clay free rocks shows experimentally three interesting features [1–7]:

(1) large values of the real dielectric permittivities $\epsilon'(0)$ at zero frequencies, sometimes exceeding those of both constituent phases;

(2) dielectric dispersion in the low frequency regime where the constituent phases are dispersion free; and

(3) Archie's phenomenological law for the real conductivity $\sigma'(0) \propto \phi^m$ where ϕ is the porosity of the medium defined as the volume of pore space divided by the total volume.

The exponent m is called the cementation index and it ranges experimentally between $m \approx 1$ and $m \approx 4$. In the absence of electrochemical effects all three features arise from the randomness in the pore space geometry [6, 7].

My objective in this paper is to discuss a recent mean field approach [27, 28] inside which the above experimental facts can be understood. The approach is based on a novel geometric characterization of porous media via local porosity distributions (LPD's) and local percolation probabilities (LPP's) which will be defined below. These geometric quantities are conceptually well defined and experimentally observable. The dielectric properties are calculated directly from the geometric characteristics using effective medium theory and scaling theory. The results are intended to be applicable to the physics of water or brine saturated clay free rocks where electrochemical effects are absent.

Despite numerous efforts over many years the theoretical understanding of the experimental situation is not coherent and usually limited to individual aspects. Theoretical

approaches during the last decade [8] can be divided into two categories: The first class [9–17] attempts to calculate the dielectric properties starting from highly simplified or indirect geometric models for the pore space. The second category [18–23] does not attempt to incorporate the pore space geometry, but concentrates instead on a sometimes sophisticated phenomenological approach. This shows that the fundamental problem of the physics of porous media lies in their geometric characterization.

Geometrical characterization of porous media is in practice often limited to a determination of the porosity ϕ . Clearly ϕ alone, being just a single number, cannot suffice to characterize the complex pore space geometry. On the other hand a complete mathematical description of the pore space boundary considered as a random two dimensional continuous manifold contains too much (possibly irrelevant) geometrical information. It is often suggested to use a "pore size distribution" as a possible geometric characterization for porous media, and mercury injection is suggested to measure it. It is well known, however, that the pore size distribution is mathematically ill defined [24, 25].

Let me therefore take a different approach by considering the porosity itself instead of the "pore size" as the fundamental random variable. This leads immediately to suggest local porosity distributions (LPD's) and local percolation probabilities (LPP's) as partial geometric characterizations of the pore space, and these quantities will be defined next. Starting from these new geometric characterizations the validity and theoretical origin of static and dynamic scaling laws for the dielectric response of porous media (e.g. Archie's law) will be discussed.

The geometric characterization is based on viewing the local geometry and its porosity on a mesoscopic scale as the fundamental random quantity. To define "local geometries" consider a porous medium with a homogeneously and isotropically disordered pore space. The points of Bravais lattice (in practice a simple cubic lattice) are superimposed on the porous medium and an arbitrary (in practice cubic) primitive cell is chosen. The local geometry around the lattice point R is defined as the intersection of the pore space and the primitive cell at R . The volume of the primitive (or measurement) cells is $V_{MC} = 1/\rho$ where ρ is the density of Bravais lattice points. This defines the length scale of resolution L as $L = \rho^{-1/3} = (V_{MC})^{1/3}$. For the simple cubic lattice with cubic primitive cell L is the lattice constant. The preceding definition of local geometries is valid for topologically and continuously disordered pore spaces. For a porous medium with substitutional disorder the measurement lattice is given by the underlying lattice.

The local geometry inside the measurement cell will become increasingly complex as the length scale of

resolution L is increased. A full geometric characterization at arbitrary L is difficult. However, at every L the local geometry may be partially characterized by two simple properties. One is the cell porosity, the second is whether the pore space percolates or not.

Consider first the local (or cell) porosity. The local porosity $\phi(\mathbf{R}, L)$ at the lattice position \mathbf{R} and length scale L is defined as $\phi(\mathbf{R}, L) = \rho \int \chi_{MC}(\mathbf{r}; \mathbf{R}, L) \chi_{PS}(\mathbf{r}) d^3r$ where $\chi_{MC}(\mathbf{r}; \mathbf{R}, L)$ is the characteristic function of the measurement cell at \mathbf{R} having size L , $\chi_{PS}(\mathbf{r})$ is the characteristic function of the pore space and the integration extends over the porous medium. The characteristic (or indicator) function $\chi_A(\mathbf{r})$ of an arbitrary set A is defined as $\chi_A(\mathbf{r}) = 1$ if $\mathbf{r} \in A$ and $\chi_A(\mathbf{r}) = 0$ if $\mathbf{r} \notin A$. One can now define LPD functions in analogy to atomic distribution functions. Thus, $\mu(\phi, \mathbf{R}; L)$ measures the probability density to find the local porosity ϕ in the range from ϕ to $\phi + d\phi$ in a cell of linear dimension L at the point \mathbf{R} . The assumption of homogeneity implies that $\mu(\phi, \mathbf{R}; L) = \mu(\phi; L)$ must be independent of \mathbf{R} . The function $\mu(\phi; L)$ will be called the LPD at scale L . The bulk porosity $\bar{\phi}$ can be thought of as the integral over a large volume or as the average over a statistical ensemble of measurement cells, and thus

$$\bar{\phi} = \phi(\mathbf{R}, L \rightarrow \infty) = \int_0^1 \phi \mu(\phi; L) d\phi \quad (1)$$

independent of \mathbf{R} and L . Higher order distribution functions can be defined similarly. The n -cell local porosity distribution function $\mu_n(\phi_1, \mathbf{R}_1; \phi_2, \mathbf{R}_2; \dots; \phi_n, \mathbf{R}_n; L)$ at scale L measures the probability density to find ϕ_1 in the cell at \mathbf{R}_1 , ϕ_2 in the cell at \mathbf{R}_2 , etc. The full information about the statistical properties of the porosity distribution at scale L is contained in the local porosity probability functional $\mu\{\phi; L\}$ at scale L which is obtained as the limit $n \rightarrow \infty$ of μ_n .

The local porosity distribution $\mu(\phi; L)$ depends strongly on L . There are two competing effects. At small L the local geometries are simple, but they are highly correlated with each other, and the one-cell-function $\mu(\phi; L)$ does not contain these complex geometric correlations. At large L the local geometries are statistically uncorrelated but each one of them is nearly as complex as the geometry of the full pore space. There must then exist an intermediate length scale L^* at which on the one hand the local geometries are relatively simple, and on the other hand the single cell distribution function has sufficient nontrivial geometric content to be a good first approximation. In this paper this length will be taken as a length of the order of the characteristic pore or grain size of the porous medium. More precisely, a porosity correlation length ξ is determined from the 2-cell distribution function $\mu_2(\phi_1, \mathbf{R}_1; \phi_2, \mathbf{R}_2; L)$ as described in Refs [27] and [29]. In the following the "local porosity distribution" is defined as $\mu(\phi) = \mu(\phi; \xi)$ the single cell LPD at scale ξ . Simultaneously with this convention it will be assumed, that the local geometries at scale ξ are "simple". This is called the "hypothesis of local simplicity", and it will be made more precise below. For systems with an underlying lattice symmetry the length ξ has to be replaced by the lattice constant.

The definition of LPD's as $\mu(\phi; \xi)$ is optimal in the sense that it contains the maximum amount of information based purely on the porosity concept. If the cells were chosen

much larger than ξ then the simple form $\mu(\phi; L \gg \xi) = \delta(\phi - \bar{\phi})$ is expected to result. The geometric information in this case is reduced to $\bar{\phi}$, and the local geometries are nearly as complex as the bulk geometry. If on the other hand the cells are chosen very small, i.e. $L \ll \xi$, then the measurement is expected to yield $\mu(\phi; L \ll \xi) = \bar{\phi} \delta(\phi - 1) + (1 - \bar{\phi}) \delta(\phi)$. Again the geometrical information in $\mu(\phi)$ is reduced to a single number.

The most important aspect of $\mu(\phi) = \mu(\phi; \xi)$ is that it is readily measurable using modern image processing equipment. It can be measured from photographs of two-dimensional thin sections through the pore space, as discussed elsewhere in more detail [29].

The local porosity distribution $\mu(\phi)$ is easily calculated for ordered or substitutionally disordered porous media, but difficult to obtain for topological or continuum disorder. For ordered or substitutionally disordered cases the measurement lattice is given by the underlying lattice, and ξ is the lattice constant. For the ordered case one finds immediately $\mu(\phi) = \delta(\phi - \phi_{\text{cell}})$ where ϕ_{cell} is the porosity in the unit cell. For substitutional disorder the LPD follows directly from the distribution of the individual geometrical elements which occupy the lattice sites.

The second geometric property to characterize local geometries is whether the pore space percolates or not. For cubic cells each cell is classified as percolating or non-percolating according to whether or not there exists at least one face of the cube which can be connected to any of the other faces via a path contained completely inside the pore space. For noncubic cells the classification has to be modified appropriately. Let $\lambda(\phi)$ denote the fraction of percolating cells with local porosity ϕ . $\lambda(\phi)$ will be called "local percolation probability" (LPP). It is an important geometric quantity for all physical properties of porous media such as conduction or fluid flow because it determines whether volume elements are permeable or not.

Based on the geometric characterization of the pore space through $\mu(\phi)$ and $\lambda(\phi)$ it is now possible to study physical problems such as the dielectric response. Consider again the subdivision of the porous medium into cells of length ξ . Because all cells are statistically independent standard one cell effective medium theory [26] can be employed to write a selfconsistency equation for the effective dielectric constant $\varepsilon(\omega)$ of the medium which reads

$$\int_0^1 \frac{\varepsilon_C(\omega; \phi) - \varepsilon(\omega)}{\varepsilon_C(\omega; \phi) + 2\varepsilon(\omega)} \lambda(\phi) \mu(\phi) d\phi + \int_0^1 \frac{\varepsilon_B(\omega; \phi) - \varepsilon(\omega)}{\varepsilon_B(\omega; \phi) + 2\varepsilon(\omega)} [1 - \lambda(\phi)] \mu(\phi) d\phi = 0. \quad (2)$$

Here $\varepsilon(\omega) = \varepsilon'(\omega) + i[\sigma'(\omega)/\omega]$ where ε' is the effective real dielectric constant, and σ' is the effective conductivity, and $\varepsilon_C(\omega; \phi)$ and $\varepsilon_B(\omega; \phi)$ are the frequency-dependent, complex, local, effective dielectric constants. The index C stands for conducting (percolating) local geometries, and the index B for blocking (non-percolating) cells. To use eq. (2) it is assumed that the local geometries are "simple" and therefore the local dielectric constants can be obtained from simple model geometries. This "hypothesis of local simplicity" underlies every effective medium approach. It will be assumed that this hypothesis can be cast into mathematical form by requiring that $\sigma'(\omega = 0; \phi)$ can be expanded into a

power series as $\sigma'_c(0; \phi) = \phi(C_1 + C_2 \phi + \dots)$ for small ϕ , and that the inverse real dielectric constant has the expansion $[\epsilon'_B(0; \phi)]^{-1} = (1 - \phi)[B_1 + B_2(1 - \phi) + \dots]$ around $\phi = 1$. The divergence of ϵ'_B for blocking geometries of large porosity is the well known thin plate effect.

The most interesting consequence of eq. (2) is that it implies the existence of a percolation transition underlying porous media. This does not contradict the fact that most porous media appear to remain connected to arbitrarily small $\bar{\phi}$ (Archie's law) because the control parameter is not the porosity $\bar{\phi}$ but the total fraction of percolating local geometries

$$p = \int_0^1 \lambda(\phi)\mu(\phi) d\phi. \quad (3)$$

The percolation threshold, $p_c = \frac{1}{3}$ has its usual effective medium value. In the rest of this paper some of the consequences of the underlying percolation transition will be reviewed briefly. A more detailed account is given elsewhere [27, 28].

Using the scaling theory for the percolation transition and assuming that the effective medium approximation remains asymptotically exact in more than one dimension [30] the well known result for the d.c. conductivity $\sigma'(0) \propto \sigma'_+ |p - p_c|^t$ is obtained. Here σ'_+ is defined through

$$\frac{1}{\sigma'_+} = \int_0^1 \frac{\lambda(\phi)\mu(\phi)}{\sigma'_c(0; \phi)} d\phi \quad (4)$$

and t is the universal conductivity exponent. The real d.c. dielectric constant diverges with the superconductivity exponent s as $\epsilon'_-(0) \propto \epsilon'_- |p - p_c|^{-s}$ where

$$\epsilon'_- = \int_0^1 \epsilon'_B(0; \phi)[1 - \lambda(\phi)]\mu(\phi) d\phi. \quad (5)$$

Because of the hypothesis of local simplicity these formulas are valid whenever $\lambda(\phi)$ and $\mu(\phi)$ are such that

$$\int_0^1 \phi^{-1} \lambda(\phi)\mu(\phi) d\phi \text{ exists.}$$

The interest in these consequences lies in the fact that they shed new light on the question of universality and scaling in porous media. Most publications on the electrical properties of porous media discuss the phenomenological scaling law [8] between d.c. conductivity and bulk porosity

$$\sigma'(0) \propto \bar{\phi}^m \quad (6)$$

called "Archie's law", which is usually written in terms of the formation factor $F = [\sigma'(0)]^{-1}$. The widespread acceptance of eq. (6) as a fundamental law for the physics of porous media is rather surprising in view of the fact that most experimental data [1-3, 7, 17] rarely span more than a decade in porosity. Correspondingly the cementation exponent m is found to scatter widely between $m \approx 1$ and $m \approx 4$. Equation (6) should not be interpreted as a relation between conductivity and pore space geometry, as it is often done in the well logging literature [1]. Instead it is shown next that within the present approach eq. (6) must be viewed as a statement about physical processes which reduce the bulk porosity.

Sedimentary and related rocks arise from sedimentation and subsequent compactification and cementation processes. The bulk porosity $\bar{\phi}$ changes during the sedimenta-

tion history of the rock. The physicochemical processes during sedimentation change the local dielectric and geometric properties. Within the present formulation these processes affect primarily $\lambda(\phi)$ and $\mu(\phi)$. This implies that σ'_+ and p depend upon $\bar{\phi}$, and consequently $\sigma'(0)$ will change with $\bar{\phi}$. Without a particular physical model it is only possible to discuss some general consequences. Clearly $\sigma'_+(\bar{\phi})$ should tend to zero as $\bar{\phi} \rightarrow 0$, and it should approach the conductivity σ'_W of water filling the pore space for $\bar{\phi} \rightarrow 1$. It seems also plausible that $p(\bar{\phi})$ should decrease as $\bar{\phi}$ is lowered. If one assumes that $\sigma'_+(\bar{\phi})$ and $p(\bar{\phi})$ can be expanded around $\bar{\phi} = 0$ as $\sigma'_+(\bar{\phi}) = \dot{\sigma}'_+(0)\bar{\phi} + \frac{1}{2}\ddot{\sigma}'_+(0)\bar{\phi}^2 + \dots$ and $p(\bar{\phi}) = p(0) + \dot{p}(0)\bar{\phi} + \dots$ where superscripted dots denote derivatives then it follows that $\sigma'(\bar{\phi}) \propto \dot{\sigma}'_+(0)\bar{\phi}[p(0) - p_c + \dot{p}(0)\bar{\phi} + \dots]^t$. This resembles eq. (6). In particular if it happens that $p(0) \approx p_c$, i.e. if criticality is approached as $\bar{\phi} \rightarrow 0$ then Archie's law follows with a cementation exponent $m = 1 + t$. More generally, if $\lambda(\phi)$ and $\mu(\phi)$ change with $\bar{\phi}$ such that $\sigma'_+(\bar{\phi}) \propto \bar{\phi}^{m_\sigma}$ and $p(\bar{\phi}) \propto p_c + \dot{p}(0)\bar{\phi}^{m_p}$ then $m = m_\sigma + m_p t$. Even more complicated results for m are obtained

if the integral $\int_0^1 \phi^{-1} \lambda(\phi)\mu(\phi) d\phi$ does not exist. It is surprising that the simplest result for m , namely $m = 1 + t$, predicts an exponent in the range from $m = 2$ for $p \gg p_c$ to $m = 1 + t \approx 3$ for $p \approx p_c$ as often found in experience. Nevertheless the cementation exponent will in general be very different for different compaction processes, and without physical models for such processes even a non-monotonous behaviour of $\sigma'(0; \bar{\phi})$ is possible. In Ref. [28] a general class of compaction models called "local porosity reduction models" has been introduced. These models exhibit a one parameter family of cementation indices which depend on geometric details of the compaction process [28]. The important result here is that the present approach provides for the first time a general framework inside which the apparent phenomenological universality and scaling properties of porous media in the low porosity limit might be understood.

A second interesting consequence is that the approach predicts similar scaling laws for the dielectric constant in the high porosity limit $\bar{\phi} \rightarrow 1$. This is a consequence of the thin plate effect, implicit in the hypothesis of local simplicity, and analogous assumptions about the physical process of dilution. Dilution plays the same role for ϵ' as the compaction process for σ' . In the high porosity limit it is predicted that $\epsilon'_-(0) \propto (1 - \bar{\phi})^{-m'}$. Here the "dilution exponent" m' is given in the simplest case as $m' = 1 + s$ where s is the superconductivity exponent. In the more general case $m' = m'_\sigma + m'_p s$. The exponents m'_σ and m'_p characterize the behaviour of ϵ'_- and p as $\bar{\phi} \rightarrow 1$. The new scaling law predicted for the real d.c. dielectric constant might be experimentally observable in water filled pore casts of systems obeying Archie's law.

Finally, the frequency dependence of $\epsilon(\omega)$ reveals information about $\mu(\phi)$ and $\lambda(\phi)$. Scaling behaviour of $\sigma'(\omega)$ and $\epsilon'(\omega)$ as functions of frequency obtains whenever p is close to p_c [27].

In summary local porosity distributions and local percolation probabilities provide a partial geometric characterization of the complex pore space geometry in porous media. From these well defined and experimentally accessible geometric quantities the dielectric response of general

porous media resulting from geometrical disorder has been analysed using effective medium theory and scaling theory. It was found that the theory predicts an underlying percolation transition, which may or may not appear as a porosity threshold for conduction. This provides a theoretical framework inside which Archie's law can be understood as a statement about the sedimentation history of rocks. In particular the universal applicability of this phenomenological relationship appears as a consequence of the universality of the percolation transition. Simplest mean field theory gives the value $m = 2$ for the cementation exponent in the low porosity limit. A new scaling law for the divergence of the real dielectric constant is predicted in the high porosity limit. This law should be observable in conductor filled insulating pore casts of systems obeying Archie's law. The present theory contains no adjustable parameters or distribution functions. The dielectric response is calculated directly from geometrical quantities. Recently also the hydrodynamical flow problem has been treated by a similar approach [28]. The results corroborate the findings reported here and indicate good agreement with experimentally observed scaling relations [28]. Experimental observation of local porosity distributions and local percolation probabilities must answer the question whether they are suitable geometric characteristics for distinguishing different classes of porous media or not.

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