

## Local-porosity theory for flow in porous media

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A recently introduced geometric characterization of porous media based on local-porosity distributions and local-percolation probabilities is used to calculate dc permeabilities for porous media. The disorder in porous media is found to be intimately related to the percolation concept. The geometric characterization is shown to open a possibility for understanding experimentally observed scaling relations between permeability, formation factor, specific internal surface, and porosity. In particular, Kozeny's equation  $k \propto \bar{\phi}^b$  between effective permeability and bulk porosity and the relation  $k \propto F^{-h}$  between permeability and formation factor are analyzed. A simple and general consolidation model is introduced. It is based on the reduction of local porosities and emphasizes the general applicability and flexibility of the local-porosity concept. The theoretical predictions are compared with the experimentally observed range for  $b$  and  $h$ , and are found to be in excellent agreement.

### I. INTRODUCTION

A better theoretical understanding of flow through porous media continues to be a subject of great interest,<sup>1-23</sup> both scientifically and technologically. A particular problem of widespread practical importance is the calculation of rock permeabilities from their microstructure. Experimentally<sup>2,3,5</sup> it is found that the permeability  $k$  of a given geological formation correlates strongly with certain parameters characterizing its microgeometry. In particular the variation with porosity  $\bar{\phi}$  obeys a power-law relationship,

$$k \propto \bar{\phi}^b. \quad (1.1)$$

The porosity is defined as the ratio between the volume of the pore space and the sample volume, i.e., the volume fraction of pores. The exponent  $b$  scatters widely.<sup>2,3</sup> It ranges typically from  $b \approx 3$  to  $b \approx 6$ . Equation (1.1) will be referred to as Kozeny's equation. A similar scaling relation is found between the so-called formation factor  $F$  and porosity:<sup>1-3,5</sup>

$$F \propto \bar{\phi}^{-m}. \quad (1.2)$$

The formation factor is the inverse dimensionless conductivity of an insulating specimen filled with a conductor. The exponent  $m$  is called the cementation index, and Eq. (1.2) is known as Archie's law.<sup>16</sup> The cementation index again scatters widely between  $m \approx 1$  and  $m \approx 4$ , but most often it is close to  $m \approx 2$ .

My objective in this paper is to present a simple theoretical approach that is aimed at developing a better insight into the experimental findings above.

Despite many attempts to include microscopic statistical aspects, the theoretical understanding of fluid flow through porous media has been limited to macroscopic continuum theories or inadequate geometric modeling, such as the well-known capillary model and ramifications thereof.<sup>1-4</sup> The fundamental difficulty can be traced to the absence of a geometric characterization for the microstructure of porous media that goes beyond the poros-

ity concept.<sup>1,2,5,20</sup> Thus the approach presented here begins with a geometric characterization of porous media. The characterization is based on local-porosity distributions and local-percolation probabilities. These quantities were introduced in Refs. 21 and 22, and their definitions are repeated below. Local-porosity distributions and local-percolation probabilities are conceptually well defined and experimentally observable. References 21 and 22 have utilized these concepts to calculate an approximate effective frequency-dependent dielectric constant. The result was found to obey Archie's law [Eq. (1.2)] and simultaneously allowed for strong dielectric dispersion and enhancement. The formal similarity between dc conductivities and dc permeabilities suggests applying the same approach to fluid flow in porous media. This paper reports the results for the zero-frequency permeability obtained through scaling theory and an effective-medium treatment.

### II. GEOMETRIC CHARACTERIZATION

Geometrical characterization of porous media is often limited to a determination of the porosity  $\phi$  and specific surface area  $S$ . Clearly these two numbers cannot suffice to characterize the complex pore-space geometry. On the other hand, the complete set of coordinates for all points of the solid-fluid interface contains too much (possibly irrelevant) geometrical information. It is often suggested to use a "pore-size distribution" as a possible geometric characterization for porous media, and mercury injection is suggested to measure it. It is well known, however, that the pore-size distribution cannot be defined without ambiguity.<sup>1,2</sup> Experimental determinations have to be interpreted cautiously.

Let me present a different statistical characterization of the pore-space geometry that was introduced recently.<sup>21</sup> It is based on the concept of local (mesoscopic) geometries as the fundamental random variables. To define "local geometries" consider a porous medium with a homogeneously and isotropically disordered pore space. The points of a simple cubic lattice (or any other Bravais

lattice) are superimposed on the porous medium and an arbitrary (in practice cubic) primitive cell is selected. Let  $G_{MC}(\mathbf{R})$  denote the set of all points inside the measurement cell around the lattice point  $\mathbf{R}$ ,  $G_{PS}$  the set of points in the pore space, and  $\partial G_{PS}$  its boundary. Then  $\partial G_{PS} \cap G_{MC}(\mathbf{R})$  defines what will be called the local geometry. The volume of the primitive (or measurement) cells is  $V_{MC} = 1/\rho$ , where  $\rho$  is the density of Bravais-lattice points. This defines the length scale of resolution  $L$  as  $L = \rho^{-1/3} = (V_{MC})^{1/3}$ . For the simple cubic lattice with cubic primitive cell,  $L$  is the lattice constant. The preceding definition of local geometries is valid for topologically and continuously disordered pore spaces. For a porous medium with substitutional disorder the measurement lattice is given by the underlying lattice.

The local geometry inside the measurement cell will become increasingly complex as the length scale of resolution  $L$  is increased. A full geometric characterization at arbitrary  $L$  is difficult. However, at every  $L$  the local geometry may be partially characterized by a number of geometrical observables. In this paper these observables will be taken to be the cell porosity  $\phi$ , the specific surface area  $S$  inside the cell, and a 0-1 variable that indicates whether the pore space inside the cell percolates or not.

Consider first the local (or cell) porosity and the local specific surface area. The local porosity  $\phi(\mathbf{R}, L)$  at the lattice position  $\mathbf{R}$  and length scale  $L$  is defined as

$$\phi(\mathbf{R}, L) = \frac{1}{V_{MC}} \int_{G_{PS}(\mathbf{R})} d^3r, \quad (2.1)$$

where  $G_{PS}(\mathbf{R}) = G_{PS} \cap G_{MC}(\mathbf{R})$ . Similarly, the local specific surface area is defined as

$$S(\mathbf{R}, L) = \frac{1}{V_{MC}} \int_{\partial G_{PS}(\mathbf{R})} d^2r. \quad (2.2)$$

While  $\phi(\mathbf{R}, L)$  is dimensionless,  $S(\mathbf{R}, L)$  has dimensions of an inverse length. The integrals in (2.1) and (2.2) are assumed to exist. This constraint excludes porous media with fractal pore spaces or fractal surfaces. Local-geometry distributions can now be defined in analogy with atomic distribution functions. Thus,  $\mu(\phi, S; \mathbf{R}; L)$  measures the joint probability density to find a local porosity  $\phi$  in the range from  $\phi$  to  $\phi + d\phi$  and a local specific surface area  $S$  in the range from  $S$  to  $S + dS$  inside a cell of linear dimension  $L$  at the point  $\mathbf{R}$ . The assumption of homogeneity implies that  $\mu(\phi, S; \mathbf{R}; L) = \mu(\phi, S; L)$  must be independent of  $\mathbf{R}$ . In agreement with the nomenclature of Ref. 21, the function  $\mu(\phi, S; L)$  will be called the local-porosity distribution at scale  $L$ , although it might be referred to more generally as a local-geometry distribution. The bulk porosity  $\bar{\phi}$  is obtained by integrating over a large volume or by averaging over a statistical ensemble of measurement cells, and thus

$$\bar{\phi} = \phi(\mathbf{R}, L \rightarrow \infty) = \int_0^\infty \int_0^\infty \phi \mu(\phi, S; L) d\phi dS, \quad (2.3)$$

independent of  $\mathbf{R}$  and  $L$ . On the other hand, the averaged specific internal surface area

$$\bar{S}(L) = \int_0^\infty \int_0^\infty S \mu(\phi, S; L) d\phi dS \quad (2.4)$$

is a *local* quantity that depends on the length scale of resolution. Higher-order local-porosity distribution functions can be defined similarly.

The local-porosity distribution  $\mu(\phi, S; L)$  depends strongly on  $L$ . At small  $L$ , the local geometries are simple, but they are highly correlated with each other. The one-cell function  $\mu(\phi, S; L)$  at small  $L$  does not contain these complex geometric correlations. At large  $L$ , the local geometries are statistically uncorrelated, but each one of them is nearly as complex as the geometry of the full pore space. There must then exist an intermediate length scale  $L^*$  at which, on the one hand, the local geometries are relatively simple, and on the other hand, the single-cell distribution function has sufficient nontrivial geometric content to be a good first approximation. Several possibilities for this length scale present themselves, and a systematic discussion is given elsewhere.<sup>23</sup> Here  $L^*$  will be taken as a length of the order of the characteristic pore or grain size of the porous medium. In the following, the local-porosity distribution is always taken at scale  $L^*$ , i.e., it is defined as  $\mu(\phi, S) = \mu(\phi, S; L^*)$ . Similarly, the average local specific surface area is then  $\bar{S} = \bar{S}(L^*)$ . Its inverse is an important length scale characterizing porous media. For systems with an underlying lattice symmetry the length  $L^*$  is simply the lattice constant.

The most important aspect of  $\mu(\phi, S)$  is that it is readily measurable using modern image-processing equipment. It can be measured from photographs of two-dimensional thin sections through the pore space, as discussed elsewhere in more detail.<sup>21-23</sup> The local-porosity distribution (LPD)  $\mu(\phi, S)$  is readily calculated for ordered or substitutionally disordered porous media, but is difficult to obtain for topological or continuum disorder. For ordered or substitutionally disordered cases the measurement lattice is given by the underlying lattice, and  $L^*$  is the lattice constant. For the ordered case one finds immediately that the distribution is concentrated at a single point. For substitutional disorder the LPD follows directly from the distribution of the individual geometrical elements that occupy the lattice sites.

The third geometric property that is needed to characterize local geometries is whether the pore space percolates or not. For cubic cells each cell is classified as percolating or nonpercolating, according to whether or not there exists at least one face of the cubic primitive cell that can be connected to any of the other faces via a path contained completely inside the pore space. For noncubic cells the classification has to be modified appropriately. Let  $\lambda(\phi, S)$  denote the fraction of percolating cells with local porosity  $\phi$  and local specific surface area  $S$ . The quantity  $\lambda(\phi, S)$  has been called the "local-percolation probability."<sup>21,22</sup> It is an important geometric quantity for all physical properties of porous media such as conduction or fluid flow because it determines whether volume elements are permeable or not.

### III. EFFECTIVE PERMEABILITY

Consider low Reynolds number (creeping) flow of an incompressible Newtonian fluid inside a rigid homogene-

ous and isotropic porous medium. The time-independent microscopic equations of motion are

$$\nabla P = \eta \Delta \mathbf{v} + \mathbf{F}, \quad (3.1a)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (3.1b)$$

together with the boundary condition

$$\mathbf{v} = 0 \quad \text{on } \partial G_{PS}. \quad (3.2)$$

Here  $\mathbf{v}$  is the velocity field of the fluid,  $\eta$  is its viscosity (often denoted as  $\mu$ ),  $P$  is the pressure, and  $\mathbf{F}$  represents an additional external force field such as gravity. Let  $l$  be a length of the order of the typical pore size in the medium. Applying standard homogenization techniques to Eq. (3.1) on length scales  $L > l$  yields Darcy's law

$$\langle \mathbf{v}^0 \rangle = \mathcal{T} \frac{\phi^3}{S^2 \eta} (\mathbf{F} - \nabla P^0) \quad (3.3)$$

to leading order in the small parameter  $\epsilon = l/L$ .<sup>4,24,25</sup> Here the brackets  $\langle \rangle$  represent an average over regions of size  $L$ ,  $\mathcal{T}$  denotes the tortuosity tensor, and the fields are expanded as  $\mathbf{v}(\mathbf{x}) = \epsilon^2 \mathbf{v}^0(\mathbf{x}, \mathbf{x}/\epsilon) + \epsilon^3 \mathbf{v}^1(\mathbf{x}, \mathbf{x}/\epsilon) + \dots$  and  $P(\mathbf{x}) = P^0(\mathbf{x}) + \epsilon P^1(\mathbf{x}, \mathbf{x}/\epsilon) + \dots$ . Thus the permeability is given by

$$k = \mathcal{T} \frac{\phi^3}{S^2}, \quad (3.4)$$

reminiscent of Kozeny's equation for capillaries.

These results are used once for the whole system and once on the mesoscopic scale  $L^*$ , leading to the self-consistent equation

$$\int_0^\infty \int_0^1 \frac{3\mathcal{T}\phi^3 \lambda(\phi, S) \mu(\phi, S)}{\mathcal{T}\phi^3 + 2kS^2} d\phi dS = 1 \quad (3.5)$$

for the effective permeability  $k$  of the porous medium. Equation (3.5) is entirely analogous to the effective-medium approximation for the electrical case.<sup>21,22</sup> Similar to that case, there exists an underlying percolation transition whose control parameter

$$p = \int_0^\infty \int_0^1 \lambda(\phi, S) \mu(\phi, S) d\phi dS \quad (3.6)$$

is the total fraction of percolating local geometries.

The existence of an underlying percolation transition implies that  $k$  exhibits universal behavior for a large class of geometries. The crucial quantity is now

$$k_0 = \left[ \int_0^\infty \int_0^1 \frac{S^2}{\mathcal{T}\phi^3} \lambda(\phi, S) \mu(\phi, S) d\phi dS \right]^{-1}. \quad (3.7)$$

If  $k_0 < \infty$ , then Eq. (3.5) is approximately solved by

$$k \approx k_0 (p - p_c)^t, \quad (3.8)$$

where  $t$  is the conductivity exponent from percolation scaling theory.<sup>26</sup> Within the effective-medium approximation  $t_{\text{EMA}} = 1$ , independent of dimension. The correct values are  $t \approx 1.3$  in two dimensions and  $t \approx 2.0$  in three dimensions.<sup>26</sup> The exact value for the percolation threshold  $p_c$  depends on the choice of Bravais lattice. Within the effective-medium approximation  $p_c = 2/z$ , where  $z$  is the lattice coordination. The result (3.8) is universal in

the sense that it remains valid as long as the local-geometry distribution and the local-percolation probability are such that  $k_0 < \infty$ . The universality condition breaks down in cases where

$$\mu(\phi, S) \propto \phi^{-\alpha} f(\phi, S) \quad (3.9)$$

for  $\phi \rightarrow 0$  with  $f$  analytic in  $\phi$  and  $-2 \leq \alpha < 1$ . A breakdown of  $k_0 < \infty$  could also be caused by the specific surface area, but this case is not expected and will thus be neglected here. If Eq. (3.9) applies, the exponent  $t$  in (3.8) becomes nonuniversal and must be replaced by

$$t_\alpha = t + \frac{2+\alpha}{1-\alpha}. \quad (3.10)$$

The key result is that Eq. (3.10) is already valid for negative  $\alpha$  in the range  $-2 \leq \alpha \leq 0$ .

#### IV. KOZENY'S EQUATION AND ARCHIE'S LAW

Kozeny's equation (1.1) appears to establish a definite relationship between bulk porosity and permeability. Such a relationship cannot be expected to hold in general because  $\bar{\phi}$  is not sufficient to characterize the pore space. In fact, the main content of the present paper is to propose  $\mu(\phi, S)$  and  $\lambda(\phi, S)$  as partial geometric characterization. The general relation between permeability and geometrical properties of a porous system is thus given by Eq. (3.8).

Nevertheless, Kozeny's scaling law can be understood within the present approach if it is interpreted as relating the change in permeability that results from a change in porosity. In other words, Eq. (1.1) must be viewed as a statement about physicochemical consolidation processes that change the pore space.<sup>21</sup> Sedimentary and related rocks arise from sedimentation and subsequent compactification and cementation processes. These processes, which will be referred to simply as consolidation, change the local geometry of the pore space and thus the two functions  $\mu(\phi, S)$  and  $\lambda(\phi, S)$ . If the bulk porosity  $\bar{\phi}$  is used to parametrize the consolidation processes, then it should also be used to parametrize  $\mu$  and  $\lambda$ . Using a mean-field approximation to simplify Eq. (3.8) yields the approximate result

$$k(\bar{\phi}, \bar{S}) \approx \frac{\mathcal{T}\bar{\phi}^3}{\bar{S}^2 \lambda_{\bar{\phi}}(\bar{\phi}, \bar{S})} [\lambda_{\bar{\phi}}(\bar{\phi}, \bar{S}) - p_c]^t \quad (4.1)$$

as long as  $\lambda_{\bar{\phi}}(\bar{\phi}, \bar{S}) > p_c$  and with

$$t_\alpha = \begin{cases} t & \text{for } \alpha < -2 \\ t + \frac{2+\alpha}{1-\alpha} & \text{for } -2 < \alpha < 1. \end{cases} \quad (4.2)$$

The parametrization of the consolidation process with  $\bar{\phi}$  has been indicated explicitly.

An expression similar to Eq. (4.1) can be obtained for the formation factor. The dielectric response of porous media was studied in Refs. 21 and 22, and a relation analogous to Eq. (3.8) holds in that case. Similar approximations as above give the result

$$F(\bar{\phi}) \simeq \bar{\phi}^{-1} \lambda_{\bar{\phi}}(\bar{\phi}) [\lambda_{\bar{\phi}}(\bar{\phi}) - p_c]^{-t_\alpha} \quad (4.3)$$

for  $\lambda_{\bar{\phi}}(\bar{\phi}) > p_c$  and where now

$$t_\alpha = \begin{cases} t & \text{for } \alpha < 0 \\ t - \frac{\alpha}{1-\alpha} & \text{for } 0 < \alpha < 1. \end{cases} \quad (4.4)$$

These equations are valid within a mean-field approximation to the effective-medium equation which becomes exact for strongly peaked porosity distributions.<sup>21</sup> The exponent  $\alpha$  is defined through  $\mu(\phi) \propto \phi^{-\alpha}$ , analogous to Eq. (3.9).

Equations (4.1) and (4.3) show that indeed the scaling laws (1.1) and (1.2) may arise from Eq. (3.8) if the bulk porosity is regarded as parametrizing a one-parameter family of consolidation processes. This point will now be discussed in more detail.

## V. SCALING LAWS

### A. General consideration

Before turning to a discussion of Kozeny's equation and Archie's law it is instructive to derive a more general scaling law relating  $k$ ,  $F$ ,  $\bar{\phi}$ , and  $\bar{S}$ . It is readily obtained by eliminating  $\lambda_{\bar{\phi}}$  between Eqs. (4.1) and (4.3) with the result

$$k \propto \frac{\mathcal{T} \bar{\phi}^2}{\bar{S}^2} F^{-1}. \quad (5.1)$$

This scaling relation establishes a correlation between permeability, conductivity, and the geometrical parameters  $\mathcal{T}$ ,  $\bar{\phi}$ , and  $\bar{S}$ . Although similar relations are well known for capillary models,<sup>1</sup> Eq. (5.1) is an entirely new result because  $\bar{S}$  is the average *local* specific surface area, and *not* the bulk specific surface area. This invalidates a well-known argument<sup>19</sup> against hydraulic radius theories where the quantity  $1/S$  is a macroscopic length. In hydraulic radius theories the total specific surface area  $S$  includes dead ends in which little or no flow occurs. Thus  $S$  should not correlate well with  $k$ . In the present approach dead ends are automatically included because of the underlying percolation transition. The local geometric quantity  $\mathcal{T}$  can be assumed to be independent of the bulk porosity as long as the consolidation processes do not create large amounts of dead ends *on the local scale*, i.e., inside the cells. The virtue of the scaling law (5.1) is that it does not depend explicitly on either  $\lambda$  or  $\mu$ . Section V C will verify Eq. (5.1) within a simple consolidation model.

Equation (5.1) bears an interesting relation with a conjecture by Johnson, Koplik, and Schwartz.<sup>19</sup> These authors propose the scaling relation

$$k \propto \frac{\Lambda^2}{F}, \quad (5.2)$$

where the length  $\Lambda$  is defined in terms of the microscopic electric field  $E_0$  of the dielectric problem as

$$\frac{2}{\Lambda} = \frac{\int_{\partial G_{PS}} |E_0|^2 d^2r}{\int_{G_{PS}} |E_0|^2 d^3r}. \quad (5.3)$$

The "electrical" length  $\Lambda$  is generally referred to as a measure for the so-called "dynamically connected pore size."<sup>18</sup> If the conjecture of Johnson, Koplik, and Schwartz holds true, then the present approach suggests that  $\Lambda$  scales as

$$\Lambda \propto \frac{\mathcal{T}^{1/2} \bar{\phi}}{\bar{S}}. \quad (5.4)$$

### B. $p_c$ -approaching consolidation processes

In this section it will be shown that certain classes of formally defined consolidation processes give rise to the empirical scaling laws (1.1) and (1.2). Assume that the consolidation processes are such that they approach  $p_c$  for  $\bar{\phi} \rightarrow 0$  and such that  $\lambda_{\bar{\phi}}(\bar{\phi}, \bar{S})$  can be expanded into a Taylor series for  $\bar{\phi} \rightarrow 0$  as  $\lambda_{\bar{\phi}}(\bar{\phi}, \bar{S}) = p_c + a_1(\bar{S})\bar{\phi} + \dots$ . Such a consolidation process will be called " $p_c$  approaching." Consolidation processes that are  $p_c$  approaching give rise to the scaling law (1.1) with exponent

$$b = \begin{cases} 3+t & \text{for } \alpha < -2 \\ 3+t + \frac{2+\alpha}{1-\alpha} & \text{for } -2 < \alpha < 1. \end{cases} \quad (5.5)$$

This implies  $b \approx 4$  if  $\alpha \approx -2$  or less,  $b \approx \frac{9}{2}$  if  $\alpha \approx -1$ , and  $b \approx 6$  if  $\alpha \approx 0$ , in remarkable agreement with the experimental range for  $b$ . It should be emphasized, however, that not only  $p_c$ -approaching consolidation processes lead to the scaling law (1.1). For example, processes for which  $\lambda \approx \text{const}$  exhibit the exponent  $b=3$ . A whole class of consolidation processes with continuously variable exponents will be obtained from a general local-porosity reduction model in Sec. V C.

For the dielectric response the class of  $p_c$ -approaching consolidation processes yields Archie's law, Eq. (1.2), with

$$m = \begin{cases} 1+t & \text{for } \alpha < 0 \\ 1+t + \frac{\alpha}{1-\alpha} & \text{for } 0 < \alpha < 1 \end{cases} \quad (5.6)$$

for the cementation index. Again, the numerical values for  $m$  found from these relations are in striking agreement with experiment. The universal result  $m = 1+t$  implies  $m = 2$  as a mean-field result. Note that the universal result holds over a much wider range of local-porosity distributions than the corresponding result  $b = 3+t$  for the flow problem. This reflects the intuitive expectation that permeabilities are more sensitive to low-porosity areas than conductivities.

For  $p_c$ -approaching consolidation processes, Eqs. (5.5) and (5.6) give rise to the exponent relation

$$b = \begin{cases} m+2 & \text{for } \alpha < -2 \\ m+2 + \frac{2+\alpha}{1-\alpha} & \text{for } -2 < \alpha < 0 \\ m+2 + \frac{2}{1-\alpha} & \text{for } 0 < \alpha < 1 \end{cases} \quad (5.7)$$

between  $b$  and  $m$ . This is a special case of the general result (5.1).

A more interesting scaling law appears if Eqs. (1.1) and (1.2) are combined into a relation between permeability and formation factor

$$k \propto F^{-h}, \quad (5.8)$$

where  $h = b/m$ . Indeed, experiments on fused glass beads<sup>15</sup> have confirmed Eq. (5.8). The experimental support for this power law extends over two to three decades in  $F$  and roughly four decades in  $k$ . This is a wider range than that for Kozeny's equation or Archie's law. In (1.1) and (1.2) the porosity variation is usually limited to less than one decade. The data of Ref. 15 are replotted in Fig. 1. The authors of Ref. 15 state that their data are best fitted with  $h = 2$  in support of their theoretical model. Least-squares fits of their data, however, yield the values  $h \approx 1.85$  for samples containing 177–210- $\mu\text{m}$  beads (symbol  $Y$ ),  $h \approx 1.93$  for 88–106- $\mu\text{m}$  beads (symbol  $X$ ), and  $h \approx 2.23$  for 44–53- $\mu\text{m}$ -sized beads (symbol  $O$ ). These fits are indicated as dashed lines in Fig. 1.

How do these findings compare to the present approach? From Eqs. (5.5) and (5.6) the exponent  $h$  is readily calculated for  $p_c$ -approaching consolidation processes as

$$h = \begin{cases} \frac{3+t}{1+t} & \text{for } \alpha < -2 \\ \frac{3+t}{1+t} + \frac{2+\alpha}{(1+t)(1-\alpha)} & \text{for } -2 < \alpha < 0 \\ \frac{5+t-2\alpha-t\alpha}{1+t-t\alpha} & \text{for } 0 < \alpha < 1. \end{cases} \quad (5.9)$$

Logarithmic corrections obtain at  $\alpha = -2$  and  $\alpha = 0$ . The exponent  $t$  crosses over from  $t = 1$  for  $p$  well outside the critical region to  $t \approx 2.0$  close to  $p_c$ . In Fig. 2 the exponent  $h$  is plotted as a function of  $\alpha$ . The upper solid curve corresponds to  $t = 1$ , the lower to  $t = 2$ . Experimental results for systems generated by  $p_c$ -approaching consolidation processes should fall between the two curves. The experimental results of Ref. 15 are consistent with this requirement. This finding could be interpreted as an indication that fused glass bead systems are created by a  $p_c$ -approaching consolidation process. For such systems Fig. 2 establishes a quantitative correlation between permeability and formation factor, on one hand, and statistical geometrical properties of the pore space expressed by  $\alpha$ , on the other hand. It can be read in two ways: If the exponent  $h$  has been determined experimentally, Fig. 2 can be used to predict a possible range for  $\alpha$ . Thus the fused glass bead sample with the smallest bead size is expected to have a local-porosity distribution of the form (3.9) with  $\alpha$  somewhere between 0 and  $-1.2$ . The second

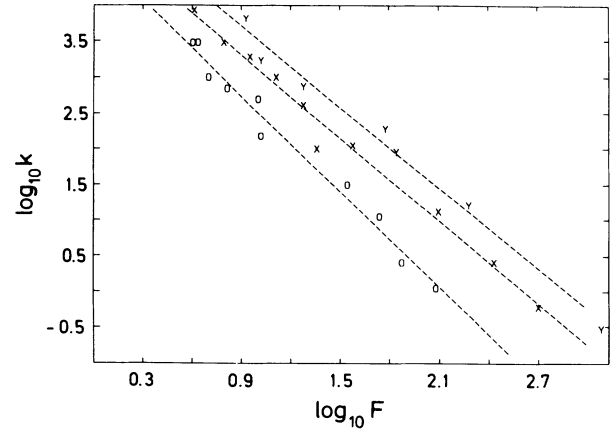


FIG. 1. Experimental results for permeability as a function of formation factor for fused glass beads with three different bead diameters, from Wong, *et al.* in Ref. 15. The dashed lines are least-squares fits giving the exponent  $h$ . The numerical values are for 177–210- $\mu\text{m}$  beads (symbol  $Y$ )  $h \approx 1.85$ , for 88–106- $\mu\text{m}$  beads (symbol  $X$ )  $h \approx 1.93$ , and for 44–54- $\mu\text{m}$  beads (symbol  $O$ )  $h \approx 2.23$ .

way to use Fig. 2 becomes possible once the geometrical exponent  $\alpha$  [or the function  $\alpha(\phi)$ ] has been determined for a particular porous medium. In that case Fig. 2 can be used to predict a possible range for  $h$ .

Whether or not fused glass bead systems are created by a  $p_c$ -approaching consolidation can only be decided by direct observation of  $\mu(\phi, S)$  and  $\lambda(\phi, S)$ . In fact, the experimental results for  $h$  are also consistent with a very different type of consolidation, which will be discussed next.

### C. Local-porosity reduction model

Consider a simple regular lattice (taken to be cubic) that is initially without disorder. Each unit cell of the lattice contains initially the same local geometry. The initial local geometry has porosity  $\phi_0$  and specific surface area  $S_0$ . The consolidation process consists of picking at random a particular cell and reducing its porosity by a

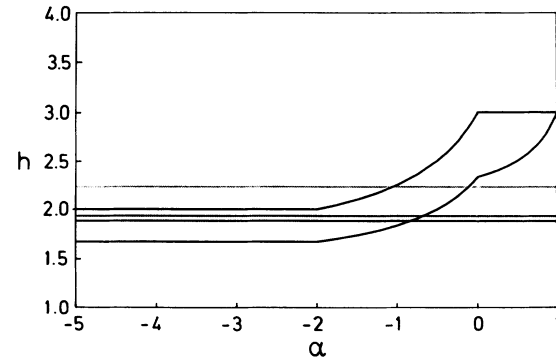


FIG. 2. Plot of the physical exponent  $h$  as a function of the geometrical exponent  $\alpha$  for  $t = 1$  (upper curve) and  $t = 2$  (lower curve). The universal regime extends below  $\alpha < -2$ . The gray lines indicate the least-squares fits to the experimental data shown in Fig. 1.

factor  $r$  and its specific surface by a factor  $s$ . These steps are repeated until the desired average porosity  $\bar{\phi}$  is achieved. The local-percolation probabilities are assumed to be given by a function  $\lambda(\phi, S)$ , which remains unchanged during the porosity reduction. This process creates a random pore space in which each cell is characterized by a random variable  $n$  giving the number of times it was picked for a reduction step. If a particular cell has been chosen  $n$  times, then its porosity and specific surface area are, respectively,

$$\phi_n = r^n \phi_0, \quad (5.10a)$$

$$S_n = s^n S_0. \quad (5.10b)$$

The definition of the local-porosity reduction model given above is very general, and it encompasses many different consolidation processes.

In general, the reduction factors  $r$  and  $s$  might be chosen independent of each other. In many cases, however, they will be related. Four generic possibilities for their relation may illustrate this point.

1. *Crack compaction.* Suppose the local geometry consisted initially of crack planes separated a distance  $w$  and the consolidation reduces this width by a factor  $x$ . Then the surface area of the crack planes and thus the specific internal surface area remain unchanged. The porosity, on the other hand, is reduced proportional to  $x$ , and thus  $r = x$  and  $s = 1 = x^0$ . Therefore,

$$s = r^0 \quad (5.11)$$

for this case.

2. *Shrinkage of capillaries.* In this case the local geometry is assumed to consist of cylindrical tubes of diameter  $w$ , and each consolidation step reduces the tube diameter by a factor  $x$ . This implies  $r = x^2$  and  $s = x$ , and thus

$$s = r^{1/2}. \quad (5.12)$$

Note that the bond shrinkage model of Ref. 15 is a special case obeying (5.12).

3. *Shrinkage of voids.* Now the local pore space is formed by a central void (such as a sphere) that is con-

nected to neighboring cells through extremely narrow capillaries. The capillaries are so small that their contribution to  $\phi$  and  $S$  can be neglected. Each consolidation step reduces the sphere radius by a factor  $x$ . In this case  $r = x^3$  and  $s = x^2$ , giving

$$s = r^{2/3}. \quad (5.13)$$

4. *Filling of voids.* Whenever the local pore space is filled up or otherwise eliminated the relation

$$s = r \quad (5.14)$$

holds.

These considerations suggest that for a large number of consolidation processes  $s$  and  $r$  will not be independent. If the consolidation process is combined out of the four alternatives above, one might expect a power-law relationship

$$s = r^a \quad (5.15)$$

between  $s$  and  $r$  with an exponent  $a$  ranging from 0 to 1.

Because the pore space created in the local-porosity reduction models is substitutionally disordered, its local-porosity distribution can be obtained explicitly. Noting that the random variable  $n$  is Poisson distributed, one finds for the local-porosity distribution

$$\mu(\phi) = \frac{\left(\frac{\phi}{\phi_0}\right)^{\ln \bar{n} / \ln r} \exp(-\bar{n})}{\Gamma(\ln(r\phi/\phi_0)/\ln r)}, \quad (5.16)$$

where  $\Gamma(x)$  denotes the gamma function and

$$\bar{n} = \frac{\ln \bar{\phi} - \ln \phi_0}{r - 1}. \quad (5.17)$$

The bulk porosity  $\bar{\phi}$  was defined in Eq. (2.3). The quantity  $\bar{n}$  is the average number of times each cell is selected. The local-porosity density (5.16) no longer depends upon  $S$  because  $S$  is determined through  $\phi$  by Eq. (5.15).

Writing the effective-medium equation (3.5) in terms of  $n$  and using again a mean-field treatment gives the mean-field result

$$k = \frac{3}{2} \frac{T\phi_0^3}{S_0^2} \left[ \frac{\bar{\phi}}{\phi_0} \right]^{[(3-2a)\ln r]/(r-1)} \left\{ \lambda \left[ \phi_0 \left[ \frac{\bar{\phi}}{\phi_0} \right]^{\ln r/(r-1)} \right], S_0 \left[ \frac{\bar{\phi}}{\phi_0} \right]^{a \ln r/(r-1)} \right\}^{-\frac{1}{3}}. \quad (5.18)$$

The approximations become exact in the  $\bar{\phi} \rightarrow 0$  limit.

A particularly interesting special case of this result is obtained for uniformly conducting models discussed in Ref. 21. A uniformly conducting model is defined through  $\lambda = \text{const}$ , i.e., it has a constant fraction of percolating geometries. This includes the case  $\lambda \equiv 1$  where all local geometries are percolating. In this case one finds Kozeny's equation for  $k$  with exponent

$$b = (3-2a) \frac{\ln r}{r-1} \quad (5.19)$$

for the flow problem. For the electrical problem Archie's

law is obtained with cementation index

$$m = \frac{\ln r}{r-1}. \quad (5.20)$$

Note that  $m \rightarrow 1$  for  $r \rightarrow 1$ . Combining Eqs. (5.19) and (5.20) yields the exponent  $h$  as

$$h = 3 - 2a \quad (5.21)$$

for the uniformly conducting local-porosity reduction model. While the result for  $b$  involves both, the reduction factor  $r$  and the exponent  $a$ , their influence appears separated in the results for  $m$  and  $h$ . Incidentally, Eq.

(5.21) combined with the range  $0 \leq a \leq 1$  predicts a similar range  $1 \leq h \leq 3$  for  $h$  as Eq. (5.9) for  $p_c$ -approaching consolidation processes. Thus the experimental results for fused glass beads<sup>15</sup> discussed in the last section can also be interpreted within the local-porosity reduction model. One finds  $a(44\text{--}53 \mu\text{m}) \approx 0.38$ ,  $a(88\text{--}106 \mu\text{m}) \approx 0.53$ , and finally  $a(177\text{--}210 \mu\text{m}) \approx 0.57$ . This could be interpreted as an indication that the sintering process for large spheres deviates from simple capillary shrinkage towards void shrinkage, while that for small beads seems to cross over towards wall shrinkage.

Note also that the bond shrinkage model of Ref. 15 emerges as a special case of these results. In the tube shrinkage model one has  $r = x^2$  and  $a = \frac{1}{2}$ , as mentioned above. Thus  $h = 2$ ,  $b = 2 \ln x^2 / (x^2 - 1)$  and  $m = \ln x^2 / (x^2 - 1)$ , recovering exactly the analytical estimates for higher dimensions<sup>15</sup> as a mean-field result.

## VI. CONCLUSION

The local-porosity approach provides a theoretical framework inside which the nature of empirical scaling relations in porous media can be understood. The approach explains why scaling laws might occur at all, and

why there exists such a great variability in the observed exponents. The importance of consolidation processes is emphasized. Kozeny's equation (1.1) and Archie's law (1.2) should be viewed as a statement about the consolidation processes of the geological formation and not as a relation between permeability and bulk porosity.<sup>21</sup> The tube shrinkage model of Wong, Koplik, and Tomanic<sup>15</sup> is found to be a special case of a more general local-porosity reduction model. The theory offers a geometrical interpretation of the dynamical  $\Lambda$  parameter introduced by Johnson, Koplik, and Schwartz.<sup>19</sup> In summary, it is hoped that the present approach improves the understanding of quantitative correlations between geometrical properties of porous media and their bulk transport coefficients. Because the theory is based on direct observation, it contains no adjustable parameters or distribution functions, and can thus be tested by experiment.

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