mathematics

Communication

Maximal Domains for Fractional Derivatives and Integrals

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Received: 11 March 2020; Accepted: 6 June 2020; Published: 6 July 2020

Abstract: The purpose of this short communication is to announce the existence of fractional calculi on precisely specified domains of distributions. The calculi satisfy *desiderata* proposed above in *Mathematics* **7**, 149 (2019). For the *desiderata* (a)–(c) the examples are optimal in the sense of having maximal domains with respect to convolvability of distributions. The examples suggest to modify *desideratum* (f) in the original list.

Keywords: fractional derivatives; fractional integrals; fractional calculus

MSC: 26A33; 34A08; 34K37; 35R11; 44A40

A list of six *desiderata* was recently proposed in [1] for calling families of operators $\{D^\alpha, I^\alpha\}$ with family index $\alpha \in \mathbb{I}$ from some index set $\mathbb{I} \subseteq \mathbb{C}$ fractional derivatives (D^α) and fractional integrals (I^α) of order $\alpha \notin \mathbb{N}$. Distributional domains for $\{D^\alpha, I^\alpha\}$ seem to require a minor modification of these *desiderata*.

Multiplication of distributions is ill-defined so that for distributions *desideratum* (f) (Leibniz rule) requires generalization. A slightly modified list of *desiderata* might read as follows:

(a) Integrals I^α and derivatives D^α of fractional order α should be linear operators on linear spaces.

(b) On some subset $\mathsf{G}_{(b)} \subseteq \mathsf{D}(I^\alpha) \cap I^\beta[\mathsf{D}(I^\beta)] \cap \mathsf{D}(I^{\alpha+\beta})$, $\mathsf{G}_{(b)} \neq \emptyset$, $\mathsf{G}_{(b)} \neq \{0\}$ the index law (semigroup property)

$$(I^\alpha \circ I^\beta) f = I^{\alpha+\beta} f \tag{1}$$

holds true for $\operatorname{Re}\alpha \geq 0$ and $\operatorname{Re}\beta \geq 0$, where $\mathsf{D}(I^\alpha)$ denotes the domain of I^α.

(c) Restricted to a suitable subset $\mathsf{G}_{(c)} \subseteq \mathsf{D}(I^\alpha)$ of the domain of I^α the fractional derivatives D^α of order α operate as left inverses

$$D^\alpha \circ I^\alpha = 1_{\mathsf{G}_{(c)}} \tag{2}$$

for all α with $\operatorname{Re}\alpha \geq 0$, where $\circ$ denotes composition of operators, and $1_{\mathsf{G}_{(c)}}$ is the identity on $\mathsf{G}_{(c)}$.

(d) Each of the two limits

$$g^1 = D^1 f = \lim_{\alpha\to 1} D^\alpha f, \qquad f \in \mathsf{G}_{(d)}, \tag{3a}$$

$$g^0 = D^0 f = \lim_{\alpha\to 0} D^\alpha f, \qquad f \in \mathsf{G}_{(d)}, \tag{3b}$$

should exist in some sense on some set $\mathsf{G}_{(d)} \subseteq \mathsf{D}(D^\alpha)$, $\mathsf{G}_{(d)} \neq \emptyset$, $\mathsf{G}_{(d)} \neq \{0\}$. Moreover, the limiting maps $D^1 : \mathsf{G}_{(d)} \to \mathsf{G}_{(d)}$ and $D^0 : \mathsf{G}_{(d)} \to \mathsf{G}_{(d)}$ should be linear.

(e) $D^0 = 1_{\mathsf{G}_{(\mathrm{d})}}$ is the identity on $\mathsf{G}_{(\mathrm{d})}$, i.e., $g^0 = f$ in Equation (3b).

(f) Endowed with a suitable multiplication $\odot : \mathsf{G}_{(\mathrm{f})} \times \mathsf{G}_{(\mathrm{d})} \to \mathsf{G}_{(\mathrm{d})}$ the limiting map $D^1 = D$ obeys the Leibniz rule

$$D(f \odot g) = f \odot (D\,g) + (D\,f) \odot g \tag{4}$$

for all $f \in \mathsf{G}_{(\mathrm{f})}, g \in \mathsf{G}_{(\mathrm{d})}$ with $\mathsf{G}_{(\mathrm{f})} \neq \emptyset$, $\mathsf{G}_{(\mathrm{f})} \neq \{0\}$. If $\mathsf{G}_{(\mathrm{d})}$ consist of numerical functions, then $\odot$ is pointwise multiplication and $\mathsf{G}_{(\mathrm{f})} = \mathsf{G}_{(\mathrm{d})}$.

Given these modified *desiderata*, the objective in this short note is to introduce fractional calculi for distributions. Let us stress that the distributional domains $\mathsf{D}(I^\alpha)$, $\mathsf{D}(D^\alpha)$ given in Theorem 1 below are maximal in a precise mathematical sense. One cannot enlarge them without violating either the *desiderata* or the interpretation of fractional derivatives and integrals as convolution operators. Recall that numerous other mathematical interpretations exist [2], that may have different maximal domains. In this paper fractional operators are interpreted as convolutions with power law kernels (cf. [2], Equation (28)). A comprehensive analysis of convolutions with power law kernels on weighted spaces of continuous functions was recently given in [3].

Define the spaces of continuously differentiable functions, test functions, and smooth functions with bounded derivates

$$\mathcal{C}^m(\mathbb{R}^d) := \Big\{ f : \mathbb{R}^d \to \mathbb{C} | f \text{ is } m\text{-times continuously differentiable} \Big\} \tag{5a}$$

$$\mathcal{D}(\mathbb{R}^d) := \Big\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) | f \text{ has compact support} \Big\} \tag{5b}$$

$$\mathcal{B}(\mathbb{R}^d) := \Big\{ f \in \mathcal{C}^\infty(\mathbb{R}^d) | f \text{ has bounded derivatives} \Big\} \tag{5c}$$

in the usual way [4]. The spaces $\mathcal{C}^m, \mathcal{D}$ are endowed with the norm $\|f\|_\infty = \sup|f|$. The topology on $\mathcal{B}$ is induced by the seminorms $\|f\|_{N,g} = \sup\{\|g\partial^{n_1}...\partial^{n_d} f\|_\infty : n_i \in \mathbb{N}, \sum_i^d n_i \leq N\}$ with $N \in \mathbb{N}$ and $g \in \mathcal{C}_\mathrm{v}$, where $\mathcal{C}_\mathrm{v}$ is the space of continuous functions vanishing at infinity.

The space of distributions $\mathcal{D}'$ is the topological dual of $\mathcal{D}$. The dual space $\mathcal{B}'$ is the space of integrable distributions. The pairing $\mathcal{D} \times \mathcal{D}' \to \mathbb{C}$ is denoted $\langle \cdot, \cdot \rangle$, the pairing $\mathcal{B} \times \mathcal{B}' \to \mathbb{C}$ as $\langle \cdot, \cdot \rangle_\mathcal{B}$.

Definition 1. *Two distributions $f_1, f_2 \in \mathcal{D}'(\mathbb{R}^d)$ are called convolvable iff $\varphi(f_1 \otimes f_2) \in \mathcal{B}'(\mathbb{R}^{2d})$ for all $\phi \in \mathcal{D}(\mathbb{R}^d)$, where $\varphi(x_1, x_2) = \phi(x_1 + x_2)$. Their convolution $f_1 * f_2$ is defined by requiring that*

$$\langle \phi, f_1 * f_2 \rangle = \langle 1, \varphi(f_1 \otimes f_2) \rangle_\mathcal{B} \tag{6}$$

holds for all $\phi \in \mathcal{D}(\mathbb{R}^d)$.

Let $\mathcal{D}'_+$ denote the space of causal distributions defined as elements $f \in \mathcal{D}'(\mathbb{R})$ whose support is bounded on the left.

Definition 2. *Fractional integrals I^α_+ and derivatives D^α_+ are defined for all $\alpha \in \mathbb{C}$ and all distributions $f \in \mathcal{D}'_+$ as convolution operators*

$$I^\alpha_+ f := K_\alpha * f \tag{7a}$$

$$D^\alpha_+ f := K_{-\alpha} * f \tag{7b}$$

with kernels

$$K_\alpha(x) = \begin{cases} \dfrac{x^{\alpha-1}}{\Gamma(\alpha)} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases} \qquad \text{for } \operatorname{Re}\alpha > 0 \tag{8a}$$

$$K_\alpha(x) = \frac{\mathrm{d}^m}{\mathrm{d}x^m} K_{\alpha+m}(x) \qquad \text{for } -m < \operatorname{Re}\alpha \le 0, m \in \mathbb{N}. \tag{8b}$$

The operators I^α_+ and D^α_+ are linear and continuous on $\mathcal{D}'_+$. The kernels $\{K_\alpha : \alpha \in \mathbb{C}\}$ form a convolution group

$$K_\alpha * K_\beta = K_{\alpha+\beta} \tag{9}$$

for all $\alpha, \beta \in \mathbb{C}$. This entails the index law $I^\alpha_+(I^\beta_+ f) = I^{\alpha+\beta}_+ f$ for all $f \in \mathcal{D}'_+$ and $\alpha, \beta \in \mathbb{C}$. Clearly, all *desiderata* are fulfilled for $\{I^\alpha_+, D^\alpha_+\}$ with $\mathsf{D}(I^\alpha_+) = \mathsf{D}(D^\alpha_+) = \mathsf{G}_{(b)} = \mathsf{G}_{(c)} = \mathsf{G}_{(d)} = \mathcal{D}'_+$ and $\mathsf{G}_{(f)} = \mathcal{C}^\infty$.

The domain $\mathcal{D}'_+$ of causal distributions will now be enlarged using certain sets of lower semicontinuous functions as convolution weights. A function $f : \mathbb{R} \to \overline{\mathbb{R}}_+$, where $\overline{\mathbb{R}}_+ := [0, \infty]$, is called lower semicontinuous, if the set $\{f \le a\}$ is closed for every $a \in \overline{\mathbb{R}}_+$. The set of all lower semicontinuous functions is denoted $\mathcal{I}$, the set of lower semicontinuous functions whose support is bounded on the left is denoted $\mathcal{I}_+$. For $(p, k) \in \mathbb{R} \times \mathbb{N}$ let

$$P^{p;k} := \left\{ f \in \mathcal{I} \mid \exists C > 0\ \forall x \in \mathbb{R} : f(x) \le C(1 + |x|)^p [\log(e + |x|)]^k \right\} \tag{10}$$

be the set of lower semicontinuous functions of power-logarithmic growth of order (p, k). Then

$$P_+ := \mathcal{I}_+ \cap \left(\bigcup_{q \in \mathbb{R}} P^{q;0} \right) \tag{11a}$$

$$R_+ := \mathcal{I}_+ \cap \left(\bigcup_{k \in \mathbb{N}_0} P^{-1;k} \right) \tag{11b}$$

are the sets of interest.

Definition 3. *Let $U \subseteq \mathcal{D}'$ and let $\mathscr{B}(\mathcal{D})$ denote the set of all bounded subsets of $\mathcal{D}$. Then*

$$(U)^*_{\mathcal{D}'} := \left\{ f \in \mathcal{D}' : (f, g) \text{ are convolvable for all } g \in U \right\} \tag{12}$$

denotes the set of all distributions convolvable with the given set U. A locally convex topology $\mathscr{T}_U$ on $U \subseteq \mathcal{D}'$ is defined by the family of seminorms

$$\|f\|_{V,g} = (|f|_V * |g|_V)(0) = \int |f|_V(x) |g|_V(-x) \mathrm{d}x \tag{13}$$

with $V \subset \mathcal{D}, V \in \mathscr{B}(\mathcal{D})$ and $g \in (U)^*_{\mathcal{D}'}$. Here, the V-modulus of an element $f \in \mathcal{D}'$ is defined as

$$|f|_V(x) := \sup_{g \in V} |\langle f(\,\cdot\,), g(\,\cdot\, - x) \rangle| \tag{14}$$

for all $x \in \mathbb{R}$.

Theorem 1. *The convolution group* $\{K_\alpha : \alpha \in \mathbb{C}\}$, *resp.* $\{K_\alpha : \alpha \in i\mathbb{R}\}$, *can be extended from* $(\mathcal{D}'_+, \mathscr{T}_{\mathcal{D}'_+})$ *to operate linearly, bijectively, and continuously on the space* $(U, \mathscr{T}_U)$ *with* $U = (P_+)^*_{\mathcal{D}'}$, *resp.* $U = (R_+)^*_{\mathcal{D}'}$, *in such a way that compact sets of indices* α *map to equicontinuous sets of operators.*

Corollary 1. *The desiderata (a)–(e) are fulfilled for* $\{I^\alpha_+, D^\alpha_+\}_{\alpha\in\mathbb{C}}$ *with*

$$\mathsf{D}(I^\alpha_+) = \mathsf{D}(D^\alpha_+) = \mathsf{G}_{(b)} = \mathsf{G}_{(c)} = \mathsf{G}_{(d)} = (P_+)^*_{\mathcal{D}'} \tag{15a}$$

and for $\{I^\alpha_+, D^\alpha_+\}_{\alpha\in i\mathbb{R}}$ *with* $D^1, \mathsf{G}_{(d)}$ *as in* (15a) *and*

$$\mathsf{D}(I^\alpha_+) = \mathsf{D}(D^\alpha_+) = \mathsf{G}_{(b)} = \mathsf{G}_{(c)} = (R_+)^*_{\mathcal{D}'}. \tag{15b}$$

In both cases it is possible to choose $\mathsf{G}_{(f)} = \mathcal{B}$.

The proof of Theorem 1 and its corollary will be published elsewhere, because it is lengthy and giving it here would distract attention from the main message. The domains $\mathsf{D}(I^\alpha_+)$, $\mathsf{D}(D^\alpha_+)$, $\mathsf{G}_{(b)}$, $\mathsf{G}_{(c)}$ are maximal with respect to convolvability in both cases. The second case $\{I^\alpha_+, D^\alpha_+\}_{\alpha\in i\mathbb{R}}$ yields a (purely imaginary) "fractional calculus of order zero" in the sense that $\operatorname{Re}\alpha = 0$ for all operators in that subset.

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