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ON A NEW CLASS OF PHASE TRANSITIONS

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To Ray Orbach in appreciation of his work and what I learned from it.

ABSTRACT

A recently introduced classification theory for phase transitions characterizes each phase transition by its generalized noninteger order and a slowly varying function. Thermodynamically this characterization arises from generalizing the classification scheme of Ehrenfest. The same characterization emerges in statistical mechanics from generalizing the finite size scaling limit. The classification theory predicts an unusual class of phase transitions characterized by fractional orders less than unity. Examples are found in unstable models of statistical mechanics. Finally it is shown how the statistical classification theory gives rise to a classification of macroscopic dynamical behaviour based on a generalization of the stationarity concept.

.. Introduction

A new class of phase transitions called anequilibrium transitions has recently been introduced [18, 19, 20] on the basis of a general classification theory of phase transitions [16, 17]. This work discusses a related classification theory for the infinitesimal generators of the time evolution of macroscopic ensemble averaged observables.

Modern discussions of critical phenomena focus on the renormalization group picture. Scaling and universality are derived from renormalization group flows in infinite dimensional spaces [25, 13].

Derivations of scaling within the renormalization group picture rely on many implicit assumptions. Violations of hyperscaling or finite size scaling represent a breakdown of one of these assumptions [13]. It is therefore of interest to find alternative derivations of scaling.

Generalizing the classification scheme of Ehrenfest [9] provides an alternative derivation of scaling within classical thermodynamics [16, 17]. The generalized classification theory leads naturally to predict the existence of a new class of phase transitions [18, 19, 20].

Let me conclude the introduction by remarking that the new class of anequilibrium transitions exhibits unusual static *and* dynamic properties similar to those found for spinglass transitions in random magnets. However, much more work than what can be reported here is required to elucidate this point, and thus I will refrain from discussing it.

2. Thermodynamic Classification Scheme

The thermodynamic classification of phase transitions is dicussed in terms of the pressure $p(T,\mu)$ as a function of temperature T and chemical potential μ. $p(T,\mu)$ is the conjugate convex function to the energy density $u(s,\rho) = U(S/V, 1, N/V)/V$ as a function of entropy density $s = S/V$ and particle number density $\rho = N/V$. Here V denotes the volume, N the particle number, S the entropy and U the internal energy. The pressure is given as

$$p(T,\mu) = \sup_{s,\rho}(Ts + \mu\rho - u(s,\rho)). \tag{1}$$

Consider a thermodynamical process $\mathcal{C} : \mathbf{R} \to \mathbf{R}^2, \sigma \mapsto (T(\sigma), \mu(\sigma))$ parametrized by σ such that $\sigma = 0$ corresponds to a critical point (T_c, μ_c). The thermodynamic classification scheme [16, 17] is based on the fractional derivatives [15, 26]

$$\mathcal{F}(\mathcal{C}, q; \sigma) = \frac{d^q p_{sng}(T(\sigma), \mu(\sigma))}{d\sigma^q} =$$
$$\lim_{N\to\infty} \Gamma(-q)^{-1} \left(\frac{|\sigma|}{N}\right)^{-q} \sum_{j=0}^{N-1} \frac{\Gamma(j-q)}{\Gamma(j+1)} p_{sng}(T(\sigma - \frac{j\sigma}{N}), \mu(\sigma - \frac{j\sigma}{N})) \tag{2}$$

where p_{sng} denotes the singular part of the pressure $p = p_{reg} + p_{sng}$. Ehrenfest [9] defines the phase transition at $\sigma = 0$ to be of order n if and only if

$$\lim_{\sigma \to \pm 0} \mathcal{F}(\mathcal{C}, n; \sigma) = A^{\pm}(\mathcal{C}) \tag{3}$$

where $n \in \mathbf{N}$ is an integer. Equation (3) expresses a finite jump discontinuity in the n-th derivative of the pressure at $\sigma = 0$.

Recently [16, 17, 19] the classification of Ehrenfest has been generalized to non-integer orders $q \in \mathbf{R}$. In the refined classification scheme [19] the transition at $\sigma = 0$ is defined to be of order $\lambda^{\pm}$ if and only if

$$\lim_{\sigma \to \pm 0} \frac{\mathcal{F}(\mathcal{C}, \lambda^{\pm}; b\sigma)}{\mathcal{F}(\mathcal{C}, \lambda^{\pm}; \sigma)} = 1 \tag{4}$$

for all $b > 0$. Equation (4) is obeyed for Ehrenfests integer order transitions, i.e. for $\mathcal{F}$ obeying (3). For the specific path $\mu = \mu_c$ the generalized order is related to the specific heat critical exponent α as $\lambda = 2 - \alpha$. Similarly $\lambda = 1 + (1/\delta)$ for an approach along $T = T_c$ where δ denotes the equation of state exponent. More generally

$$\lambda_X = 2 - \alpha_X \tag{5}$$

where X is a local operator and α_X the associated fluctuation exponent [12]. Equation (4) requires the fractional derivative of order λ to be a slowly varying function [23]. A real valued positive and measurable function $\Lambda(x)$ is called slowly varying at infinity if $\lim_{x\to\infty}(\Lambda(bx)/\Lambda(x)) = 1$ for all $b > 0$. In the refined classification scheme each transition is characterized by generalized left and right orders $\lambda^{\pm}$ and corresponding slowly varying functions $\Lambda^{\pm}$.

First order and second order phase transitions play a special role. Transitions of order λ_u in $u(s)$ are Legendre-conjugate to transitions of order λ_p where

$$\lambda_p = \frac{\lambda_u}{\lambda_u - 1} \tag{6}$$

Second order phase transitions are selfconjugate in the sense that $\lambda_u = \lambda_p = 2$. First order transitions on the other hand correspond to a special limiting case of infinite conjugate order.

3. Anequilibrium Phase Transitions

The classification scheme raises the question whether transitions of order $\lambda < 1$ are thermodynamically allowed. Consider $u(s,\rho)$ at consant density ρ. If there is a critical point of order $\lambda = \lambda^+ = \lambda^-$ at s_c then $u(s) = u(s, \rho = const)$ has the form

$$u(s) = u_{reg}(s) + u^{\pm}(s)|s - s_c|^{\lambda} \tag{7}$$

where $u_{reg}(s)$ denotes the regular part and $u^{\pm}(s)$ is a slowly varying function for $s \to s_c^{\pm}$. This shows that for $s_c < \infty$ the condition $\lambda < 1$ would imply a violation of the basic convexity requirement. Thermodynamic stability seems to restrict the order of phase transitions to the range $\lambda \geq 1$. However the laws of thermodynamics do not require $s_c < \infty$ and this gives rise to the possibility of $\lambda < 1$ transitions.

To prove the possibility of fractional phase transitions with $\lambda < 1$ in thermodynamics it suffices to give an explicit example for $u(s)$ which obeys the mathematical requirements for a fundamental equation [6, 27]. Such an example is given by

$$u(s) = as + b(s^2 + c^2)^{1/2} \tag{8}$$

with $a, b, c > 0$ and $a > b$. Equation (8) defines a single-valued, continuous and differentiable energy function which is monotonically increasing, $T(s) = \partial u/\partial s > 0$, and convex, $\partial^2 u/\partial u^2 > 0$. The energy function defined in eq. (8) fulfills all requirements of classical thermodynamics and exhibits transitions of order $\lambda_u^{\pm} = 1$ at $s_c = \pm\infty$. Classically, entropy S and energy U are allowed to vary over the full range $-\infty < U, S < \infty$ of real numbers. The Legendre transform of $u(s)$ reads

$$p(T) = (c^2(b^2 - (T - a)^2))^{1/2} \tag{9}$$

and is only defined in the restricted temperature range

$$a - b = T_{min} < T < T_{max} = a + b. \tag{10}$$

Thus the pressure $p(T)$ exhibits transitions of order $\lambda_p^+ = 1/2$ at T_{min} and $\lambda_p^- = 1/2$ at T_{max}. In general an infinite entropy transition of order $0 < \lambda_u < \infty$ in $u(s)$ is related to a transition of order

$$\lambda_p = \frac{\lambda_u}{\lambda_u + 1} < 1 \tag{11}$$

less than unity in $p(T)$.

The simple example (8) demonstrates the coexistence of equilibrium transitions of integer or continuous orders $\lambda \geq 1$ and a new class of *anequilibrium transitions* of fractional orders $\lambda \leq 1$ within classical thermodynamics. Fractional anequilibrium transitions are characterized by diverging entropies and the fact that the set of equilibrium temperatures is restricted to a subset of the positive real axis. The limiting temperatures can only be approached infinitesimally but cannot be reached in a quasistatic thermodynamic process. However if in addition to classical thermodynamics the third law is assumed to hold then there exists a temperature $T_0 > T_{min}$ at which the entropy density vanishes, $s(T_0) = 0$, and T_{min} can no longer be approached arbitrarily closely.

Next it will be shown that the division into continuous equilibrium and fractional anequilibrium transitions can be found also in classical statistical mechanics.

4. The Ensemble Limit

Classical statistical mechanics is based on the law of large numbers and the central limit theorem from probability theory [22]. The statistical fluctuations near critical points and the formal similarities of the renormalization approach with semigroups studied in probability theory has long suggested a connection between the theory of stable laws and critical phenomena [8, 24]. The same idea will be followed here. The difference to previous approaches lies in the way in which the thermodynamic limit, the scaling (or continuum) limit and the approach to criticality are combined.

Consider a statistical mechanical system on a d-dimensional simple cubic lattice with lattice spacing $a > 0$. The system is finite with sidelength $L < \infty$ in all d directions. A fluctuating scalar observable X is associated with each lattice point. The fluctuations in X can be characterized by a correlation length $\xi_X(\Pi)$ depending on the parameters $\Pi = (\Pi_1, \Pi_2, ...)$ of the system. Let a, L and Π be such that the system decomposes into a large number of uncorrelated blocks of linear extension ξ_X, i.e.

$$0 < a \ll \xi_X(\Pi) \ll L < \infty. \tag{12}$$

The *ensemble limit* is defined as the simultaneous limit

$$a \to 0, L \to \infty, \Pi \to \Pi_c \qquad \text{such that} \qquad \xi_X(\Pi) \to \xi_X(\Pi_c) < \infty. \tag{13}$$

The ensemble limit is called *critical* if $0 < \xi_X(\Pi_c) < \infty$ and it is called *noncritical* if $\xi_X(\Pi_c) = 0$. The reason for this terminology is that the correlation length diverges in units of a for the critical ensemble limit, while in the noncritical limit it does not. The critical ensemble limit generates an infinite ensemble of uncorrelated blocks and this feature allows the application of standard limit theorems from probability theory.

The number of uncorrelated blocks of linear extension ξ_X is denoted by

$$N = \left(\frac{L}{\xi_X}\right)^d \tag{14}$$

while

$$M = \left(\frac{\xi_X}{a}\right)^d \tag{15}$$

is the number of lattice sites within each block. The total number of lattice sites is then $NM = (L/a)^d$. The fluctuating scalar observable X at site j ($j = 1, ..., M$) inside block i ($i = 1, ..., N$) is denoted by $X_{iN}(j)$. From these the *block variables* are defined as

$$X_{iN} = \sum_{j=1}^{M} X_{iN}(j) \tag{16}$$

and the *ensemble variables* as

$$X_N = \sum_{j=1}^{N} X_{iN}. \tag{17}$$

As usual the system will be assumed to be translationally invariant rendering the individual block variables uncorrelated and identically distributed. The probability distribution of the macroscopic observables X_N is denoted $P_N(x) = \text{Prob}\{X_N \leq x\}$.

The basic limit theorem for sums of uncorrelated (or weakly correlated) random variables [10, 14, 21] states that the weak limit

$$P(x) = \lim_{N\to\infty} P_N(xD_N + C_N) \tag{18}$$

exists for a suitable choice of centering constants C_N and norming constants D_N. The limiting distribution function $P(x)$ has a characteristic function $p(k) = \int_{-\infty}^{\infty} \exp^{ikx} dP(x)$ given by

$$\log p(k) = iCk - D|k|^{\varpi}(1 - i\zeta\frac{k}{|k|}\omega(k,\varpi)) \tag{19}$$

whose parameters obey

$$\begin{array}{l} 0 < \varpi \leq 2 \\ -1 \leq \zeta \leq 1 \\ -\infty < C < \infty \\ 0 \leq D \end{array} \tag{20}$$

and where

$$\omega(k,\varpi) = \begin{cases} \tan\left(\frac{\varpi\pi}{2}\right) & : \text{ for } \varpi \neq 1 \\ \frac{2}{\pi}\log|k| & : \text{ for } \varpi = 1 \end{cases} \tag{21}$$

For $D > 0$ the norming constants D_N have the form

$$D_N = N^{1/\varpi}\Lambda(N) \tag{22}$$

with $\Lambda(N)$ slowly varying at infinity.

From translation invariance and the definition of the block ensemble limit in (13) it follows that the block variables X_{iN} approach a common distribution belonging to the domain of attraction of a certain stable distribution. If that limiting stable law has a characteristic function whose logarithm is $-D|k|^{\varpi}(1 - i\zeta\frac{k}{|k|}\omega(k,\varpi))$ with $\varpi, \zeta, D, \omega(k,\varpi)$ as above then the distribution function for the block variables converges towards a distribution function $Q(x)$ whose characteristic function reads

$$\log q(k) = i\tilde{C}k - D|k|^{\varpi}\tilde{\Lambda}(k)(1 - i\zeta\frac{k}{|k|}\omega(k,\varpi)) \tag{23}$$

in the limit $k \to 0$[21]. Here $\tilde{C}$ is a constant and $\tilde{\Lambda}(k)$ a slowly varying function at zero. It has been shown [19] that the slowly varying function $\Lambda(N)$ in (22) and $\tilde{\Lambda}(k)$ in (23) are related to each other through

$$\Lambda(N) = \left(D\mathcal{L}^*\left(\frac{1}{N}\right)\right)^{-\frac{1}{\varpi}} \tag{24}$$

where $\mathcal{L}^*(x)$ is the conjugate slowly varying function to the slowly varying function $\mathcal{L}(x)$ defined by the relation

$$\tilde{\Lambda}(k) = \mathcal{L}(k^{\varpi}). \tag{25}$$

Using this result together with eq.(18) implies that the distribution function $P_N(x)$ for the macroscopic variables X_N has the scaling form

$$P_N(x) = P\left(\frac{x\mathcal{L}^{*1/\varpi}(N^{-1})}{N^{1/\varpi}};\varpi,\zeta,0,1\right) \tag{26}$$

in the limit of large N where $P(x;\varpi,\zeta,0,1)$ denotes the standard $(C = 0, D = 1)$ stable law with index ϖ and parameter ζ. Equation (26) will be referred to as *finite ensemble scaling* for the ensemble sums X_N. A similar finite ensemble scaling form holds for the macroscopic ensemble averages $\overline{X}_N = X_N/(NM)$, namely

$$\overline{P}_N(\overline{x}) = P\left(\frac{\overline{x}\mathcal{L}^{*1/\varpi}(N^{-1})}{N^{(1-\varpi)/\varpi}};\varpi,\zeta,0,1\right). \tag{27}$$

It is now apparent that the block ensemble limit in statistical mechanics generates a scaling property for the distribution function of macroscopic ensemble averages. The derivation is general and does not involve renormalization group arguments. It will now be shown that there exists a statistical classification scheme of phase transitions which is precisely analogous to the thermodynamic classification scheme described in the previous section.

5. Statistical Classification Scheme

Consider first the case where $D = 0$ in (19). The distribution of macroscopic variables is then degenerate, i.e. concentrated at a single point. This case corresponds to the noncritical ensemble limit, i.e. $\xi_X = 0$, where the fluctuations between different blocks vanish in the limit. In modern quantum field theory terminology this case corresponds the "boring limit" [11].

A second immediate observation is the existence of a "Gaussian limit" corresponding to $\varpi = 2$ suggesting that the thermodynamic order λ is directly related to the stable index ϖ. This relationship follows indeed from a comparison with finite size scaling. [1, 3, 7]

As stated above the distribution function of the blockvariables X_{iN} approaches a distribution within the domain of attraction of the stable law with index ϖ and is thus characterized by eq.(23). From the theory of domains of attraction of stable laws [10, 21] follows that two cases must be distinguished. If $\varpi < 2$ then the distribution of block variables has a power law tail and satisfies a scaling relationship similar to (26) with N replaced by M. If $\varpi = 2$ then the distribution may have a power law tail $\propto x^{-\alpha}$ with $\alpha \geq 2$ or not. In either case the distribution which is approached has a finite variance. The scaling relationship for individual block variables analogous to finite ensemble scaling (26),(27) is finite size scaling which holds in the limit $M \to \infty$ with a finite number of blocks, $N < \infty$. Inserting $N = (L/\xi)^d$ into eq.(27) and comparing with the finite size scaling hypothesis for the order parameter distribution $X = \Psi$ [3, 4]

$$P_L(\overline{\Psi}) = P\left(\overline{\Psi} L^{\frac{\beta d}{\gamma+2\beta}}\right) \tag{28}$$

yields the identification

$$\varpi_\Psi = \frac{\gamma + 2\beta}{\gamma + \beta} = 1 + \frac{1}{\delta} = \lambda_\Psi. \tag{29}$$

Similarly for the energy $X = E$

$$\varpi_E = 2 - \alpha = \lambda_E \tag{30}$$

is obtained suggesting that

$$\varpi_X = \lambda_X \tag{31}$$

holds generally except in those cases with $\varpi_X = 2$ where the distribution of block variables approaches a distribution within the domain of attraction of the Gaussian which does not exhibit power law tails. The main conclusion from these considerations is that violations of finite size scaling (and thus hyperscaling) [5] are related to the general inequality $\varpi \leq 2$ in (20). These conclusions are new insofar as they are obtained without using renormalization group arguments, and because they predict that hyperscaling violations should never be observed for transitions of order less than 2.

It remains to identify statistical mechanical model systems which exhibit anequilibrium transitions. Such a class of models is most easily identified from the scaling relation obeyed by the ensemble average of the block energies

$$\overline{E}_N \equiv \frac{1}{N}\sum_{i=1}^{N}\overline{E}_{iN} \stackrel{d}{=} N^{-1+(1/\varpi)}\overline{E}_{1N} \tag{32}$$

where $\stackrel{d}{=}$ indicates equality in distribution. For $\varpi < 1$ the energy of a composite system is not extensive, and this indicates to search for anequilibrium transitions among unstable systems.

An explicit example is the one dimensional Gaussian model [2]. Its exact free energy density reads

$$-\frac{f(T)}{k_B T} = \frac{1}{2}\log\pi - \frac{1}{2}\log\left(\frac{1}{2}(\sigma + \sqrt{\sigma^2 - K^2})\right) \tag{33}$$

where $K = J/(k_B T)$, k_B is the Boltzmann constant, J the coupling, T the temperature and $1/\sigma$ the standard deviation of the Gaussian single spin measure. The free energy density shows an anequilibrium transition of order $\lambda_\mathcal{E} = 1/2$ at the critical temperature $T_{min} = J/(k_B\sigma)$.

6. Dynamics near Anequilibrium Transitions

This final section discusses some aspects of the dynamics of macroscopic observables X_N near anequilibrium transitions. The divergence of energy expectation values at anequilibrium transitions indicates unusual dynamic behaviour because the energy is the infinitesimal generator of time translations.

Any sequence of block configurations or block variables X_{iN} may be embedded into time as a sequence of snapshots

$$X_N(t_i) = X_{iN} \tag{34}$$

where $t_i \in \mathbf{R}$ is the time sequence of observation instants. As $N \to \infty$ the ensemble limit becomes the long time limit. The notation $X_\infty = X$ will be used for simplicity. The stochastic process $X(t)$ is called stationary if

$$\langle X(t_1-\tau)...X(t_k-\tau)\rangle = \langle X(t_1)...X(t_k)\rangle \tag{35}$$

for all $k \geq 1$, all choices of t_j and all $\tau \in \mathbf{R}$. Here $\langle ... \rangle$ denotes averaging over the random realizations of X. Stationarity expresses invariance of the correlation functions under under the semigroup $T(\tau)$

$$T(\tau)\, f(t_1, ..., t_n) = f(t_1 - \tau, ..., t_n - \tau) \tag{36}$$

of right translations. The embedding defined in (34) allows to interpret the sequence

$$\tau_i = t_{i-1} - t_i > 0 \tag{37}$$

as decorrelation or decoupling times because the X_{iN} are uncorrelated by virtue of the ensemble limit. These decorrelation times are themselves random variables whose distribution is determined by the microscopic time evolution and the decorrelation criterion. The system will be called ergodic if the random decorrelation times have a finite average,

$$\langle \tau \rangle = \lim_{N \to \infty} \frac{1}{N}(\tau_1 + ... + \tau_N) < \infty. \tag{38}$$

There are three reasons for adopting this definition: Firstly it ensures the equality of ensemble averages and long-time averages. Secondly it allows to observe their convergence within a finite observation time. Thirdly, and most importantly, it gives rise to the semigroup (36) of time shifts which in turn implies that the only functions left invariant by the time evolution are constant.

The intimate relationship between ergodicity and stationarity appears if one studies the possible limit distributions for the decorrelation sums

$$T_N = \tau_1 + \tau_2 + ... + \tau_N \tag{39}$$

in the ensemble limit $N \to \infty$. By the same considerations as in the static case the decorrelation times must belong to the domain of attraction of some stable law characterized by the parameters $\varpi_{X,T}$ and $\zeta_{X,T}$. However now the possible limiting stable distributions are restricted by the requirement of positivity. The limiting distribution $P(t) = \lim_{N \to \infty} P_N(tD_N + C_N)$ must vanish for $t < 0$. Here $P_N(t) = \mathrm{Prob}\{T_N \leq t\}$. The positivity requirement restricts $P(t)$ to be a stable distribution with parameters $0 < \varpi_{X,T} < 1$ and $\zeta_{X,T} = 1$ or to be degenerate, $P(t) = \Theta(t-1)$, where $\Theta(x)$ denotes the Heaviside step function. The degenerate distribution arises from the case $0 < \varpi_{X,T} < 1$ as the limit $\varpi_{X,T} \to 1^-$.

The important result is then that the only limiting distribution compatible with ergodicity as defined in eq. (38) is the degenerate distribution $\varpi = 1$ whose convolution semigroup [10] coincides with the semigroup of time translations (36). The convolution semigroup of $\Theta(t-1)$ is defined by

$$T_1(t')f(t) = \int_{-\infty}^{\infty} f(t-s)\, d\Theta(st'^{-1}-1) = f(t-t') \tag{40}$$

and is thus identical with eq. (36). Using $T(t) = T_1(t)$ the definition (35) of stationarity may be rewritten for the one-point function as $(\mathbf{1} - T(\tau))\langle X(t)\rangle = 0$ where $\mathbf{1}$ denotes the identity operator. Defining the infinitesimal generator A of a semigroup $T(t)$ by

$$(-A)f(x) = \lim_{t\to 0^+} \frac{1}{t}(\mathbf{1} - T(t))f(x) \tag{41}$$

this becomes

$$(\mathbf{1} - T(\tau))\langle X(t)\rangle = \int_0^{\tau} T(s)\,(-A)\,\langle X(t)\rangle\, ds = 0. \tag{42}$$

Using the well known fact that the generator A for the time translations is the temporal derivative, $A = d/dt$, yields

$$\frac{d}{dt}\langle X(t)\rangle = 0 \tag{43}$$

the familiar formulation of stationarity for $\langle X(t)\rangle$.

The divergence of energies and entropies at anequilibrium transitions requires to reconsider the notions of stationarity and ergodicity. The required generalization of these concepts will be based on the convolution semigroup (40) for the equilibrium case. In the general case the convolution semigroup is defined by replacing $\Theta(x)$ in (40) through the one sided limiting laws for T as [10]

$$T_\varpi(t')X(t) = \int_{-\infty}^{\infty} X(t-s)dP(st'^{-1/\varpi}) \tag{44}$$

with $t > 0$. For $\varpi = 1$ this reduces to equation (40). The generalization of equation (42) takes the form

$$0 = (\mathbf{1} - T_\varpi(\tau))^{\varpi}\langle X(t)\rangle =$$

$$\frac{1}{\Gamma(\varpi)\Gamma(-\varpi)}\int_0^\infty \sum_{0\leq j<(s/\tau)}\left(\frac{\Gamma(j-\varpi)}{\Gamma(j+1)}(s-j\tau)^{\varpi-1}\right)T_\varpi(s)\,\frac{d^\varpi}{dt^\varpi}\langle X(t)\rangle\,ds \quad (45)$$

with the fractional time derivatives of order ϖ as infinitesimal generators of the macroscopic time evolution. The usual definition of stationarity (43) now becomes

$$\frac{d^\varpi}{dt^\varpi}\langle X(t)\rangle = 0. \quad (46)$$

The solution to this equation is not a constant but

$$\langle X(t)\rangle = C_0 t^{\varpi-1} \quad (47)$$

where C_0 is a constant. The temporal evolution of macroscopic variables may be called stationary if the decay is algebraic with exponents smaller than unity.

Another interesting consequence [20] from identifying fractional derivatives as the generators of the macroscopic time evolution follows from generalizing the equations of motion for macroscopic observables as

$$\frac{d^\varpi X(t)}{dt^\varpi} = BX(t) \quad (48)$$

where B denotes a generalized Liouville operator. Laplace Transformation and using $X_0 = X(t=0)$ then yields

$$X(u) = u^{\varpi-1}(u^\varpi - B)^{-1}X_0. \quad (49)$$

Inverting the Laplace transform gives [20]

$$X(t) = \left(\sum_{k=0}^{\infty}\frac{t^{k\varpi}}{\Gamma(k\varpi+1)}B^k\right)X_0 \quad (50)$$

as the final result. Evidently this solution represents a slow nonexponential decay approaching algebraic decay in the long time limit.

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