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Random walks with short memory in a disordered environment

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The backward-jump model is investigated for the case of a bond-disordered lattice. The backward-jump model is a correlated nearest-neighbor random-walk model in which the walker has a different transition rate for jumps to its previously visited site than for jumps to all other nearest-neighbor sites. The standard formulation of the model must be modified if the disorder is introduced at the level of the usual master equation. The difficulties with the standard formulation are discussed in the paper. The first-order master equation for the disordered backward-jump model is established, and a symmetrized second-order equation that was suggested previously is derived from it.

A well-known generalization of uncorrelated random walks on regular lattices is random walks with short-term memory. In the so-called backward-jump model the memory extends one time step into the past. In that model the walker has a different probability for back jumps than for all other (forward or sideward) jumps. This type of correlated random walk was first studied in one dimension by Fürth[1] and subsequently by many authors.[2-6] The model has found application to different physical situations including polymer statistics, exciton transport, superionic conductors, and cellular automata for Lorentz lattice gases. For a review the reader is referred to Ref. 7.

My purpose here is to elucidate certain new aspects of the backward-jump model when the underlying lattice is not regular but a bond-diluted disordered lattice. The study of the backward-jump model in a disordered environment was initiated in Refs. 8 and 9.

Disordering the backward-jump model turns out to be a more subtle problem than one might think at first. The difficulties arise from an inconsistency in the formulation of the model for ordered lattices. This inconsistency will be discussed first. Afterwards it is shown how to reformulate the model in a consistent manner.

Given that the random walker remembers only its previously occupied site the standard formulation considers the probability density $P(i,j,t)$ to find the walker at site i at time t given that site j was occupied immediately prior to the last jump (and that the walker started at the origin at $t=0$). It is well known[7] that one can obtain a Markovian description by considering the history of the walker, i.e., its current position together with its previous position instead of just its current position. The standard formulation is given by the following master equation for $P(i,j,t,)$:

$$\frac{d}{dt}P(i,j,t)=w_b[P(j,i,t)-P(i,j,t)] \qquad (1)$$
$$+w\sum_{k\neq i}[P(j,k,t)-P(i,j,t)].$$

In Eq. (1) w_b is the transition rate for back jumps, i.e., jumps to the previously occupied site, while the rate w applies to all other transitions. The sum runs over all nearest-neighbor sites k of site j (except i). Equation (1) is valid on a regular lattice with coordination number z.

Let me now consider Eq. (1) for a bond-disordered lattice. Each bond of the underlying regular lattice has a probability p of being present, and is missing with probability $1-p$. Only bonds which are present can be crossed by the walker, transitions over missing bonds are forbidden. Note that in this model the coordination number z_i of site i is a random quantity varying between 0 and z.

In the disordered case there will always be lattice sites with $z_i=1$. Obviously, if w_b approaches 0 then the walker acquires an infinite memory at such points. This is inconsistent with the original assumption of a finite length memory.

Another manifestation of the problem is the absence of a model parameter related to the length of memory. The rate w determines the units of time as $1/w$. The rate w_b, on the other hand, sets the strength of the correlations, i.e., it determines the average number of executed or avoided back jumps, and w_b/w might be called a correlation time. The length of memory is fixed implicitly as the length of the history, i.e., the number of previous sites that influence the transition rate. But there is no model parameter to change the memory length at fixed length of the history. As long as the lattice is regular this presents no problems because on average there will be a transition within a time $1/w$, and therefore the length of history is roughly equal to the length of memory. In the disordered case, however, the length of the memory becomes position dependent as well as w_b dependent.

In Ref. 8 these problems have been avoided by introducing the disorder after symmetrizing Eq. (1). The symmetrization consists in writing a closed second-order equation for the probability densities $P(i,t)=\sum_{j\{i\}}P(i,j,t)$ where the summation runs over all nearest neighbors of site i. The probability density $P(i,t)$ is the usual quantity known from uncorrelated random walks. It is the probability density to find the random walker at site i at time t if it started from the origin at time 0. The resulting symmetrized equation can be easily disordered[8] to give

$$\frac{d^2}{dt^2}P(i,t)+(\gamma+w_b-w)\frac{d}{dt}P(i,t)=w\sum_{j\{i\}}A_{ij}\frac{d}{dt}[P(j,t)-P(i,t)]+w\gamma\sum_{j\{i\}}A_{ij}[P(j,t)-P(i,t)] \tag{2}$$

where $\gamma=w_b+w(z-1)$, z is the coordination number of the underlying regular lattice, and the A_{ij} are defined as

$$A_{ij}=A_{ji}=\begin{cases}1 & \text{if the bond } [ij] \text{ is present}\\ 0 & \text{if the bond } [ij] \text{ is absent .}\end{cases}$$

Equation (2) is symmetric, and the possibility of an infinite memory at $z_i=1$ sites no longer exists.

In the rest of this paper it will be shown that Eq. (2) can be obtained more directly than in Ref. 8. The disorder can be introduced directly into Eq. (1) after some small modifications in the formulation which eliminate the position dependence of the memory length.

The inconsistency in the formulation of Eq. (1) can be eliminated by allowing transitions from a site to itself. In Eq. (1) it was assumed that $i\neq j$. This restriction is now lifted. Taking this modification into account and introducing disorder via a factor A_{ij} for each bond one arrives at the following master equation:

$$\frac{d}{dt}P(i,j,t)=\sum_k A_{ij}w_{ijk}A_{jk}P(j,k,t) -\sum_l A_{li}w_{lij}A_{ij}P(i,j,t). \tag{3a}$$

Here the A_{ij} are defined as $A_{ii}=1$ for all i, and

$$A_{ij}=A_{ji}=\begin{cases}1 & \text{if the bond } [ij] \text{ is present}\\ 0 & \text{if the bond } [ij] \text{ is absent}\end{cases} \tag{3b}$$

as before. The rates w_{ijk} are defined as

$$w_{ijk}=\begin{cases} w & \text{for } i\neq j\neq k\\ w_b & \text{for } i=k, i\neq j\\ w_b & \text{for } i=j=k\\ w & \text{for } j=k, i\neq j\\ (M+z-z_i)w & \text{for } i=j, i\neq k\\ 0 & \text{otherwise ,}\end{cases} \tag{3c}$$

where M is a constant. The first sum in Eq. (3) runs over all k that are nearest neighbors of j as well as $k=j$. Similarly the second sum runs over $l=i$ as well as all l that are nearest neighbors of i. Equation (3) represents the formulation of correlated random walks in a bond-disordered environment.

The parameter M determines the length of the memory as $1/M$ in units of $1/w$. A finite value for M means that the previously visited site is forgotten on the average after a time $1/M$ even if no jump occurred during that time interval. For $M=0$ the previously visited site is only forgotten when the walker jumps. In that case it will be seen below that Eq. (3) reduces to Eq. (1) if the lattice is regular.

The master equation, Eq. (3), implies that $P(i,j,t)=0$ if the bond $[ij]$ is absent because such bonds cannot be crossed. This implies the relation $A_{ij}P(i,j,t)=P(i,j,t)$. Using it Eq. (3) can be rewritten for the case $i\neq j$ as

$$\frac{d}{dt}P(i,j,t)=wA_{ij}P(j,t)+(w_b-w)P(j,i,t)-\gamma P(i,j,t) , \tag{4a}$$

while for $i=j$ one has

$$\frac{d}{dt}P(i,i,t)=(M+z-z_i)wP(i,t) +(w_b-w)P(i,i,t)-\gamma P(i,i,t) . \tag{4b}$$

In Eq. (4) $\gamma=w_b+w(M+z-1)$ and

$$P(i,t)=\sum_j P(i,j,t) \tag{5}$$

where the sum runs over $j=i$ and all nearest neighbors j of site i.

For the case of the regular lattice, i.e., $z_i=z$, and $M=0$ one recovers Eq. (1) from Eq. (4) as noted above. In the disordered case, however, Eq. (4b) remains coupled to Eq. (4a) for all values of M. This is the reason for the absence of the inconsistency discussed above.

In order to derive the symmetrized form of Eq. (3) for the quantities $P(i,t)$ of Eq. (5) one differentiates Eq. (4) and sums over all j to obtain

$$\frac{d^2}{dt^2}P(i,t)+[\gamma-(M+z-z_i)w]\frac{d}{dt}P(i,t) =w\sum_{i\neq i}A_{ij}\frac{d}{dt}P(j,t)+(w_b-w)\sum_j\frac{d}{dt}P(j,i,t) . \tag{6}$$

Next one writes Eq. (4) with i and j interchanged. Again summing over j yields

$$\sum_j\frac{d}{dt}P(j,i,t)=\gamma P(i,t)-\gamma\sum_j P(j,i,t).$$

Inserting this into Eq. (6) leads to

$$\frac{d^2}{dt^2}P(i,t)+[\gamma-(M+z-z_i)w]\frac{d}{dt}P(i,t)-(w_b-w)\gamma P(i,t)=w\sum_{j\neq i}A_{ij}\frac{d}{dt}P(j,t)+(w_b-w)\gamma\sum_j P(j,i,t). \tag{7}$$

Finally Eq. (4) is summed over j and then solved for $\sum_j P(j,i,t)$. The result is

$$(w_b-w)\sum_j P(j,i,t)=\frac{d}{dt}P(i,t)+[\gamma-(M+z-z_i)w]P(i,t)-w\sum_{j\neq i}A_{ij}P(j,t) .$$

Together with Eq. (7) one obtains

$$\frac{d^2}{dt^2}P(i,t)+[\gamma+w_b+w(z_i-1)]\frac{d}{dt}P(i,t)+\gamma w z_i P(i,t)=w\sum_{j\neq i}A_{ij}\frac{d}{dt}P(j,t)+w\gamma\sum_{j\neq i}A_{ij}P(j,t)\ . \tag{8}$$

This result is equivalent to Eq. (2) in the limit $M=0$.

The foregoing derivation establishes the equivalence of the model defined by Eqs. (3) and (2) which is the system treated in Ref. 8. In Ref. 8 Eq. (2) was studied using an effective-medium approach. New and interesting effects from the interplay between memory correlations and disorder effects were found in the generalized frequency-dependent diffusion coefficient. As an example the real part of the frequency-dependent diffusion coefficient can exhibit a maximum while the imaginary part shows a zero for certain choices of the parameter w_b. The reader is referred to Ref. 8 for a complete discussion.

In summary this paper has investigated the problem of correlated random walks in a bond-disordered lattice. It was shown that the usual formulation leads to problems because it fixes the length of the history but not the length of the memory. The inconsistencies can be avoided by reformulating the problem such that transitions from a site to itself are admissible. The corresponding master equation was established, and it was shown to be equivalent to the formulation which was used as the starting point for the investigations in Ref. 8. The model discussed here represents one of the simplest systems for studying correlation effects on transport in a disordered environment.

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