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Remarks on Fractional Time

Rudolf Hilfer

In dankbarer Erinnerung an die Jahre 1983–1986
Herrn Prof. Dr. Carl Friedrich von Weizsäcker
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1 Introduction

It is not possible to repeat an experiment in the past. The underlying philosophical truth in this observation is the difference between certainty of the past and potentiality of the future. This difference is discussed, for example, in C.F. von Weizsäcker's papers [1, 2], and it was often pointed out by him in our discussions in the years 1983–1986 in the Starnberg institute. The perennial philosophical problem related to this difference between past and future is the question whether time is real or not. Compared to the difference between past and future, the dogmatic time reversibility of mechanical processes in physics appears as a secondary and derived property. Let us therefore assume for the purposes of this paper that the asymmetry of time is more fundamental than the reversibility of time implied by the limited validity of mechanical equations.

If the asymmetry of time is placed above all other laws of physics (e.g. also above the law of energy conservation) then the so called problem of irreversibility becomes reversed (see [3]): Instead of explaining how irreversible behaviour (such as diffusion) arises from reversible mechanical laws, one must now explain why time reversible equations appear so frequently in physics. In this paper I intend to show that this "reversed irreversibility problem" can be solved, and that, surprisingly, the answer has nothing to do with statistical mechanics or the second law, thereby avoiding the ongoing debate [4] about Boltzmann's views.

Let me therefore postulate the following statement as an empirical and fundamental law of nature:

Every time evolution of a physical system is irreversible.

The mathematical structures corresponding to irreversible time evolutions are abstract Cauchy problems and operator semigroups [5]. In fact postulating irreversibility as the starting point of physics stands in stark contrast to most fundamental theories of physics including Weizsäcker's "Urtheorie" [6, 7]. In these theories one starts from a symmetry group containing the time translations as a subgroup. In this way one postulates, by Noether's theorem, energy conservation and reversibility as the fundamental starting point.

In the following I want to show that there is no contradiction between time translation invariance of physical observations and the fundamental nature of time irreversibility. On the contrary, both facts can be reconciled. Let me first discuss basic requirements for time evolutions in physics. Next some mathematical consequences will be drawn that have been discussed earlier in [3, 8]. Finally I shall attempt to interpret the consequences in the light of the deliberations of C.F. von Weizsäcker exposed e.g. in [1, 2].

2 Requirements for time evolution operators

2.1 Semi-group property

A physical time evolution $\{\mathrm{T}(\Delta t) : 0 \leq \Delta t < \infty\}$ is defined as a one-parameter family (with time parameter Δt) of bounded linear time evolution operators $\mathrm{T}(\Delta t)$ on a Banach space B. This family of operators is a representation of the semi-group of nonnegative real numbers $(\mathbb{R}_+, +)$ (time durations), i.e. it fulfills the conditions

$$\mathrm{T}(\Delta t_1)\mathrm{T}(\Delta t_2) f(t_0) = \mathrm{T}(\Delta t_1 + \Delta t_2) f(t_0) \tag{1}$$

$$\mathrm{T}(0) f(t_0) = f(t_0) \tag{2}$$

for all $\Delta t_1, \Delta t_2 \geq 0$, $t_0 \in \mathbb{R}$ and $f \in B$. The elements $f \in B$ represent time dependent physical observables, i.e. functions on the time axis $\mathbb{R}$. In the following B is chosen as the set $B = L^1(\mathbb{R})$ of Lebesgue-integrable functions on the time axis in a vector space (e.g. $\mathbb{R}^n$) with norm $\|f\| = \int |f(t)|\mathrm{d}t$. Note that the argument $\Delta t \geq 0$ of $\mathrm{T}(\Delta t)$ has the meaning of a time duration, while $t \in \mathbb{R}$ in $f(t)$ means a time instant.

Equations (1) and (2) define a physical time evolution as a representation of the semigroup $(\mathbb{R}, +)$ of real numbers, not as a group. The inverse elements $\mathrm{T}(-\Delta t)$ are absent. This reflects the fundamental difference between past and future, in the sense that it is not possible to evolve into the past.

The linear operator A defined by

$$D(\mathrm{A}) = \left\{ f \in B : \operatorname*{s-lim}_{\Delta t \to 0+} \frac{\mathrm{T}(\Delta t) f - f}{\Delta t} \text{ exists} \right\} \tag{3}$$

and

$$\mathrm{A} f = \operatorname*{s-lim}_{\Delta t \to 0+} \frac{\mathrm{T}(\Delta t) f - f}{\Delta t} \tag{4}$$

is called the infinitesimal generator of the semigroup. Here s-lim $f = g$ is the strong limit and means $\lim \|f - g\| = 0$ as usual.

2.2 Continuity

A physical time evolution should be continuous. This requirement is realized mathematically by the assumption that

$$\operatorname*{s\text{-}lim}_{\Delta t \to 0} \mathrm{T}(\Delta t) f = f \tag{5}$$

holds for all $f \in B$, where s-lim is again the strong limit. Semigroups of operators satisfying this condition are called strongly continuous or C_0-semigroups.

2.3 Homogeneity

Homogeneity of time is used here to mean two requirements: First it requires that observations are independent of the initial instant or position in time, and secondly it requires arbitrary divisibility of time durations and self-consistency for the transition between time scales. This second condition will be discussed in detail below in Sect. 2.5.

Independence of physical processes from the position of initial instant on the time axis means that physical experiments and processes are not changed if they are *ceteris paribus* shifted in time. Sometimes one rephrases this as: All time instants are equivalent. Or one says that there are no preferred or distinguished time instants. This formulation can easily be misunderstood. The requirement that the difference between past and future is fundamental implies that the present is special, and in this sense the start of an experiment is always special, even though it can be shifted to another position. Mathematically this distinction of the starting instant t_0 is a consequence of the semigroup properties in (1) and (2). The special character of the initial instant is familiar in biological systems where it corresponds to the time instant of birth.

The requirement that the initial instant can be shifted to any other instant is expressed mathematically as the requirement of invariance under time translations. As a consequence one demands commutativity of the time evolution with time translations. One demands

$$\mathcal{T}(\tau)\mathrm{T}(\Delta t) f(t_0) = \mathrm{T}(\Delta t)\mathcal{T}(\tau) f(t_0) = \mathrm{T}(\Delta t) f(t_0 - \tau) \tag{6}$$

for all $\Delta t \geq 0$ und $t_0, \tau \in \mathbb{R}$. Here the translation operator $\mathcal{T}(t)$ is defined by

$$\mathcal{T}(\tau) f(t_0) = f(t_0 - \tau). \tag{7}$$

Note that τ can also be negative. This means that physical experiments in the past have the same outcome as in the present. This can be checked in the present with the help of documents (e.g. a video recording), irrespective of the fact that the experiment cannot be repeated in the past.

2.4 Causality

Causality of the physical time evolution requires that the values of the image function $g(t) = (\mathrm{T}(\Delta t) f)(t)$ depend only upon values $f(s)$ of the original function with time instants $s < t$.

2.5 Homogenous Divisibilty

The semigroup property (1) implies that for $\Delta t > 0$

$$\mathrm{T}(\Delta t)\ldots\mathrm{T}(\Delta t) = [\mathrm{T}(\Delta t)]^n = \mathrm{T}(n\Delta t) \tag{8}$$

holds. Homogeneous divisibility of a physical time evolution is the requirement that there exist rescaling factors D_n for Δt such that with $\Delta t = \overline{\Delta t}/D_n$ the limit

$$\lim_{n\to\infty} \mathrm{T}(n\overline{\Delta t}/D_n) = \overline{\mathrm{T}}(\overline{\Delta t}) \tag{9}$$

exists und defines a time evolution $\overline{\mathrm{T}}(\overline{\Delta t})$. The limit $n \to \infty$ corresponds to two simultaneous limits $n \to \infty, \Delta t \to 0$, and it corresponds to the passage from a microscopic time scale Δt to a macroscopic time scale $\overline{\Delta t}$.

3 Consequences from the requirements

The requirement (6) of homogeneity implies that the operators $\mathrm{T}(\Delta t)$ are convolution operators. Let T be a bounded linear operator on $L^1(\mathbb{R})$ that commutes with time translations, i.e. that fulfills (6). Then there exists a finite Borel measure μ such that

$$(\mathrm{T}f)(s) = (\mu * f)(s) = \int f(s-x)\mu(\mathrm{d}x) \tag{10}$$

holds [9, p. 26]. Applying this theorem to physical time evolution operators $\mathrm{T}(\Delta t)$ yields a convolution semigroup $\mu_{\Delta t}$ of measures $\mathrm{T}(\Delta t)f(t) = (\mu_{\Delta t} * f)(t)$

$$\mu_{\Delta t_1} * \mu_{\Delta t_2} = \mu_{\Delta t_1+\Delta t_2} \tag{11}$$

with $\Delta t_1, \Delta t_2 \geq 0$. For $\Delta t = 0$ the measure μ_0 is the Dirac-measure concentrated at 0.

The requirement of causality implies that the support $\operatorname{supp}\mu_{\Delta t} \subset \mathbb{R}_+ = [0,\infty)$ of the semigroup is contained in the positive half axis.

The convolution semigroups with support in the positive half axis $[0,\infty)$ can be characterized completely by Bernstein functions [10]. An arbitrarily often differentiable function $b : (0,\infty) \to \mathbb{R}$ with continuous extension to $[0,\infty)$ is called Bernstein function if for all $x \in (0,\infty)$

$$b(x) \geq 0 \tag{12}$$

$$(-1)^n \frac{\mathrm{d}^n b(x)}{\mathrm{d}x^n} \leq 0 \tag{13}$$

holds for all $n \in \mathbb{N}$. Bernstein functions are positive, monotonously increasing and concave. The set of all Bernstein functions forms a convex cone containing the functions that are positive and constant.

The characterization is given by the following theorem [10, p. 68]. There exists a one-to-one mapping between the convolution semigroups $\{\mu_t : t \geq 0\}$ with support on $[0, \infty)$ and the set of Bernstein functions $b : (0, \infty) \to \mathbb{R}$ [10]. This mapping is given by

$$\int_0^\infty e^{-ux} \mu_{\Delta t}(\mathrm{d}x) = e^{-\Delta t b(u)} \tag{14}$$

with $\Delta t > 0$ and $u > 0$.

The requirement of homogeneous divisibility further restricts the set of admissible Bernstein functions. It leaves only those measures μ that can appear as limits

$$\lim_{n\to\infty, \Delta t \to 0} \underbrace{\mu_{\Delta t} * \ldots * \mu_{\Delta t}}_{n \text{ factors}} = \lim_{n\to\infty} \mu_{n\overline{\Delta t}/D_n} = \overline{\mu}_{\overline{\Delta t}} \tag{15}$$

Such limit measures $\overline{\mu}$ exist if and only if $b(x) = x^\alpha$ with $0 < \alpha \leq 1$ and $D_n \sim n^{1/\alpha}$ holds [3, 11, 12].

The remaining measures define the class of fractional time evolutions $\mathrm{T}_\alpha(\Delta t)$ that depend only on one parameter, the fractional order α. These remaining fractional measures have a density and they can be written as [3, 8, 13–15]

$$\mathrm{T}_\alpha(\Delta t) f(t_0) = \int\limits_0^\infty f(t_0 - s) h_\alpha\left(\frac{s}{\Delta t}\right) \frac{\mathrm{d}s}{\Delta t} \tag{16}$$

where $\Delta t \geq 0$ and $0 < \alpha \leq 1$. The density functions $h_\alpha(x)$ are the one-sided stable probability densities [3, 8, 13–15]. They have a Mellin transform [16–18]

$$\mathcal{M}\{h_\alpha(x)\}(s) = \frac{1}{\alpha} \frac{\Gamma((1-s)/\alpha)}{\Gamma(1-s)} \tag{17}$$

allowing to identify

$$h_\alpha(x) = \frac{1}{\alpha x} H^{10}_{11}\left(\frac{1}{x} \middle| \begin{matrix} (0, 1) \\ (0, 1/\alpha) \end{matrix}\right) \tag{18}$$

in terms of H-functions [17–20]. Their Laplace transform is

$$\mathcal{L}\{h_\alpha(x)\}(u) = e^{-u^\alpha}. \tag{19}$$

The infinitesimal generators of the fractional semigroups $\mathrm{T}_\alpha(\Delta t)$

$$\mathrm{A}_\alpha f(t) = -(\mathrm{D}^\alpha f)(t) = -\frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{f(t-s) - f(t)}{s^{\alpha+1}} \,\mathrm{d}s \tag{20}$$

are fractional time derivatives of Marchaud–Hadamard type [21, 22]. This fundamental and general result suggests to generalize physical equations of motion by replacing the integer order time derivative with a fractional time derivative as the generator of time evolution [3, 13].

For $\alpha = 1$ one gets $h_1(x) = \delta(x-1)$, the Dirac-measure at $x = 1$. In this case the fractional semigroup $\mathrm{T}_1(\Delta t)$ reduces to the conventional translation semigroup $\mathrm{T}_1(\Delta t) f(t_0) = f(t_0 - \Delta t)$. This observation provides an answer to the question from the introduction why ordinary first order time derivatives appear so frequently in the equations of physics. The theory sketched here can be made more precise and detailed. From the detailed theory one finds that the special case $\alpha = 1$ occurs more frequently in the limit (15) than the cases $\alpha < 1$, in the sense that it has a larger domain of attraction. The fact that the semigroup $\mathrm{T}_1(\Delta t)$ can often be extended to a group on all of $\mathbb{R}$ provides an explanation for the seemingly fundamental reversibility of mechanical laws and equations.

4 Philosophical remarks

The homogenous divisibility of time could perhaps be viewed as a mathematical expression of the philosophical idea that C.F. von Weizsäcker calls "umfassende Gegenwart" [2, p. 612ff]. Homogeneous divisibility formalizes the fact that a verbal statement in the present tense presupposes always a certain time scale. For this reason the present should not be thought of as being a time instant but rather as a certain time period or duration, no matter how short it is [3, 8].

Fractional time evolutions seem to be related also to the subjective human experience of time. In physics the time duration is measured with periodic clocks by counting the number of periods. Contrary to this the subjective human experience of time corresponds to the comparison with an hour glass, i.e. with a nonperiodic reference. Correspondingly most humans experience a fixed physical time interval (e.g. an hour, a day, a year) during senescence as being much shorter than in their infancy or adolescence. The reason is that the reference process to which the fixed interval is compared is one's own life time. It seems that a time duration is considered "long" if it is comparable to the time interval that has passed since birth. In other words the rapidity of perception decreases with age. It seems that this phenomenon is reflected in fractional stationary states defined as solutions of the stationarity condition $\mathrm{T}_\alpha(\Delta t) f(t) = f(t)$. Hence the existence of fractional time evolutions requires a generalization of concepts such as "stationarity" or "equilibrium". This outlook could be of interest for nonequilibrium and biological systems [3, 8, 13–15].

Finally, also the special case $\alpha \to 0$ challenges philosophical remarks. In the limit $\alpha \to 0$ the time evolution operator degenerates into the identity. This could be expressed verbally by saying that for $\alpha = 0$ "becoming" and "being" coincide. For $\alpha = 0$ physical states evolve in time by being mapped to themselves, hence without change. This paradoxical situation reminds me of C.F. von Weizsäcker's thoughts in [2, Kap. 13] concerning certainty of the future and potentiality of the past.

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