J. Phys. A: Math Gen. 17 (1984) L783–L789. Printed in Great Britain

LETTER TO THE EDITOR

Renormalisation on symmetric fractals

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Received 9 July 1984

Abstract. We introduce and investigate new classes of Sierpinski-type fractals. We determine their fractal and spectral dimensions using renormalisation procedures and, for particular classes, we give these dimensions in closed form. The spectral dimensions densely fill the interval [1, 2], allowing us to choose flexibly models for applications.

The importance of dilatational invariance as a physical symmetry principle has been well recognised since the introduction of scaling and renormalisation methods in the study of critical phenomena. In the area of condensed matter physics it was suggested that many disordered structures are dilatationally invariant (Mandelbrot 1982). Examples for such structures, called fractals, are linear and branched polymers (Havlin and Ben Avraham 1982), amorphous and porous materials (Pfeifer and Avnir 1983, Even *et al* 1984), epoxy resins (Alexander *et al* 1983), diffusion limited aggregates (Witten and Sander 1981) and percolation clusters at criticality (Gefen *et al* 1981, Mandelbrot 1982, Alexander and Orbach 1982). According to the way in which matter and voids are distributed these fractals can be viewed as *stochastic*.

On the other hand there exist *deterministic* fractals in which the distribution of sites is determined by an unambiguous, non-random prescription. Examples for such structures are the Sierpinski gaskets (Urysohn 1927) and their extensions (Dhar 1977, Hilfer and Blumen 1984). Evidently, the deterministic prescriptions standardise the fractals and make them very useful as model systems. Hence many recent analyses have centred on Sierpinski type fractals (Dhar 1977, 1978, Gefen *et al* 1980, 1984, Alexander and Orbach 1982, Rammal and Toulouse 1983, Blumen *et al* 1983, Given and Mandelbrot 1983, Klafter *et al* 1984, Zumofen *et al* 1984).

Fractals are described by (at least) three distinct dimensions: the spatial dimension d of the embedding Euclidean space, the fractal dimension $\bar{d}$, which is connected to the density of sites and the spectral (fracton) dimension $\tilde{d}$, related to dynamical processes on the fractal (*vide infra*). It should be emphasised that $\bar{d}$ and $\tilde{d}$ are amenable to experimental observation (Dhar 1977, Alexander *et al* 1983, Mandelbrot 1982): recent measurements of the electronic energy transfer allow us to determine these dimensions with high accuracy (Klafter and Blumen 1984, Even *et al* 1984). For modelling purposes it then becomes desirable to construct deterministic fractals with prescribed $\bar{d}$ and $\tilde{d}$ values.

In a recent work (Hilfer and Blumen 1984), hereafter denoted by I, we considered Sierpinski-type structures, obtained by using a large class of generators (*vide infra*). The spectral dimensions $\tilde{d}$ for these structures lie between that of a 2D Sierpinski

0305-4470/84/140783+07$02.25

gasket, $\tilde{d}_s = 2\ln 3/\ln 5 = 1.36$ and the value 2. In this letter we extend this study by constructing additional classes of fractals, whose $\tilde{d}$ are dense in the interval [1, 2], i.e. which approach any value to given accuracy. Our main concern is the region below $\tilde{d}_s$, since in the neighbourhood of 1 scaling effects and long-time tails in energy decay are most pronounced (Klafter *et al* 1984, Zumofen *et al* 1984).

Sierpinski-type fractals may be built iteratively from a generator G. Here we take as generators d-dimensional hypertetrahedrons (HT) of sidelength b. A particular generator $G = G(b, d)$ is obtained by filling such a HT with smaller HTS of unit sidelength. We restrict ourselves to generators which have tetrahedral symmetry and which furthermore form connected graphs. Figure 1 displays four examples for allowed $G(14, 2)$. In I we have analysed generators of type A, whereas here we extend our study to all symmetric G.

From G the fractal is iteratively constructed: The structure at stage $n+1$ is obtained by enlarging G by b^n and then filling all upward pointing HTS with the stage-n structure. Figure 2 depicts the first step of the iteration for a particular $G(5, 2)$ of type C.

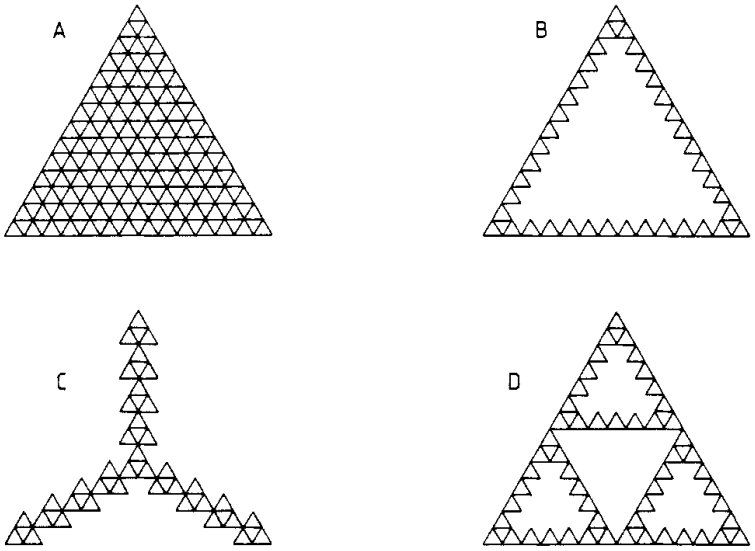

Figure 1. Various symmetric tetrahedral generators for $d = 2$ and $b = 14$.

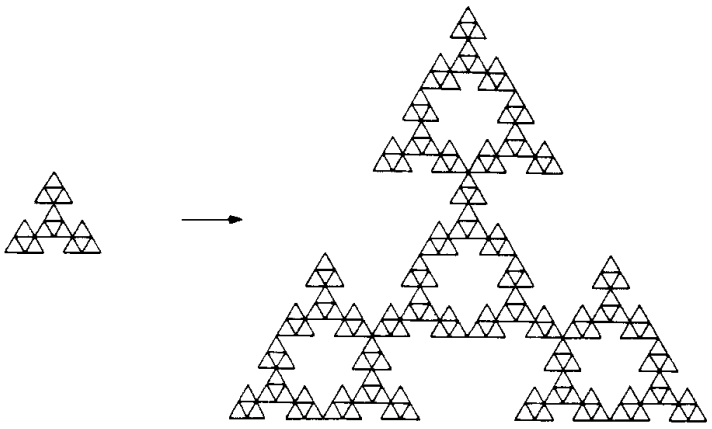

Figure 2. One step in the iterative construction of the fractal.

Let us first focus on the fractal dimension $\bar{d}$, which is related to the density of sites. The fractal dimension mirrors geometrical properties of the lattice and can be monitored experimentally through the direct energy transfer (Klafter and Blumen 1984, Even *et al* 1984); it is given through

$$b^{\bar{d}} = \lim_{n\to\infty} N(n+1)/N(n) \tag{1}$$

where $N(n)$ is the number of sites of the structure at stage n. Here the $N(n)$ fulfil the recursion relation

$$N(n+1) = N(1) + N[N(n) - (d+1)] \tag{2}$$

where N is the number of small HTS inside G. Therefore

$$\bar{d} = \ln N / \ln b. \tag{3}$$

We further remark that $E(n)$ the number of edges of the structure at stage n obeys:

$$E(n+1) = NE(n) \tag{4}$$

which, in conjunction with (3), allows to express $\bar{d}$ through $E(n)$.

The spectral (fracton) dimension $\tilde{d}$ governs dynamical properties. First introduced by Dhar (1977), $\tilde{d}$ gives the low-frequency behaviour of the density of normal modes, $\rho(\omega) \sim \omega^{\tilde{d}-1}$ (Dhar 1977, Alexander and Orbach 1982, Rammal and Toulouse 1983). In the energy transfer case, $\tilde{d}$ appears in the expressions for the indirect (multistep) transfer, where the energy performs a random walk on the fractal (Blumen *et al* 1983, Klafter and Blumen 1984). Here we exemplify the determination of $\tilde{d}$ for the fractal through a random walk model. Thus the probability $P(\boldsymbol{r}_i, t)$ to find the walker at site $\boldsymbol{r}_i$ at time t is governed by the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t} P(\boldsymbol{r}_i, t) = \sum_{j(i)} [w_{ij} P(\boldsymbol{r}_j, t) - w_{ji} P(\boldsymbol{r}_i, t)] \tag{5}$$

where the sum runs over all $\boldsymbol{r}_j$ that are nearest neighbours to $\boldsymbol{r}_i$ and the w_{ij} are the transition probabilities per unit time from $\boldsymbol{r}_j$ to $\boldsymbol{r}_i$. Motivated by the correspondence to the discrete-time analogue, in which the walker steps at fixed times to one of the $z(\boldsymbol{r}_j)$ neighbours of the occupied site $\boldsymbol{r}_j$, we specify the w_{ij} as in I through:

$$z(\boldsymbol{r}_j) w_{ij} = w = \text{constant}. \tag{6}$$

In general the coordination number $z(\boldsymbol{r}_j)$ is not constant over our lattices. In I, where generators $G(b, d)$ were completely filled with small HTS, we had $\min z(\boldsymbol{r}_j) = 2d$ and $\max z(\boldsymbol{r}_j) = \eta d$, with $\eta = \min(b, d+1)$. The difference here is that most generators are only partly filled with small HTS, see e.g. figure 1. Therefore there are edges belonging to a single HT and for these $z(\boldsymbol{r}_j) = d$.

This difference, however, does not affect the general renormalisation procedure for equation (5), which carries through exactly as in I. One considers the $P(\boldsymbol{r}_i, t)$ values at longer times, when the situation inside each generator G appearing in the fractal is quasistationary. One iteration of the real-space renormalisation procedure consists in decimating all interior points of G and keeping only its corners; this decimation inverts the process by which the fractal was generated. Since in equation (5) the corners of G are coupled to their nearest neighbours, which are in general internal sites, and thus get decimated, one has to re-express these couplings in terms of corner

sites only. In the Laplace domain this expression is, equation (I.13) of I:

$$\sum_{i=1}^{d} Q(\boldsymbol{d}_i, u) = dg(\alpha)Q(\boldsymbol{s}_0, u) + h(\alpha) \sum_{i=1}^{d} Q(\boldsymbol{s}_i, u). \tag{7}$$

Here we set $Q(\boldsymbol{r}_i, u) = P(\boldsymbol{r}_i, u)/z(\boldsymbol{r}_i)$ and the two sums extend over all nearest neighbours $\boldsymbol{d}_i$ of the corner $\boldsymbol{s}_0$ and over all other corners $\boldsymbol{s}_i$ of G, respectively. At long times, conservation of probability requires that $P(\boldsymbol{s}_i, t)$ renormalises to $NP(\boldsymbol{s}_i, t)$, where N is again the number of small HTS inside G. The rates w renormalise to $wh(0)/N$, where $h(0)$ stems from equation (7), and one has $g(0) + h(0) = 1$. In I we have related $h(0)$ to the renormalisation mapping $\phi'(0) = \kappa = N/h(0)$—the fractal Einstein relation of Given and Mandelbrot (1983)—so that the renormalised rate is w/κ.

We pause to note that these results hold for generators G with tetrahedral symmetry which form connected graphs. If the generator divides into disconnected pieces, then the random walker cannot explore the whole structure, but is confined to the connectedness component of the starting site. The case that the corners are disconnected is uninteresting, since then $h(0) = 0$, because of symmetry. If the corners are connected to each other but parts of the graph are disconnected from the corners then equation (7) is non-trivial, but the HTS belonging to the disconnected islands do not affect the solution and should not be counted when establishing N.

For the probability to be at the origin $\boldsymbol{r}_0$ at long times (Alexander and Orbach 1982)

$$P(\boldsymbol{r}_0, t) \sim wt^{-\tilde{d}/2} \tag{8}$$

the spectral dimension $\tilde{d}$ is obtained by renormalising both sides of equation (8)

$$\tilde{d} = 2 \ln N/\ln \kappa = 2(1 - \ln h(0)/\ln N)^{-1}. \tag{9}$$

To establish $\tilde{d}$ according to equation (9) one has only to obtain $h(0)$ or equivalently $g(0)$. This may be done (see I, equation (10)) by inverting the matrix $(\boldsymbol{D} - \boldsymbol{A}_1)$ which pertains to the interior sites $\boldsymbol{d}_i$ of G: Specifically, $(\boldsymbol{D})_{ii} = z(\boldsymbol{d}_i)$ and $\boldsymbol{A}_1$ is the submatrix of the adjacency matrix $\boldsymbol{A}$ of G, obtained by eliminating all rows and columns of $\boldsymbol{A}$ which correspond to the corners of G. For connected generators the inverse exists and leads to a non-trivial form for equation (7), as can be demonstrated using the matrix-tree theorem of graph theory (Kirchhoff 1847, Harary 1969). From a close analysis we derive that $g(0) \geqslant 0$ and $h(0) \geqslant 0$, relations which are intuitively clear, since one may view equation (7) as a way to express probabilities for the inner sites $\boldsymbol{d}_i$ in terms of weighted probabilities for the boundary sites of G. Since, as above, $h(0) + g(0) = 1$ we conclude that $h(0), g(0) \in [0, 1]$. Thus, from equation (9) it follows that for all our fractals $\tilde{d} \leqslant 2$.

Instead of proceeding in a general way, we now consider generators of chainlike structure, as exemplified by the cases B and C of figure 1. Type B is obtained by placing small HTS only on the edges of the generator, whereas type C has its HTS mainly (but, for connectedness purposes, not exclusively) along the tetrahedral symmetry axes. These structures are particularly useful: due to their chainlike form they allow us to obtain $\tilde{d}$ values which are close to unity, an interesting region for asymptotic forms and scaling properties (Klafter *et al* 1984, Zumofen *et al* 1984, Anlauf 1984).

The number of small HTS in a B-type d-dimensional generator of sidelength b, $G(b, d)$ is

$$N = (d+1)(1 - d + bd/2) \tag{10}$$

and thus, from equation (3):

$$\bar{d} = \ln[(d+1)(1-d+bd/2)]/\ln b. \tag{11}$$

To calculate $\tilde{d}$ one may invert the $(\boldsymbol{D}-\boldsymbol{A}_1)$ matrix. For the B-type generator it is, however, much more expedient to use the electric circuit analogy (Kirkpatrick 1973, Gefen *et al* 1981), where the bonds are replaced by equal resistances R and to perform a star–HT transformation to calculate the renormalised resistance $\hat{R}$ of G. In general the analogy also requires the sites to be grounded via site-dependent capacitors, but in the limit $\alpha \to 0$ these capacitors can be neglected.

We obtain

$$\hat{R} = R/h(0) = R[2(b-4)(d+1)+(d+3)^2]/(d+1)^2$$

and thus

$$\tilde{d} = 2\ln[(d+1)(1-d+bd/2)]/\ln\{(d+1)^{-1}(1-d+bd/2)[2(b-4)(d+1)+(d+3)^2]\}. \tag{12}$$

Here we have assumed $b \geqslant 5$, in order to have a truly B-type form. The cases $b=2$ and $b=4$ generate the Sierpinski gaskets and $b=3$ is identical to the A-type generator $G(3, d)$. From equation (12) we thus have $\lim_{b\to\infty} \tilde{d} = 1$ for fixed d and $\lim_{d\to\infty} \tilde{d} = 2$ for b held fixed. Thus for B-type generators, the $\tilde{d}$ span the $[1, 2]$-interval, whose boundaries are limit points of the $\tilde{d}$ set.

For the C-type generator of figure 1, $d=2$, we obtain

$$\bar{d} = \ln[3(b-1)]/\ln b \tag{13}$$

a value identical to equation (11), and, again with a star–HT transformation:

$$\tilde{d} = 2\ln[3(b-1)]/\ln[5b(b-1)/2]. \tag{14}$$

In order to display the range of values obtainable and the influence of the sidelength b of the generators, we give in table 1 several numerical values for B- and C-type generators of different dimensions.

We present now a procedure to build generators G for more complex fractals. Starting from two generators $G_1(b_1, d)$ and $G_2(b_2, d)$ we construct $G_{12}(b_1 b_2, d)$ by

Table 1. Fractal ($\bar{d}$) and spectral ($\tilde{d}$) dimensions for fractals obtained from the B- and C-type generators of figure 1. The generator side length is b and d is the embedding spatial dimension.

	Type B ($d=2$)		Type B ($d=3$)		Type C ($d=2$)	
b	$\bar{d}$	$\tilde{d}$	$\bar{d}$	$\tilde{d}$	$\bar{d}$	$\tilde{d}$
5	1.543 96	1.335 37	1.920 57	1.506 85	1.543 96	1.270 39
8	1.464 11	1.284 85	1.773 98	1.436 54	1.464 11	1.232 19
11	1.418 41	1.257 68	1.693 34	1.397 83	1.418 41	1.211 09
20	1.349 60	1.217 49	1.575 07	1.339 38	1.349 60	1.179 34
50	1.275 67	1.174 18	1.451 10	1.274 88	1.275 67	1.144 58
101	1.235 89	1.150 53	1.385 36	1.239 01	1.235 89	1.125 39

first enlarging G_1 by b_2 and then inserting G_2 into all the upward-pointing HTS. By construction G_{12} is again connected and symmetric and may be thus used to build a fractal. The generator of type D in figure 1 is an example: it was obtained from the Sierpinski gasket, generator $G_1(2,2)$, and from a B-type generator, $G_2(7,2)$. The building procedure may be used repeatedly, and leads after m steps to $G_{1\ldots m}(b,d)$, with $b = b_1 \ldots b_m$. The number of HTS of unit length inside $G_{1\ldots m}$ is $N = N_1 \ldots N_m$, where N_i is the corresponding value for G_i. In the same way, repeated decimations of sites in $G_{1\ldots m}$ show that the coefficient κ equals $\kappa_1 \ldots \kappa_m$. Hence, for $G_{1\ldots m}$:

$$\bar{d} = \sum_{i=1}^{m} \ln N_i \Big/ \sum_{i=1}^{m} \ln b_i \tag{15}$$

and

$$\tilde{d} = 2 \sum_{i=1}^{m} \ln N_i \Big/ \sum_{i=1}^{m} \ln \kappa_i. \tag{16}$$

Each of the G_i leads to fractals with dimensions $\bar{d}_i$ and $\tilde{d}_i$. Relations (15) and (16) allow us now to obtain intermediate values in these ranges, and to adjust $\bar{d}$ or $\tilde{d}$ to preassigned values with high accuracy.

To summarise, we have introduced and analysed a wide class of fractal lattices, which generalise the Sierpinski gaskets. We have presented methods for evaluating the fractal and spectral dimensions exactly: for particular lattice types we have obtained these dimensions in closed form. The spectral dimensions span the interval [1, 2]. Furthermore, we have shown how to construct fractals whose dimensionalities interpolate between given values. This allows us to obtain models with prescribed dimensions, tailored according to the experimental findings.

We thank Professor G L Hofacker, Dr J Klafter and Dr G Zumofen for fruitful discussions and gratefully acknowledge grants from the Fonds der Chemischen Industrie and the Deutsche Forschungsgemeinschaft.

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