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Sequential generalized Riemann–Liouville derivatives based on distributional convolution

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Abstract

Sequential generalized fractional Riemann–Liouville derivatives are introduced as composites of distributional derivatives on the right half axis and partially defined operators, called Dirac-function removers, that remove the component of singleton support at the origin of distributions that are of order zero on a neighborhood of the origin. The concept of Dirac-function removers allows to formulate generalized initial value problems with less restrictions on the orders and types than previous approaches to sequential fractional derivatives. The well-posedness of these initial value problems and the structure of their solutions are studied.

Keywords Sequential derivatives · Generalized initial value problems · Convolution of distributions

Mathematics Subject Classification 26A33 · 33E12 · 34A08 · 34K37 · 35R11 · 60G22

1 Introduction

An important issue in applications of fractional differential equations are initial conditions, as emphasized already in [8], [9, p.115] or [10] for fractional relaxation and fractional diffusion. Derivatives $\mathrm{D}_{0+}^{\alpha,\beta}$ of fractional order α and type β were introduced precisely because their type β parametrizes different types of initial conditions.

Most investigations of fractional initial value problems were until recently concerned with the simplest fractional derivatives $\mathrm{D}_{0+}^{\alpha,0}$ of type 0 (Riemann–Liouville) or $\mathrm{D}_{0+}^{\alpha,1}$ of type 1 (Liouville–Caputo) [7,12,16,22,31]. A number of recent works

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considered initial value problems for sequential fractional derivatives [3,24,25,30] thereby continuing the classical investigations in [5] and [31, Sec. 42.2]. In [3,24,25] generalized fractional derivatives of various types were studied all of which were based on the standard kernel $t^{\alpha-1}/\Gamma(\alpha)$. Other recent works studied generalized fractional derivatives where the kernel $t^{\alpha-1}/\Gamma(\alpha)$ is replaced by a Sonine kernel [19,20,26,27,29]. Riemann–Liouville derivatives and their generalizations $\mathrm{D}_{0+}^{\alpha,\beta}$ have β-dependent domains and null spaces, but they all coincide on a specific complement of the null space of $\mathrm{D}_{0+}^{\alpha,1}$ [10, Sec.7]. Exactly on this complement they also coincide with the causal distributional fractional derivative D_{+}^{α} of order α defined as [32]

$$\mathrm{D}_{+}^{\alpha}\colon \mathscr{D}'_{0+} \to \mathscr{D}'_{0+}, \qquad f \mapsto Y_\alpha * f, \tag{1.1}$$

where $\mathscr{D}'_{0+}$ is the space of distributions supported on the right half axis and $Y_\alpha \in \mathscr{D}'_{0+}$ are causal power distributions that satisfy $Y_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha)$ for $t > 0$ (see Equation (2.7)), provided the function spaces for $\mathrm{D}_{0+}^{\alpha,\beta}$ are canonically identified via restriction and zero extrapolation. Motivated by this observation and the recent research activity in sequential fractional derivatives, the main purpose of this work is to generalize and unify the theory of fractional initial value problems that involve several sequential derivatives.

Distributional convolution operators on the full axis modified by certain partially defined operators will be used in this work to reinterpret sequential fractional derivatives

$$\mathrm{D}_{0+}^{\alpha_0,\beta_0} \circ \cdots \circ \mathrm{D}_{0+}^{\alpha_n,\beta_n} \qquad \text{with} \quad \alpha_0,\ldots,\alpha_n,\beta_0,\ldots,\beta_n \in [0,1] \tag{1.2}$$

on the right half axis where $n \in \mathbb{N}_0$. Explicitly, composite operators of the form

$$\mathrm{D}_{+}^{\alpha_0} \circ \mathrm{R} \circ \mathrm{D}_{+}^{\alpha_1} \circ \cdots \circ \mathrm{R} \circ \mathrm{D}_{+}^{\alpha_n} \qquad \text{with} \quad \alpha_0,\ldots,\alpha_n \in \mathbb{R} \tag{1.3}$$

with D_{+}^{α} from (1.1) are investigated, where each first-order derivative in Eq. (1.2) gives rise to an operator R in Eq. (1.3). In this representation the operator R is a partially defined operator on $\mathscr{D}'_{0+}$ that is called δ-*eliminator*, because it removes the δ-part of a distribution at the origin.

Given a generalized sequential fractional derivative of the form (1.3), it can be rewritten in the normal form

$$\mathrm{D}_{0+}^{\alpha|\gamma_1,\ldots,\gamma_m} = \mathrm{D}_{+}^{\alpha} \circ \mathrm{R}^{\gamma_m} \circ \cdots \circ \mathrm{R}^{\gamma_1} \qquad \text{with} \quad \gamma_1 < \cdots < \gamma_m, \tag{1.4}$$

where $\alpha = \alpha_0 + \alpha_1 + \ldots + \alpha_n \in \mathbb{R}$ is its fractional order, $(\gamma_1,\ldots,\gamma_m)$ is called its *sequential type*, and $\mathrm{R}^\gamma := \mathrm{D}_{+}^{-\gamma} \circ \mathrm{R} \circ \mathrm{D}_{+}^{\gamma}$ is called Y_γ-*eliminator*. Linear combinations of generalized sequential fractional derivatives can then be written in the form

$$D = \sum_{l=0}^{m} D_l \circ \mathrm{R}^{\gamma_m} \circ \cdots \circ \mathrm{R}^{\gamma_1} \qquad \text{with} \quad \gamma_1 < \cdots < \gamma_m,\ m \in \mathbb{N}_0, \tag{1.5}$$

where the operators D_l are linear combinations of distributional fractional derivatives $\mathrm{D}_+^\alpha, \alpha \in \mathbb{R}$ (Theorem 2, p. 21). Our Theorem 3 on p. 23 derives a set of basis functions $K_1, \ldots, K_m$ such that every distribution from the null space of D is a linear combination of the $K_1, \ldots, K_m$. Ranges and maximal domains such that D becomes a bijective operator are calculated in Theorem 4, p. 24. If a distribution g belongs to the range of D, then it is an admissible inhomogeneity for the equation $Df = g$. And a distribution f with $Df = 0$ is a linear combination of the basis functions $K_1, \ldots, K_m$ that solves an initial value problem with initial values obtained by applying the operators $\mathrm{V} \circ \mathrm{D}_+^{\gamma_l} \circ \mathrm{R}^{\gamma_{l-1}} \circ \cdots \circ \mathrm{R}^{\gamma_1}$ to K_k for $k, l = 1, \ldots, m$, where the δ-value operator V extracts the coefficient multiplying a δ-distribution.

While Theorem 3 provides a useful result, a complete characterization of the null space of D requires an analysis of the value and well defined-ness of $\mathrm{V} \circ \mathrm{D}_+^{\gamma_l} \circ \mathrm{R}^{\gamma_{l-1}} \circ \cdots \circ \mathrm{R}^{\gamma_1}(K)$ with K a linear combination of the $K_1, \ldots, K_m$. In the case where D is a linear combination of sequential derivatives with distinct orders Theorem 5, p. 25, gives a helpful simplification for this characterization. This leads to a generalization of the existence and uniqueness results from [3] and [5, Theorem 4]. In the general case however, a complete characterization is quite complicated. For the case with only two sequential types, γ_1 and γ_2, the null space of D is fully characterized here in Section 5.4 below.

Analogous to the generalized fractional derivatives from [19,20,29] the operators (1.4) can be generalized to $\mathrm{C}_K \circ \mathrm{R}^{\gamma_n} \circ \cdots \circ \mathrm{R}^{\gamma_1}$ where C_K denotes the convolution operator $f \mapsto K * f$ on $\mathscr{D}'_{0+}$ with convolution kernel $K \in \mathscr{D}'_{0+}$. Theorems 2, 3 and 4 will be proved for this more general class of operators.

Section 2 summarizes some basic mathematical notations on causal distributional convolution operators, the convolution field generated by causal power distributions [17] and partially defined linear operators. Section 3 studies coefficient operators, projectors and eliminators and their application to certain series of distributions. The generalized sequential fractional derivatives are introduced in Section 4. Their fundamental properties are established and their relation to classical Fractional Calculus operators is elucidated. In the final Section 5, the kernels and maximal injective domains of linear combinations of sequential derivatives are studied.

2 Preliminaries and notations

Subsection 2.1 recalls some basic properties of convolution of distributions with support on the right half axis. Subsection 2.2 summarizes some results from [17] about the field of convolution operators that arises naturally from L. Schwartz' approach to Fractional Calculus. Subsection 2.3 summarizes definitions for partially defined linear operators and forms.

2.1 Convolution of distributions with support on the right half axis

The following describes some important properties of convolution of distributions with support contained in the right half axis. The details can be found in [32, Chap. IV, §5] or [15]. Convolution of distributions is a bilinear continuous operation on

$$\mathscr{D}'_{0+} := \left\{ f \in \mathscr{D}'(\mathbb{R}) \, ; \, \operatorname{supp} f \subseteq [0, +\infty[\right\}, \tag{2.1}$$

the space of *causal distributions*. In other words, $\mathscr{D}'_{0+}$ is a convolution algebra with continuous convolution operation. It follows that any distribution $U \in \mathscr{D}'_{0+}$ gives rise to a continuous linear translation-invariant endomorphism by means of the definition

$$\mathrm{C}_U : \mathscr{D}'_{0+} \to \mathscr{D}'_{0+}, \quad f \mapsto U * f. \tag{2.2}$$

Conversely, any continuous linear translation-invariant endomorphism of $\mathscr{D}'_{0+}$ is a convolution operator C_U for some kernel distribution $U \in \mathscr{D}'_{0+}$.

The associative law for $(\mathscr{D}'_{0+}, *)$ entails the composition rule

$$\mathrm{C}_U \circ \mathrm{C}_V = \mathrm{C}_{U*V} \qquad \text{for all} \quad U, V \in \mathscr{D}'_{0+}. \tag{2.3}$$

In particular, the convolution operator C_δ coincides with the identity operator E. A convolution operator C_U with $U \in \mathscr{D}'_{0+}$ is continuously invertible if and only if there exists a distribution $V \in \mathscr{D}'_{0+}$ such that $U * V = \delta$, where δ denotes the "Dirac-function". Note, that V is unique because the convolution algebra $\mathscr{D}'_{0+}$ has no zero divisors.

A sequence $(f_n) \subseteq \mathscr{D}'_{0+}$ is called *absolutely summable* if and only if

$$\sum_{n=1}^{\infty} |\langle f_n, \varphi \rangle| < \infty \qquad \text{for all} \quad \varphi \in \mathscr{D}(\mathbb{R}). \tag{2.4}$$

Summation of sequences is compatible with convolution:

Lemma 1 *Let $(f_n), (g_n) \subseteq \mathscr{D}'_{0+}$ be absolutely summable sequences. Then the double sequence $(f_n * g_m)$ is absolutely summable as well and*

$$\left(\sum_{n=1}^{\infty} f_n \right) * \left(\sum_{m=1}^{\infty} g_m \right) = \sum_{n,m=1}^{\infty} f_n * g_m. \tag{2.5}$$

Proof The space $\mathscr{D}'_{0+}$ is complete. In $\mathscr{D}'_{0+}$ sequences converge weakly if and only if they converge strongly. Thus, the lemma follows from continuity of convolution [32, Thm. XIII]. □

An immediate consequence of Lemma 1 is the following.

Lemma 2 *Let $f, g \in \mathscr{D}'_{0+}$ such that $f = \delta - g$. If the expression*

$$h := \sum_{p=0}^{\infty} g^{*p} \qquad \textit{with} \quad g^{*p} := g * \overset{p-\text{times}}{\cdots} * g \quad \textit{and} \quad g^{*0} = \delta \tag{2.6}$$

is well defined as an absolutely convergent series in $\mathscr{D}'_{0+}$, then $f * h = \delta$.

2.2 Causal distributional fractional calculus

In his book [32] L. Schwartz considered fractional integrals and derivatives, I_{+}^{α} and D_{+}^{α}, with orders $\alpha \in \mathbb{C}$, as convolution operators with distributional kernels operating on the space $\mathscr{D}'_{+}$ of distributions with support bounded on the left. In this work their restrictions to $\mathscr{D}'_{0+}$ are used. That is, one defines $\mathrm{I}_{+}^{\alpha} := \mathrm{C}_{Y_\alpha}$ and $\mathrm{D}_{+}^{\alpha} := \mathrm{C}_{Y_{-\alpha}}$ with the kernels $Y_\alpha \in \mathscr{D}'_{0+}$ defined as

$$Y_\alpha(t) := \begin{cases} t^{\alpha-1}/\Gamma(\alpha) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases} \qquad \text{for } \alpha \in \mathbb{H}, \tag{2.7a}$$

$$Y_\beta := \mathrm{D}^m\, Y_\alpha, \qquad \text{for } \alpha \in \mathbb{H},\ \beta = \alpha - m,\ m \in \mathbb{N}. \tag{2.7b}$$

Due to Equation (2.3) the semigroup property $Y_\alpha * Y_\beta = Y_{\alpha+\beta}$ automatically translates to the well known index laws

$$\mathrm{I}_{+}^{\alpha} \circ \mathrm{I}_{+}^{\beta} = \mathrm{I}_{+}^{\alpha+\beta}, \qquad \mathrm{D}_{+}^{\alpha} \circ \mathrm{D}_{+}^{\beta} = \mathrm{D}_{+}^{\alpha+\beta} \qquad \text{for all} \quad \alpha, \beta \in \mathbb{C}. \tag{2.8}$$

The operators I_{+}^{α} and D_{+}^{α} are continuous, linear and inverse to each other.

The distributions Y_α with orders $\alpha \in \mathbb{R}$ generate a subalgebra $\mathscr{A}[Y_\mathbb{R}]$ of $\mathscr{D}'_{0+}$. The quotient field $\mathscr{Q}[Y_\mathbb{R}]$ of $\mathscr{A}[Y_\mathbb{R}]$ can be realized as a subalgebra of $\mathscr{D}'_{0+}$ [17]. The non-zero elements of $\mathscr{Q}[Y_\mathbb{R}]$ were used in [17] to define translation-invariant linear systems that correspond to fractional differential equations. An explicit description of the quotient field $\mathscr{Q}[Y_\mathbb{R}]$ was obtained from Lemma 2. It was found [17, Sec. 3] that the distribution

$$U = \delta + \sum_{k=1}^{n} \lambda_k Y_{\alpha_k} \tag{2.9}$$

with $n \in \mathbb{N}_0, 0 < \alpha_1 < \cdots < \alpha_n$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$ possesses a unique convolutional inverse given by

$$U^{*-1} := \sum_{p=0}^{\infty} (-1)^p \left(\sum_{k=1}^{n} \lambda_k Y_{\alpha_k} \right)^{*p} \in \mathscr{D}'_{0+}. \tag{2.10}$$

Moreover, using the notation of convolution quotients

$$\frac{U}{V}\!\!\!\!\!\!{*}\quad := U * V^{*-1}, \tag{2.11}$$

the set $\mathscr{Q}[Y_\mathbb{R}] \setminus \{0\}$ coincides with the set of convolution quotients

$$*\!\!\frac{\lambda_1 Y_{\alpha_1} + \cdots + \lambda_n Y_{\alpha_n}}{\delta + \mu_1 Y_{\beta_1} + \cdots + \mu_m Y_{\beta_m}} \tag{2.12}$$

with uniquely determined numbers $\alpha_1 < \cdots < \alpha_n$, $0 < \beta_1 < \cdots < \beta_m$, $\lambda_k, \mu_l \in \mathbb{C}^\times$ and $n, m \in \mathbb{N}_0$. Maximal domains for the convolution operators defined by the kernels (2.12) were discussed in [17, Sec. 7]. However, for the purposes of this work it suffices to consider them as convolution operators acting on $\mathscr{D}'_{0+}$.

Remark 1 The presented approach to Fractional Calculus resembles operational calculus approaches [13,23,28] in the sense that a quotient field construction is used in both cases. However, the restriction to convolution quotients of a certain class of special distributions, whose inverses are well understood distributions (see [17]), has the advantage that the concrete interpretation of the abstract quotients is known very early on.

2.3 Partially defined linear operators and forms

A partially defined linear operator $A\colon \operatorname{dom} A \subseteq X \to X$ is a linear operator $A\colon \operatorname{dom} A \to X$ with $\operatorname{dom} A \subseteq X$ a linear subspace. Similarly, a partially defined linear form $L\colon \operatorname{dom} L \subseteq X \to \mathbb{K}$ is a linear form $L\colon \operatorname{dom} L \to X$ with $\operatorname{dom} L \subseteq X$ a linear subspace. The following recalls basic operations on partially defined linear operators and forms.

Let A, B partially defined linear operators on X. One denotes

$$A \subseteq B \quad \Leftrightarrow \quad \operatorname{dom} A \subseteq \operatorname{dom} B \text{ and } Ax = Bx \quad \text{for all } x \in \operatorname{dom} A. \tag{2.13}$$

The sum $A + B$ of A and B is defined as

$$\operatorname{dom}(A + B) := \operatorname{dom} A \cap \operatorname{dom} B \tag{2.14a}$$
$$(A + B)x := Ax + Bx \qquad \text{for } x \in \operatorname{dom}(A + B). \tag{2.14b}$$

The scalar multiple $\lambda \cdot A$ with $\lambda \in \mathbb{K}$ is defined as

$$\operatorname{dom}(\lambda \cdot A) := \operatorname{dom} A \tag{2.15a}$$
$$(\lambda \cdot A)x := \lambda \cdot (Ax) \qquad \text{for } x \in \operatorname{dom} A. \tag{2.15b}$$

The composition $A \circ B$ of A and B is defined as

$$\operatorname{dom}(A \circ B) := \{ x \in \operatorname{dom} A ;\ Bx \in \operatorname{dom} A \} \tag{2.16a}$$
$$(A \circ B)x := A(Bx) \qquad \text{for } x \in \operatorname{dom}(A \circ B). \tag{2.16b}$$

Addition (2.14) respectively composition (2.16) defines a semigroup with neutral element given by the zero operator $\mathbf{0}$ with $\operatorname{dom} \mathbf{0} = X$ respectively the identity operator E with $\operatorname{dom} \mathrm{E} = X$. The distributive laws read

$$(A + B) \circ C = A \circ C + B \circ C, \tag{2.17a}$$
$$C \circ (A + B) \supseteq C \circ A + C \circ B. \tag{2.17b}$$

Equality holds in Equation (2.17b) as well if $\operatorname{dom} C = X$.

Addition of partially defined linear forms is defined analogous to (2.14). The composition $L \circ A$ of a partially defined linear form L and a partially defined linear operator A is defined analogous to (2.16). This composition defines a semigroup operation of the partially defined linear operators on the partially defined linear forms on X.

The operations for partially defined operators and forms exhibit strange behaviour. For instance, $A - A = \mathbf{0}_{\operatorname{dom} A} \subseteq \mathbf{0}$ and $0 \cdot A = \mathbf{0}_{\operatorname{dom} A} \subseteq \mathbf{0}$, but equality holds if and only if $\operatorname{dom} A = X$. For every linear subspace $Y \subseteq X$ one obtains a linear space of partially defined linear operators $A\colon\ \operatorname{dom} A \subseteq X \to X$ with $\operatorname{dom} A = Y$, but the set of *all* partially defined linear operators or linear forms does not define a linear space.

3 Eliminators, initial values and series expansions

This section introduces and studies Y_α-coefficient operators, -projectors and -eliminators. These are defined for $\alpha = 0$ in Subsection 3.1. Via fractional integration and differentiation the definition is transported to general α in Subsection 3.2. Then, in Subsection 3.3, the operators are applied to represent the coefficients of a series of distributions Y_α.

3.1 The δ-limit operator, the δ-projector and the δ-eliminator

Roughly speaking, the δ-elminator, which is denoted by R, removes the "δ-*component*" of a distribution. This "surgery" does not go well without careful preparations. Therefore, the δ-eliminator needs to be introduced as a partially defined operator on $\mathscr{D}'_{0+}$.

Let $I \subseteq \mathbb{R}$ be open. The space of *distributions of order zero on* I, denoted by $\mathscr{D}'^0(I)$, is the set of distributions $\mu \in \mathscr{D}'(I)$ satisfying

$$\forall J \subseteq I \text{ compact } \exists C \in \mathbb{R}_+ \ \forall \varphi \in \mathscr{D}_J \quad : \quad |\mu(\varphi)| \leq C\|\varphi\|_\infty, \tag{3.1}$$

where $\mathscr{D}_J = \{\, \varphi \in \mathscr{D}(\mathbb{R})\,;\ \operatorname{supp}\varphi \subseteq J \,\}$. Any distribution $f \in \mathscr{D}'^0(I)$ can be identified with a Radon measure on I, see [14, Def. 2.1.1] or [15, 4, §4].

Definition 1 Define the space of distributions

$$\mathscr{D}'_{0+,\delta} := \left\{ f \in \mathscr{D}'_{0+}\,;\ \exists \varepsilon > 0 : \ f|_{]-\varepsilon,\varepsilon[} \in \mathscr{D}'^0(]-\varepsilon, \varepsilon[) \right\}. \tag{3.2}$$

The δ-*value operator* is the partially defined linear form V given by

$$\operatorname{dom} \mathrm{V} := \mathscr{D}'_{0+,\delta} \tag{3.3a}$$

$$\mathrm{V} f := \lim_{\varepsilon \searrow 0} \langle f, 1_{]-\varepsilon,\varepsilon[} \rangle \qquad \text{for all } f \in \operatorname{dom} \mathrm{V}, \tag{3.3b}$$

where the right hand side of (3.3b) is defined for $\varepsilon > 0$ small enough by

$$\langle f, 1_{]-\varepsilon,\varepsilon[} \rangle := \lim_{n \to \infty} \langle f, \varphi_n \rangle \tag{3.3c}$$

for any sequence $(\varphi_n) \subseteq \mathscr{D}(\mathbb{R})$ such that $\varphi_n \nearrow 1_{]-\varepsilon,\varepsilon[}$ for $n \to \infty$.
The δ-*projector* is the partially defined operator

$$\mathrm{P}\colon \mathscr{D}'_{0+,\delta} \subseteq \mathscr{D}'_{0+} \to \mathscr{D}'_{0+}, \quad f \mapsto \mathrm{P}\, f := \delta \cdot \mathrm{V}\, f. \tag{3.4a}$$

The δ-*eliminator* is the complementary projection of P, that is

$$\mathrm{R} := \mathrm{E} - \mathrm{P}. \tag{3.4b}$$

The δ-*residual space of causal distributions* is the range of the δ-remover

$$\mathscr{D}'_{0+,\not\delta} := \operatorname{ran} \mathrm{R} = \mathrm{R}\left(\mathscr{D}'_{0+,\delta}\right). \tag{3.5}$$

It is immediate from the definition of the space $\mathscr{D}'_{0+,\delta}$ that, for any $\varepsilon > 0$ and any $f \in \mathscr{D}'_{0+,\delta}$, the distribution f can be written as

$$f = \mu + g, \qquad \mu \in \mathscr{D}'^{0},\ \operatorname{supp} \mu \subseteq [0, \varepsilon],\ g \in \mathscr{D}',\ \inf \operatorname{supp} g > 0. \tag{3.6}$$

The δ-coefficient can be defined equivalently as $\mathrm{V}\, f = \mu(\{0\})$ for $f = \mu + g$ with μ, g as in (3.6), due to the continuity of Radon measures [1,6]. Here $\mu(\{0\})$ denotes the μ-measure of the Borel set $\{0\} \subseteq \mathbb{R}$.

Proposition 1 *The space $\mathscr{D}'_{0+,\delta}$ is a unitary convolution subalgebra of $\mathscr{D}'_{0+}$, that is $\mathscr{D}'_{0+,\delta} * \mathscr{D}'_{0+,\delta} = \mathscr{D}'_{0+,\delta}$.*

Proof The proposition is immediate from Equation (3.6) and the following three facts: It holds $\inf \operatorname{supp}(f * g) \geq \inf \operatorname{supp} f + \inf \operatorname{supp} g$ for all $f, g \in \mathscr{D}'_{0+}$, the set $\mathscr{D}'_{0+} \cap \mathscr{D}'^{0}$ is closed with respect to convolution, and $\delta \in \mathscr{D}'_{0+,\delta}$. □

Proposition 2 *The operators* P *and* R *are complementary linear projection operators on $\mathscr{D}'_{0+,\delta}$, that is* $\mathrm{P} \circ \mathrm{P} = \mathrm{P}$, $\mathrm{R} \circ \mathrm{R} = \mathrm{R}$, $\mathrm{R} \circ \mathrm{P} = \mathrm{P} \circ \mathrm{R} = 0$ *and* $\mathrm{P} + \mathrm{R} = \mathrm{E}_{\mathscr{D}'_{0+,\delta}}$. *Every distribution $f \in \mathscr{D}'_{0+,\delta}$ has a unique representation*

$$f = a \cdot \delta + g \qquad \text{with} \quad a \in \mathbb{C},\ g \in \mathscr{D}'_{0+,\not\delta}, \tag{3.7}$$

where a and g are given by $a = \mathrm{V}\, f$ and $g = \mathrm{R}\, f$. In addition,

$$\mathrm{V}\,(f * g) = \mathrm{V}\,(f) \cdot \mathrm{V}\,(g) \qquad \text{for all} \quad f, g \in \mathscr{D}'_{0+,\delta}. \tag{3.8}$$

Proof It is immediate from the Definitions (3.3) that $\mathrm{P}\,\delta = \delta$. Thus, $\mathrm{P} \circ \mathrm{P} = \mathrm{P}$ and the first part of the proposition follows from basic linear algebra. For Equation (3.8), recall the remarks below Equations (3.1) and (3.6). Then use that

$$(\mu * \nu)(\{0\}) = (\mu \otimes \nu)(\{\, (t, -t)\,;\ t \in \mathbb{R}\,\}) = \mu(\{0\}) \cdot \nu(\{0\}) \tag{3.9}$$

for all $\mu, \nu \in \mathscr{D}'^{0} \cap \mathscr{D}'_{0+}$.

The latter follows from Equation (1) in [2, Ch. VIII, §1, No. 1]. □

Equation (3.8) means, that the operator V defines an "augmentation" of the convolution algebra $\mathscr{D}'_{0+,\delta}$. That is, the operator V defines a linear homomorphism $\mathrm{V}\colon \mathscr{D}'_{0+,\delta} \to \mathbb{C}$. Equation (3.8) implies that

$$\mathscr{D}'_{0+,\delta} * \mathscr{D}'_{0+,\not\delta} = \mathscr{D}'_{0+,\not\delta}. \tag{3.10}$$

In particular, $\mathscr{D}'_{0+,\not\delta}$ is a non-unitary convolution subalgebra of $\mathscr{D}'_{0+,\delta}$.

3.2 Generalized initial value operators, eliminators and projectors

Composing the operators V, R and P with the distributional fractional integrals and derivatives from Subsection 2.2 yields operators that act on the Y_γ-part of a distribution in an analogous way. This section studies these operators and, for the case of real orders, the generated algebra of partially defined linear operators.

Definition 2 Let $\gamma \in \mathbb{C}$. The Y_γ*-coefficient operator*, the Y_γ*-projector* and the Y_γ-*eliminator* are defined as the composite operators

$$\mathrm{V}^\gamma := \mathrm{V} \circ \mathrm{D}^\gamma_+, \qquad \mathrm{P}^\gamma := \mathrm{I}^\gamma_+ \circ \mathrm{P} \circ \mathrm{D}^\gamma_+, \qquad \mathrm{R}^\gamma := \mathrm{I}^\gamma_+ \circ \mathrm{R} \circ \mathrm{D}^\gamma_+ . \tag{3.11}$$

Further, one defines the distribution spaces

$$\mathscr{D}'_{\mathrm{Y},\gamma} := \mathrm{I}^\gamma_+ \left(\mathscr{D}'_{0+,\delta}\right), \qquad \mathscr{D}'_{\not\mathrm{Y},\gamma} := \mathrm{I}^\gamma_+ \left(\mathscr{D}'_{0+,\not\delta}\right), \tag{3.12}$$

where $\mathscr{D}'_{0+,\delta}$ and $\mathscr{D}'_{0+,\not\delta}$ are from eqs. (3.2) and (3.5).

It is immediate from the definitions, that the operators I^γ_+ and D^γ_+ induce bijections between spaces $\mathscr{D}'_{\mathrm{Y},\gamma}$ and $\mathscr{D}'_{\not\mathrm{Y},\gamma}$ of different orders. For instance, one has a bijection

$$\mathrm{I}^{\delta-\gamma}_+ \colon \mathscr{D}'_{\mathrm{Y},\gamma} \to \mathscr{D}'_{\mathrm{Y},\delta} \qquad \text{with } \mathrm{I}^{\delta-\gamma}_+ \left(\mathscr{D}'_{\not\mathrm{Y},\gamma}\right) = \mathscr{D}'_{\not\mathrm{Y},\delta} \qquad \text{for all } \gamma, \delta \in \mathbb{C}. \tag{3.13}$$

The domains of the operators from Definition 2 are

$$\operatorname{dom} \mathrm{V}^\gamma = \operatorname{dom} \mathrm{P}^\gamma = \operatorname{dom} \mathrm{R}^\gamma = \mathscr{D}'_{\mathrm{Y},\gamma}. \tag{3.14}$$

Ranges and kernels of these operators are given by

$$\ker \mathrm{V}^\gamma = \ker \mathrm{R}^\gamma = \operatorname{ran} \mathrm{P}^\gamma = \langle Y_\gamma \rangle , \tag{3.15a}$$

$$\ker \mathrm{P}^\gamma = \operatorname{ran} \mathrm{R}^\gamma = \mathscr{D}'_{\not\mathrm{Y},\gamma}. \tag{3.15b}$$

The brackets $\langle - \rangle$ denote the (complex) linear span.

There hold the composition and commutation rules

$$\mathrm{V}^\gamma \circ \mathrm{D}^\delta_+ = \mathrm{V}^{\gamma+\delta}, \qquad \mathrm{R}^\gamma \circ \mathrm{D}^\delta_+ = \mathrm{D}^\delta_+ \circ \mathrm{R}^{\gamma+\delta} \qquad \text{for all } \gamma, \delta \in \mathbb{C}. \tag{3.16}$$

Analogous to the operators V, P and R, one has the relations

$$\mathrm{P}^{\gamma} = Y_{\gamma} \cdot \mathrm{V}^{\gamma}, \qquad \mathrm{R}^{\gamma} = \mathrm{E}_{\mathscr{D}'_{0+}} - \mathrm{P}^{\gamma} \qquad \text{for all } \gamma \in \mathbb{C}. \tag{3.17}$$

The operators P^{γ} and R^{γ} are complementary projections, just as P and R, and a statement analogous to (3.7) holds. However, the operator V^{γ} does not define an augmentation. From Equation (2.8) and Equation (3.10), one obtains the convolution inclusions

$$\mathscr{D}'_{\mathrm{Y},\gamma} * \mathscr{D}'_{\mathrm{Y},\delta} = \mathscr{D}'_{\mathrm{Y},\gamma+\delta}, \qquad \mathscr{D}'_{\mathrm{Y},\gamma} * \mathscr{D}'_{\not{\mathrm{Y}},\delta} = \mathscr{D}'_{\not{\mathrm{Y}},\gamma+\delta} \qquad \text{for all } \gamma, \delta \in \mathbb{C}. \tag{3.18}$$

Lemma 3 *It holds* $\mathrm{I}^{\gamma}_{+}\left(\mathscr{D}'_{0+,\delta}\right) = \mathscr{D}'_{\mathrm{Y},\gamma} \subseteq \mathscr{D}'_{0+,\not\delta}$ *for all* $\gamma \in \mathbb{H}$.

Proof It holds $\mathrm{V}\left(Y_{\gamma} * \mathscr{M}_{0+}\right) = \{0\}$ for $\gamma \in \mathbb{H}$ due to $Y_{\gamma} \in L^1_{\mathrm{loc}}$. □

Corollary 1 *Let* $\gamma, \delta \in \mathbb{C}$. *If* $\Re\gamma < \Re\delta$, *then* $\mathrm{P}^{\gamma}|_{\mathscr{D}'_{\mathrm{Y},\delta}} = \mathbf{0}_{\mathscr{D}'_{\mathrm{Y},\delta}}$ *and* $\mathrm{R}^{\gamma}|_{\mathscr{D}'_{\mathrm{Y},\delta}} = \mathrm{E}_{\mathscr{D}'_{\mathrm{Y},\delta}}$.

Lemma 4 *Every composition of the form*

$$\mathrm{R}^{\gamma_1} \circ \cdots \circ \mathrm{R}^{\gamma_n} \qquad \textit{with } \ \gamma_1, \ldots, \gamma_n \in \mathbb{R}, \ n \in \mathbb{N}_0 \tag{3.19}$$

is equal to a composition of the form

$$\mathrm{R}^{\delta_m} \circ \cdots \circ \mathrm{R}^{\delta_1} \qquad \textit{with } \ \delta_1 < \cdots < \delta_m, \ m \in \mathbb{N}_0, \tag{3.20}$$

where $\delta_k = \gamma_{\sigma(k)}$ *for some function* $\sigma : \{1, \ldots, m\} \to \{1, \ldots, n\}$. *For any partially defined linear operator on* $\mathscr{D}'_{0+}$ *of the form* (3.20) *the parameters* $\delta_1, \ldots, \delta_m$, $m \in \mathbb{N}_0$ *are uniquely determined.*

Proof Corollary 1 implies $\mathrm{R}^{\gamma} \circ \mathrm{R}^{\delta} = \mathrm{R}^{\delta}$ whenever $\delta \geq \gamma$. Using this repeatedly, the expression (3.19) can be reduced to an expression of the form (3.20) without changing the resulting operator. Because every set of the form $\{Y_{\delta_1}, \ldots, Y_{\delta_m}\}$ is linearly independent and the kernel of the operator from Equation (3.20) is equal to $\langle Y_{\delta_1}, \ldots, Y_{\delta_m} \rangle$, it follows that the representation (3.20) is unique. □

Definition 3 Let $\gamma_1 < \cdots < \gamma_n$ and $n \in \mathbb{N}$. The $Y_{\gamma_1,\ldots,\gamma_n}$-*eliminator* and the $Y_{\gamma_1,\ldots,\gamma_n}$-*projector* are defined as

$$\mathrm{R}^{\gamma_1,\ldots,\gamma_n} := \mathrm{R}^{\gamma_n} \circ \cdots \circ \mathrm{R}^{\gamma_1}, \qquad \mathrm{P}^{\gamma_1,\ldots,\gamma_n} := \mathrm{E} - \mathrm{R}^{\gamma_1,\ldots,\gamma_n}. \tag{3.21}$$

The $Y_{\gamma_n|\gamma_1,\ldots,\gamma_{n-1}}$-*coefficient operator* is defined as

$$\mathrm{V}^{\gamma_n|\gamma_1,\ldots,\gamma_{n-1}} := \mathrm{V}^{\gamma_n} \circ \mathrm{R}^{\gamma_1,\ldots,\gamma_{n-1}}. \tag{3.22}$$

The operators $\mathrm{R}^{\gamma_1,\ldots,\gamma_n}$, $\mathrm{P}^{\gamma_1,\ldots,\gamma_n}$ and $\mathrm{V}^{\gamma_n|\gamma_1,\ldots,\gamma_{n-1}}$ have the joint domain

$$\mathscr{D}'_{\Upsilon,\gamma_n} + \langle Y_{\gamma_1}, \ldots, Y_{\gamma_{n-1}} \rangle = \mathscr{D}'_{\not\Upsilon,\gamma_n} + \langle Y_{\gamma_1}, \ldots, Y_{\gamma_n} \rangle . \tag{3.23}$$

For kernels and ranges one has

$$\ker \mathrm{P}^{\gamma_1,\ldots,\gamma_n} = \operatorname{ran} \mathrm{R}^{\gamma_1,\ldots,\gamma_n} = \mathscr{D}'_{\not\Upsilon,\gamma_n}, \tag{3.24a}$$

$$\operatorname{ran} \mathrm{P}^{\gamma_1,\ldots,\gamma_n} = \ker \mathrm{R}^{\gamma_1,\ldots,\gamma_n} = \langle Y_{\gamma_1}, \ldots, Y_{\gamma_n} \rangle . \tag{3.24b}$$

Lemma 5 *Let $\gamma_1 < \cdots < \gamma_n$ and $n \in \mathbb{N}$. Then*

$$\mathrm{R}^{\gamma_1,\ldots,\gamma_n} = \mathrm{E} - \sum_{k=1}^{n} \mathrm{P}^{\gamma_k} \circ \mathrm{R}^{\gamma_1,\ldots,\gamma_{k-1}}, \tag{3.25a}$$

$$\mathrm{P}^{\gamma_n} \circ \mathrm{R}^{\gamma_1,\ldots,\gamma_{n-1}} = \mathrm{R}^{\gamma_1,\ldots,\gamma_{n-1}} - \mathrm{R}^{\gamma_1,\ldots,\gamma_n} . \tag{3.25b}$$

In particular

$$\begin{aligned} &\langle \mathrm{R}^{\gamma_1,\ldots,\gamma_n} ;\ \gamma_1 < \cdots < \gamma_n,\ n \in \mathbb{N}_0 \rangle \\ &= \langle \mathrm{E}, \mathrm{P}^{\gamma_n} \circ \mathrm{R}^{\gamma_1,\ldots,\gamma_{n-1}} ;\ \gamma_1 < \cdots < \gamma_n,\ n \in \mathbb{N} \rangle \end{aligned} \tag{3.26}$$

with the convention $\mathrm{R}^{\gamma_1,\ldots,\gamma_n} = \mathrm{E}$ *for* $n = 0$.

Proof The distributive law (2.17a) and Eq. (3.17) imply Eq. (3.25b). Let $k \in \{1, \ldots, n\}$. Using (3.17) and (2.17a) again one obtains

$$\mathrm{R}^{\gamma_1,\ldots,\gamma_k} = \left[\mathrm{E} - \mathrm{P}^{\gamma_k}\right] \circ \mathrm{R}^{\gamma_1,\ldots,\gamma_{k-1}} = \mathrm{R}^{\gamma_1,\ldots,\gamma_{k-1}} - \mathrm{P}^{\gamma_k} \circ \mathrm{R}^{\gamma_1,\ldots,\gamma_{k-1}} . \tag{3.27}$$

Equation (3.25a) follows from a repeated application of (3.27). □

Proposition 3 *Let $\gamma_1 < \cdots < \gamma_n$, $\delta_1 < \cdots < \delta_m$ and $n, m \in \mathbb{N}$. Then, with $l = \min\{ k = 1, \ldots, n ;\ \gamma_k > \delta_m \}$, it holds*

$$\mathrm{R}^{\gamma_1,\ldots,\gamma_n} \circ \mathrm{R}^{\delta_1,\ldots,\delta_m} = \mathrm{R}^{\delta_1,\ldots,\delta_m,\gamma_l,\ldots,\gamma_n} . \tag{3.28}$$

Proof One applies the cancelation rule $\mathrm{R}^{\gamma} \circ \mathrm{R}^{\delta} = \mathrm{R}^{\delta}$ for $\delta \geq \gamma$. □

3.3 Series expansions in causal power distributions

Absolutely convergent series over expressions $c_\gamma Y_\gamma$, with $c_\gamma \in \mathbb{C}$ and $\gamma \in \mathbb{R}$, are now considered as a kind of fractional Taylor series expansion. Theorem 1 shows how to extract the coefficients c_γ using coefficient operators V^{γ} and composite eliminators $\mathrm{R}^{\gamma_1,\ldots,\gamma_n}$.

Definition 4 Let $U \in \mathscr{D}'_{0+}$ and $\gamma \in \mathbb{R}$. Then U is said to *possess a non-trivial Y_γ-part* if and only if

$$U \in \mathscr{D}'_{\mathrm{Y},\gamma} \text{ and } \mathrm{V}^{\gamma}(U) \neq 0. \tag{3.29}$$

The statement $\mathrm{O}(U) = \gamma$ will be used as equivalent notation for conditions (3.29). The notation $\mathrm{O}(U) \in \mathbb{R}$ means that there exists $\gamma \in \mathbb{R}$, such that conditions (3.29) hold true.

Let $k \in \mathbb{N}_0$. One defines $\mathrm{O}_k(U) := \mathrm{O}(U_k)$, where U_k is recursively defined by the initial condition $U_0 := U$ and

$$U_{l+1} := \mathrm{R}^{\gamma_l}(U_l) \text{ with } \gamma_l := \mathrm{O}(U_l) \qquad \text{for } l = 0, \ldots, k-1, \tag{3.30}$$

whenever $\mathrm{O}(U_0), \ldots, \mathrm{O}(U_k) \in \mathbb{R}$.

Lemma 6 *Let (γ_n) be a sequence of strictly increasing positive numbers and (c_n) a sequence of complex numbers. If the series $\sum_{n=0}^{\infty} c_n Y_{\gamma_n}$ converges absolutely in $\mathscr{D}'$, then it converges absolutely in $L^1_{\mathrm{loc}} \cap \mathscr{E}(\,]0,\infty[\,)$.*

Proof Let $K \subseteq \mathbb{R}$ compact, $\chi \in \mathscr{D}$, $\chi \geq 0$ and $\chi(K) \subseteq \{1\}$. Then

$$\|Y_\gamma\|_{1,K} = \int_{K \cap \mathbb{R}_+} \left| \frac{t^{\gamma-1}}{\Gamma(\gamma)} \right| \mathrm{d}t \leq \langle Y_\gamma, \chi \rangle \qquad \text{for all} \quad \gamma > 0. \tag{3.31a}$$

Let $L \subseteq \,]0,+\infty[$ be compact. There exists $\varepsilon > 0$ such that $L \subseteq \,]\varepsilon,+\infty[$ and a function $\theta \in \mathscr{D}(]0,+\infty[)$, $\theta \geq 0$ and $\theta(L) \subseteq \{1\}$. It holds

$$\|\mathrm{D}^m Y_\gamma\|_{1,L} = \int_L \left| \frac{t^{\gamma-m-1}}{\Gamma(\gamma-m)} \right| \mathrm{d}t \leq |\langle Y_{\gamma-m}, \theta \rangle| = |\langle Y_\gamma, \mathrm{D}^m \theta \rangle| \tag{3.31b}$$

for all $\gamma > 0$ and $m \in \mathbb{N}$ that satisfy $m - \gamma \notin \mathbb{N}_0$. (If $m - \gamma \in \mathbb{N}_0$, then $\mathrm{D}^m Y_\gamma$ vanishes on $]0,+\infty[$.) □

Corollary 2 *If the series from Lemma 6 converges absolutely in $\mathscr{D}'$, then its limit belongs to $\mathscr{D}'_{0+,\mathrm{Y}}$.*

Theorem 1 *Let $(\gamma_k) \subseteq \mathbb{R}$ and $(c_k) \subseteq \mathbb{C}$ such that (γ_k) is strictly increasing and the series $\sum_{k=0}^{\infty} c_k Y_{\gamma_k}$ converges absolutely to the limit U. Then $\mathrm{O}_k(U) = \gamma_k$ and $\mathrm{V}^{\gamma_k} \circ \mathrm{R}^{\gamma_1,\ldots,\gamma_{k-1}}(U) = c_k$ for all $k \in \mathbb{N}_0$.*

Proof For all $k \in \mathbb{N}_0$ it holds $U_k = \sum_{n=k}^{\infty} c_k Y_{\gamma_n} = c_k Y_{\gamma_k} + U_{k+1}$ with $U_{k+1} \in \mathscr{D}'_{\mathrm{Y},\gamma_{k+1}}$ due to Corollary 2. □

Proposition 4 *Let $U, V \in \mathscr{D}'_{0+}$ such that $\mathrm{O}(U)$ and $\mathrm{O}(V)$ exist. Then $\mathrm{O}(U * V)$ exists as well and $\mathrm{O}(U * V) = \mathrm{O}(U) + \mathrm{O}(V)$.*

Proof Equation (3.18) implies

$$(c \cdot Y_\gamma + \mathscr{D}'_{\mathbb{Y},\gamma}) * (d \cdot Y_\delta + \mathscr{D}'_{\mathbb{Y},\delta}) = (c \cdot d) \cdot Y_{\gamma+\delta} + \mathscr{D}'_{\mathbb{Y},\gamma+\delta} \tag{3.32}$$

for all $\gamma, \delta, c, d \in \mathbb{C}$, which proves the proposition. □

Lemma 7 *Let* $\alpha_1 < \cdots < \alpha_n, 0 < \beta_1 < \cdots < \beta_m, \lambda_1, \ldots, \lambda_n \in \mathbb{C}^\times, \mu_1, \ldots, \mu_m \in \mathbb{C}^\times$, $m \in \mathbb{N}_0$ *and* $n \in \mathbb{N}$. *Then*

$$\mathrm{O}\left(\ast \frac{\lambda_1 Y_{\alpha_1} + \cdots + \lambda_n Y_{\alpha_n}}{\delta + \mu_1 Y_{\beta_1} + \cdots + \mu_m Y_{\beta_m}} \right) = \alpha_1. \tag{3.33}$$

Proof The limit U^{*-1} of the series in (2.10) satisfies $\mathrm{O}\left(U^{*-1}\right) = 0$ and one has $\mathrm{O}\left(\lambda_1 Y_{\alpha_1} + \cdots + \lambda_n Y_{\alpha_n}\right) = \alpha_1$ according to Theorem 1. Thus, Equation (3.33) follows follows from Proposition 4. □

4 Generalized sequential fractional derivatives

The generalized sequential fractional derivatives are introduced and their basic properties summarized in Subsection 4.1. Then, in Subsection 4.2, restrictions and extrapolations of measures and distributions are defined. Subsection 4.3 discusses examples of generalized sequential fractional derivatives and their relations to other Fractional Calculus operators on the right half axis from the literature.

4.1 Definition and fundamental properties

Generalized sequential fractional derivatives are introduced as catenations of a distributional fractional derivative and a composite of eliminators.

Definition 5 Let $\alpha \in \mathbb{R}, \gamma_1 < \cdots < \gamma_n$ and $n \in \mathbb{N}$. The *sequential fractional derivative of order* α *and sequential type* $(\gamma_1, \ldots, \gamma_n)$ is defined as

$$\mathrm{D}_{0+}^{\alpha|\gamma_1,\ldots,\gamma_n} := \mathrm{D}_+^\alpha \circ \mathrm{R}^{\gamma_1,\ldots,\gamma_n} \tag{4.1}$$

where $\mathrm{D}_{0+}^{\alpha|} := \mathrm{D}_+^\alpha$ is used to link the notation for sequential fractional derivatives with that for simple fractional derivatives.

The operators defined in Equation (4.1) coincide with alternating compositions of eliminators and fractional derivatives with real orders. This means operators

$$\mathrm{D}_+^{\alpha_0} \circ \mathrm{R} \circ \cdots \circ \mathrm{R} \circ \mathrm{D}_+^{\alpha_n} \qquad \text{with} \quad \alpha_0, \ldots, \alpha_n \in \mathbb{R}. \tag{4.2}$$

Note, that $\mathrm{R} \circ \mathrm{D}_+^\alpha \circ \mathrm{R} = \mathrm{D}_+^\alpha \circ \mathrm{R}$ holds for all $\alpha < 0$, so that $\alpha_1, \ldots, \alpha_{n-1} > 0$ may be assumed without loss of generality. The expression on the right hand side of Equation (4.1) constitutes a normal form of the operators from Equation (4.2). The advantage of the normal form from Equation (4.1) is that domain and kernel are given by

$$\operatorname{dom} \mathrm{D}_{0+}^{\alpha|\gamma_1,\dots,\gamma_n} = \operatorname{dom} \mathrm{R}^{\gamma_1,\dots,\gamma_n}, \qquad \ker \mathrm{D}_{0+}^{\alpha|\gamma_1,\dots,\gamma_n} = \ker \mathrm{R}^{\gamma_1,\dots,\gamma_n}, \tag{4.3}$$

because the operator D_{+}^{α} is bijective. Thus, domain and kernel are given by Equation (3.23) and (3.24a).

Another normal form is obtained when the operators D_{+}^{α} and $\mathrm{R}^{\gamma_1,\dots,\gamma_n}$ are commuted. Let $\alpha_0, \dots, \alpha_n \in \mathbb{R}$ with $\alpha_1, \dots, \alpha_{n-1} > 0$. Using the commutation rules (3.16) the relations between different normal forms can be derived. They read

$$\mathrm{D}_{+}^{\alpha_n} \circ \mathrm{R} \circ \cdots \circ \mathrm{R} \circ \mathrm{D}_{+}^{\alpha_0} = \mathrm{D}_{+}^{\alpha} \circ \mathrm{R}^{\gamma_1,\dots,\gamma_n} = \mathrm{R}^{\delta_1,\dots,\delta_n} \circ \mathrm{D}_{+}^{\alpha} \tag{4.4a}$$

with the parameters

$$\alpha := \alpha_0 + \cdots + \alpha_n, \qquad \gamma_k := \sum_{l=0}^{k-1} \alpha_l, \qquad \delta_k := -\sum_{l=n+1-k}^{n} \alpha_l \tag{4.4b}$$

for $k = 1, \dots, n$. Using Equations (3.17) and (3.25a) one obtains

$$\mathrm{D}_{0+}^{\alpha|\gamma_1,\dots,\gamma_n} = \mathrm{D}_{+}^{\alpha} - \sum_{k=1}^{n} Y_{\gamma_k - \alpha} \cdot \mathrm{V}^{\gamma_k|\,\gamma_1,\dots,\gamma_{k-1}} \tag{4.4c}$$

for $\alpha \in \mathbb{R}$, $\gamma_1 < \cdots < \gamma_n$, $n \in \mathbb{N}$.

Proposition 5 *Let $\alpha, \beta \in \mathbb{R}$, $\gamma_1 < \cdots < \gamma_n$, $\delta_1 < \cdots < \delta_m$ and $m, n \in \mathbb{N}_0$. Then the composition law*

$$\mathrm{D}_{0+}^{\alpha|\gamma_1,\dots,\gamma_n} \circ \mathrm{D}_{0+}^{\beta|\delta_1,\dots,\delta_m} = \mathrm{D}_{0+}^{\alpha+\beta|\delta_1,\dots,\delta_m,\gamma_l+\beta,\dots,\gamma_n+\beta} \tag{4.5}$$

holds with $l = \min\{\, k \in 1, \dots, n\,;\ \gamma_k + \beta > \delta_m \,\}$.

Proof The commutation rules from Eq. (3.16) give

$$\mathrm{D}_{0+}^{\alpha|\gamma_1,\dots,\gamma_m} \circ \mathrm{D}_{0+}^{\beta|\delta_1,\dots,\delta_n} = \mathrm{D}_{+}^{\alpha+\beta} \circ \mathrm{R}^{\gamma_1+\beta,\dots,\gamma_m+\beta} \circ \mathrm{R}^{\delta_1,\dots,\delta_n} \tag{4.6}$$

and an application of Proposition 3 yields Equation (4.5). □

Remark 2 The normal form (4.4c) exists as well for composite operators of the form $\mathrm{R}^{\gamma} \circ \mathrm{C}_K$ with $\gamma \in \mathbb{R}$ and $K \in \mathscr{Q}[Y_{\mathbb{R}}]$. It holds

$$\mathrm{R}^{\gamma} \circ \mathrm{C}_K = \mathrm{C}_K - Y_{\gamma} \cdot \mathrm{V}^{\mathrm{O}(K)}(K) \cdot \mathrm{V}^{\gamma - \mathrm{O}(K)}. \tag{4.7}$$

Equation (4.7) holds also when $K \in \mathscr{D}'_{\mathrm{Y},\mathrm{O}(K)}$. However, even for $K \in \mathscr{A}[Y_{\mathbb{R}}]$ a similar description of composites of the form $\mathrm{R}^{\gamma_1,\dots,\gamma_n} \circ \mathrm{C}_K$ leads to a more complicated expression that involves multiple case distinctions. Let us remark, that for future studies of the operators $\mathrm{C}_K \circ \mathrm{R}^{\beta_1,\dots,\beta_n}$, it seems plausible to generalize the eliminators as well.

4.2 Restriction and extrapolation of distributions

Let $I \subseteq \mathbb{R}$ be an interval and $J \subseteq \mathbb{R}$ an *open* interval. The restriction of a distribution $f : \mathscr{D}(\mathbb{R}) \to \mathbb{C}$ to J is defined as $f|_J := f|_{\mathscr{D}_J} : \mathscr{D}_J \to \mathbb{C}$ with $\mathscr{D}_J = \{\varphi \in \mathscr{D}\,;\ \operatorname{supp}\varphi \subseteq J\}$. The restriction of a Radon measure $\mu : \mathscr{K}(\mathbb{R}) \to \mathbb{C}$ to I is defined by $\mu|_I(\varphi) := \mu(\varphi|_{\mathbb{R}}^{\text{zero}})$ with $\varphi|_{\mathbb{R}}^{\text{zero}}(x)$ equal to 0 for $x \in I$ and equal to $\varphi(x)$ for $x \notin I$. Here, the Borel measurable function $\varphi|_{\mathbb{R}}^{\text{zero}}$ is the zero extrapolation of $\varphi \in \mathscr{K}(I)$.

The restriction of a distribution $f \in \mathscr{D}'_{0+,\delta}$ to $\mathbb{R}_{0+}$ is defined as the linear form $f : \mathscr{D}(\mathbb{R}_{0+}) \to \mathbb{C}$ such that

$$f|_{\mathbb{R}_{0+}}(\varphi) := \mu|_{\mathbb{R}_{0+}}(\varphi) + g|_{\mathbb{R}_+}((\chi\varphi)|_{\mathbb{R}}^{\text{zero}}) \qquad \text{for } \varphi \in \mathscr{D}(\mathbb{R}_{0+}), \tag{4.8}$$

where μ and g are defined as in (3.6), $\mathscr{D}(\mathbb{R}_{0+}) = \{\varphi|_{\mathbb{R}_{0+}}\,;\ \varphi \in \mathscr{D}(\mathbb{R})\}$ and $\chi \in \mathscr{E}(\mathbb{R}_{0+})$ is such that $\chi(\operatorname{supp} g) \subseteq \{1\}$ and $\chi([0,\varepsilon]) \subseteq \{0\}$ for some $\varepsilon > 0$. This is well defined due to $0 \notin \operatorname{supp} g$.

For $I = \mathbb{R}_+$, $\alpha \in \mathbb{R}$, or, $I = \mathbb{R}_{0+}$, $\alpha \geq 0$, define the space

$$\mathscr{D}'_{\Upsilon,\alpha}(I) := \left\{ f|_I\,;\ f \in \mathscr{D}'_{\Upsilon,\alpha} \right\}, \qquad \mathscr{D}'_{\cancel{\Upsilon},\alpha}(I) := \left\{ f|_I\,;\ f \in \mathscr{D}'_{\cancel{\Upsilon},\alpha} \right\}. \tag{4.9}$$

with $\mathscr{D}'_{\Upsilon,\alpha}$, $\mathscr{D}'_{\cancel{\Upsilon},\alpha}$ defined in Eq. (3.12).

Define the *zero extrapolation of a Radon measure* $\mu \in \mathscr{K}(I)$ *to* $\mathbb{R}$ as

$$\mu|_{\mathbb{R}}^{\text{zero}}(\varphi) := \mu(\varphi|_I) \qquad \text{for all } \varphi \in \mathscr{K}(\mathbb{R}) \text{ and } I \subseteq \mathbb{R} \text{ an interval}. \tag{4.10}$$

The *zero extrapolation* $f|_{\mathbb{R}}^{\text{zero}}$ *of* $f \in \mathscr{D}'_{0+,\delta}(\mathbb{R}_+)$ is defined as

$$\left\langle f|_{\mathbb{R}}^{\text{zero}}, \varphi \right\rangle := \mu(\varphi|_{\mathbb{R}_+}) + g(\theta\varphi), \tag{4.11}$$

where μ and g are defined as in (3.6) and $\theta \in \mathscr{E}(\mathbb{R}_+)$ is such that $\theta^{-1}(1) \supseteq \operatorname{supp}\varphi$ and $\operatorname{supp}\theta \subseteq \mathbb{R}_+$. Define the *zero extrapolation of a distribution* $f \in \mathscr{D}'_{0+,\delta}(\mathbb{R}_{0+})$ *to* $\mathbb{R}$ as

$$f|_{\mathbb{R}}^{\text{zero}} := \mu|_{\mathbb{R}}^{\text{zero}} + g|_{\mathbb{R}}^{\text{zero}}, \tag{4.12}$$

with μ resp. g as in the representation formula (3.6) and their zero extrapolations defined by Equations (4.10) resp. (4.11). The *continuous constant extrapolation* $f|_{\mathbb{R}}^{\text{c.c.}}$ *of* f is defined as

$$f|_{\mathbb{R}}^{\text{c.c.}} := 1_{\mathbb{R}_-} \cdot \mathrm{V}^1(f) + f|_{\mathbb{R}}^{\text{zero}} \qquad \text{for } f \in \mathscr{D}'_{\Upsilon,1}(\mathbb{R}_+) \cup \mathscr{D}'_{\Upsilon,1}(\mathbb{R}_{0+}), \tag{4.13}$$

where $1_{\mathbb{R}_-}$ denotes the indicator function for the open left half axis.

Restricting a distribution from $\mathscr{D}'_{\mathrm{Y},0}$ to $\mathbb{R}_+$ and extending to $\mathbb{R}$ by zero afterwards has the same effect as applying the eliminator. Restricting to $\mathbb{R}_{0+}$ and zero-extrapolating to $\mathbb{R}$ has no effect. This can be expressed as

$$\left(f|_{\mathbb{R}_+}\right)\big|_{\mathbb{R}}^{\text{zero}} = \mathrm{R}(f) \qquad \text{for all } f \in \mathscr{D}'_{\mathrm{Y},0}, \tag{4.14a}$$

$$\left(f|_{\mathbb{R}_{0+}}\right)\big|_{\mathbb{R}}^{\text{zero}} = f \qquad \text{for all } f \in \mathscr{D}'_{\mathrm{Y},0}. \tag{4.14b}$$

Because R reduces to the identity on $\mathscr{D}'_{0+,\not{g}}$, the mappings

$$\mathscr{D}'_{0+,\not{g}}(\mathbb{R}_{0+}) \to \mathscr{D}'_{0+,\not{g}}(\mathbb{R}_+), \qquad f \mapsto f|_{\mathbb{R}_+} := \left(f|_{\mathbb{R}}^{\text{zero}}\right)\big|_{\mathbb{R}_+}, \tag{4.15a}$$

$$\mathscr{D}'_{0+,\not{g}}(\mathbb{R}_+) \to \mathscr{D}'_{0+,\not{g}}(\mathbb{R}_{0+}), \qquad f \mapsto f|_{\mathbb{R}_{0+}}^{\text{zero}} := \left(f|_{\mathbb{R}}^{\text{zero}}\right)\big|_{\mathbb{R}_{0+}}, \tag{4.15b}$$

define mutually inverse isomorphisms that restrict to isomorphisms between $\mathscr{D}'_{\mathrm{Y},\alpha}$ and $\mathscr{D}'_{\mathrm{Y},\alpha}$ for $\alpha > 0$.

4.3 Examples

Numerous operators can be reinterpreted as generalized sequential fractional derivatives on subspaces of $\mathscr{D}'_{0+}$ in the context of restriction to the right half axis and extrapolation to the full real axis. This section collects some examples.

Distributional derivatives on the right half axis

The distributional derivative is well defined as an operator acting on distributions defined on the open right half axis $\mathbb{R}_+$. Using restriction and zero extrapolation operators the sequential derivative $\mathrm{D}_{0+}^{1|1}$ can be related to the operator $\mathrm{D}\colon \mathscr{D}'(\mathbb{R}_+) \to \mathscr{D}'(\mathbb{R}_+)$ via

$$\mathrm{D}\, f = \mathrm{D}\left(f|_{\mathbb{R}}^{\text{zero}}\right)\big|_{\mathbb{R}_+} \qquad \text{for all } f \in \mathscr{D}'_{\mathrm{Y},1}(\mathbb{R}_+), \tag{4.16a}$$

$$\mathrm{D}_{0+}^{1|1} f = \mathrm{D}\left(f|_{\mathbb{R}_+}\right)\big|_{\mathbb{R}}^{\text{zero}} \qquad \text{for all } f \in \mathscr{D}'_{\mathrm{Y},1}(\mathbb{R}). \tag{4.16b}$$

The sequential derivative $\mathrm{D}_{0+}^{1|1} f$ of $f \in \mathscr{D}'_{\mathrm{Y},1}(\mathbb{R})$ is a modification of the distributional derivative of f that ignores jumps at the origin.

According to (4.16b), this is equivalent to interpreting f as a distribution on the open right half axis and extrapolating by zero after the calculation of the derivative.

Unfortunately, the case of the *closed* right half axis is more involved, despite the existence of the isomorphism $\mathscr{D}'_{\mathrm{Y},1}(\mathbb{R}_+) \to \mathscr{D}'_{\mathrm{Y},1}(\mathbb{R}_{0+})$. The reason is, that the definition of the derivative of a function $f \in \mathscr{D}'_{\mathrm{Y},1}(\mathbb{R}_{0+})$ at the origin depends on its extrapolation to a neighborhood of the origin. In particular, it holds

$$\left(\mathrm{D}\left(f|_{\mathbb{R}}^{\mathrm{zero}}\right) - \mathrm{D}\left(f|_{\mathbb{R}}^{\mathrm{c.c.}}\right)\right)\Big|_{\mathbb{R}_{0+}} = \left(\delta \cdot \mathrm{V}^1\left(f|_{\mathbb{R}}^{\mathrm{zero}}\right)\right)\Big|_{\mathbb{R}_{0+}} \tag{4.17a}$$

$$\mathrm{D}_{0+}^{1|1}\left(f|_{\mathbb{R}}^{\mathrm{zero}}\right)\Big|_{\mathbb{R}_{0+}} = \mathrm{D}\left(f|_{\mathbb{R}}^{\mathrm{c.c.}}\right)\big|_{\mathbb{R}_{0+}} \tag{4.17b}$$

for all $f \in \mathscr{D}'_{\mathrm{Y},1}(\mathbb{R}_{0+})$. These relations reflect the fact that it is always necessary give a precise interpretation of the derivatives when a boundary is involved.

Riemann–Liouville fractional integrals

Riemann–Liouville fractional integrals with orders $\alpha \in \mathbb{H}$, that act on functions on the closed or open right half axis, can be represented using the sequential derivatives $\mathrm{D}_{0+}^{-\alpha|} = \mathrm{I}_{+}^{\alpha}$. Note, that the Riemann–Liouville integral $_{\mathrm{RL}}\,\mathrm{I}_{0+}^{\alpha}\,\mu(t)$ of a measure $\mu \in \mathscr{M}(\mathbb{R})$ or $\mu \in \mathscr{M}(\mathbb{R}_{+})$ can be defined, in the almost everywhere sense for the variable t, by applying the classical formula to μ. (In the latter case, the domains of integration must include the point zero.) This extends most common definitions of the Riemann–Liouville integral (compare [24, Remark 4.4]).

With these definitions one obtains the relations

$$\mathrm{I}_{+}^{\alpha}\left(\mu|_{\mathbb{R}}^{\mathrm{zero}}\right) = \left({}_{\mathrm{RL}}\mathrm{I}_{0+}^{\alpha}\,\mu\right)\Big|_{\mathbb{R}}^{\mathrm{zero}} \qquad \text{for all } \mu \in \mathscr{M}(\mathbb{R}_{+}) \cup \mathscr{M}(\mathbb{R}_{0+}), \tag{4.18a}$$

$$\left(\mathrm{I}_{+}^{\alpha}\,\mu\right)\big|_{\mathbb{R}_{0+}} = {}_{\mathrm{RL}}\mathrm{I}_{0+}^{\alpha}\left(\mu|_{\mathbb{R}_{0+}}\right) \qquad \text{for all } \mu \in \mathscr{D}'_{0+,\delta}(\mathbb{R}) \cap \mathscr{M}(\mathbb{R}), \tag{4.18b}$$

$$\left(\mathrm{I}_{+}^{\alpha}\,\mu\right)\big|_{\mathbb{R}_{+}} = {}_{\mathrm{RL}}\mathrm{I}_{0+}^{\alpha}\left(\mu|_{\mathbb{R}_{+}}\right) \qquad \text{for all } \mu \in \mathscr{D}'_{0+,\not\delta}(\mathbb{R}) \cap \mathscr{M}(\mathbb{R}). \tag{4.18c}$$

However, the action of the sequential derivative $\mathrm{D}_{0+}^{-\alpha|} = \mathrm{I}_{+}^{\alpha}$ on functions $f \in \mathscr{D}'_{\mathrm{Y},-\alpha} \setminus \mathscr{D}'_{0+,\delta}$ can not be described using Riemann–Liouville integrals defined on the closed or open right half axis, because neither the corresponding restriction operators nor the corresponding extrapolation operators are well defined for such functions f.

Generalized Riemann–Liouville fractional derivatives [7]

The fractional derivatives of Riemann–Liouville and their generalizations can be reinterpreted by replacing the first order derivatives with $\mathrm{D}_{0+}^{1|1}$ and $_{\mathrm{RL}}\,\mathrm{I}_{0+}^{1-\alpha}$ with $\mathrm{D}_{0+}^{\alpha-1|}$. Specifically, the generalized fractional derivative $\mathrm{D}_{0+}^{\alpha,\mu}$ of order α and type μ with $\alpha \in \,]0,1]$, $\mu \in [0,1]$ can be interpreted as a generalized sequential derivative

$$\mathrm{D}_{0+}^{\alpha;\mu} := \mathrm{D}_{0+}^{\alpha|\alpha+\mu-\alpha\mu} = \mathrm{I}_{+}^{\mu(1-\alpha)} \circ \mathrm{D}_{0+}^{1|1} \circ \mathrm{I}_{+}^{(1-\mu)(1-\alpha)} \tag{4.19}$$

of order α and sequential type $\alpha + \mu - \alpha\mu$. The sequential reinterpretation $\mathrm{D}_{0+}^{\alpha;\mu}$ is distinguished notationally by a semicolon instead of a comma from the original operator $\mathrm{D}_{0+}^{\alpha,\mu}$. The domain of $\mathrm{D}_{0+}^{\alpha;\mu}$ is

$$\mathrm{dom}\ \mathrm{D}_{0+}^{\alpha;\mu} = \mathscr{D}'_{\mathrm{Y},1-\alpha\mu}. \tag{4.20}$$

With the notations from Equation (4.19) the well-known relation [7, p.434] between two derivatives of same order $\alpha \in]0,1]$, but distinct types $0 \leq \mu_1 < \mu_2 \leq 1$ reads

$$\mathrm{D}_{0+}^{\alpha;\mu_1} = \mathrm{D}_{0+}^{\alpha;\mu_2} - Y_{\mu_2(1-\alpha)} \cdot \mathrm{V}^{\alpha+\mu_2(1-\alpha)} . \tag{4.21a}$$

The relation is immediate from the fact that

$$\mathrm{D}_{0+}^{\alpha;\mu_i} = \mathrm{D}_{+}^{\alpha} - Y_{\mu_i(1-\alpha)} \cdot \mathrm{V}^{\alpha+\mu_i(1-\alpha)} \qquad \text{for } i = 1, 2, \tag{4.21b}$$

according to Equation (4.4c), because the partially defined linear form $\mathrm{V}^{\alpha+\mu_1(1-\alpha)}$ reduces to zero on the domain of partially defined linear form $\mathrm{V}^{\alpha+\mu_2(1-\alpha)}$.

*n*th-level fractional derivatives [5,24]

The sequential fractional derivatives from Definition 5 extend earlier definitions as studied in [5], or later ones, in [24]. In the following, the operators from Definition 5 are compared with the "*n*th-level derivatives" from Definition 3.6 in [24]. Similar remarks apply to the sequential derivatives given in [31]. For the purpose of a more convenient comparison the domains X^1_{nL} of nth-level derivatives are defined as spaces of functions on the whole real line. This definition becomes equal to the definition in Equation (3.41) from [24] when the functions are restricted to $]0, 1[$.

More precisely, let $0 < \alpha \leq 1, \gamma = (\gamma_1, \ldots, \gamma_n)$ such that $\gamma_k \geq 0$ and $\alpha + s_k \leq k$ for all $k = 1, \ldots, n$, with the notation $s_k := \sum_{i=1}^{k} \gamma_i$. The indices for the corresponding generalized sequential fractional derivatives are defined as $\delta_{n-k+1} := \alpha + s_k - k + 1$. These satisfy $\delta_1 < \cdots < \delta_n$ if and only if $\gamma_k < 1$ for $k = 2, \ldots, n-1$. Further, it holds $0 < \delta_1$ if and only if $\alpha + s_n > n + 1$.

Consider now, the space $X^1_{nL,+}(\mathbb{R})$, defined as

$$X^1_{nL,+}(\mathbb{R}) := \left\{ f \in \operatorname{dom} \mathrm{D}_{0+}^{\alpha|\delta_1,\ldots,\delta_n} ;\ \mathrm{D}_{0+}^{\alpha|\delta_1,\ldots,\delta_n} f \in L^1_{\mathrm{loc}}(\mathbb{R}) \right\} . \tag{4.22}$$

The space X^1_{nL} from Definition 3.6 in [24] can be characterized as the set of restrictions $f|_{]0,1[}$ with $f \in X^1_{nL,+}(\mathbb{R})$ by using the following lemma.

Lemma 8 *For every $f \in \mathscr{D}'_{Y,1}(\mathbb{R})$ there holds the equivalence*

$$f|_{\mathbb{R}_{0+}} \in \mathrm{AC}(\mathbb{R}_{0+}) \quad \Leftrightarrow \quad f \in \mathrm{AC}(\mathbb{R}) \quad \Leftrightarrow \quad \mathrm{D}_{0+}^{1|1} f \in L^1_{\mathrm{loc}}(\mathbb{R}). \tag{4.23}$$

The operator ${}_{n\mathrm{L}}\,\mathrm{D}_{0+}^{\alpha,(\gamma)}$ with $\operatorname{dom} {}_{n\mathrm{L}}\,\mathrm{D}_{0+}^{\alpha,(\gamma)} = X^1_{nL,+}(\mathbb{R})$ and the parameters $\gamma = (\gamma_1, \ldots, \gamma_n)$ can now be defined as in [24, Def. 3.6], but for the whole right half axis. With the comparison of partially defined operators from (2.13), one has

$${}_{n\mathrm{L}}\,\mathrm{D}_{0+}^{\alpha,(\gamma)} \subseteq \mathrm{D}_{0+}^{\alpha|\delta_1,\ldots,\delta_n} . \tag{4.24}$$

The advantage of the different parameterization from Definition 5 is, that the domain and the kernel of the operators $\mathrm{D}_{0+}^{\alpha|\delta_1,\ldots,\delta_n}$ can be deducted directly from the indices

$\delta_1, \ldots, \delta_n$. Compare Equations (3.23), (3.24) and (4.3) with Equation (3.45) from [24]. Further, Definition 5 allows more general indices because the only condition is $\delta_1 < \cdots < \delta_n$ and it is always guaranteed that the derivative is "truly nth-level".

Let $0 < \gamma_k \le 1$ and define $\sigma_k := \sum_{k=1}^{n} \gamma_k - 1$, $k = 1, \ldots, n$. Similar to the nth-level derivatives, the Dzherbashian-Nersesian derivatives ${}_{\mathrm{DN}}\mathrm{D}_{0+}^{\sigma_n}$, defined in Equation (2.5) from [5], satisfy

$$ {}_{\mathrm{DN}}\mathrm{D}_{0+}^{\sigma_k} \subseteq \mathrm{D}_{0+}^{\sigma_k|\sigma_0+1,\ldots,\sigma_{(k-1)}+1} \qquad \text{for} \quad k = 0, 1, \ldots, n. \tag{4.25} $$

Generalized fundamental theorem of fractional calculus

The composition law from Proposition 5 implies:

Corollary 3 *Let $\alpha \in \mathbb{R}$ and $\gamma_1 < \cdots < \gamma_n$. Then*

$$ \mathrm{D}_{0+}^{\alpha|\gamma_1,\ldots,\gamma_n} \circ \mathrm{I}_{+}^{\alpha} = \mathrm{R}^{\gamma_1-\alpha,\ldots,\gamma_n-\alpha}, \tag{4.26a} $$

$$ \mathrm{I}_{+}^{\alpha} \circ \mathrm{D}_{0+}^{\alpha|\gamma_1,\ldots,\gamma_n} = \mathrm{R}^{\gamma_1,\ldots,\gamma_n}. \tag{4.26b} $$

Equations (4.26) correspond to the Fundamental Theorem of Fractional Calculus. More specifically, Equation (4.26b) implies the projector formula [25, Theorem 2]. The fact that the right-hand side of Equation (4.26a) is not the identity operator on a function space does *not* contradict the Fundamental Theorem of Fractional Calculus from [24, Theorem 3.4]. Because, under suitable assumptions on the parameters $\alpha, \gamma_1, \ldots, \gamma_n$ and when the distributional translation-invariant Riemann–Liouville operator I_{+}^{α} in Equation (4.26a) is restricted to a classical domain of the Riemann–Liouville integral on the half axis, then the right hand side becomes an identity operator on this restricted domain.

This can be expressed by the slightly more general relations

$$ \mathrm{D}_{0+}^{\alpha|\gamma_1,\ldots,\gamma_n} \circ \mathrm{I}_{+}^{\alpha}\big|_{\mathscr{D}'_{0+,\delta}} = \mathrm{E}_{\mathscr{D}'_{0+,\delta}} \qquad \text{if } 0 < \gamma_1 < \cdots < \gamma_n < \alpha, \tag{4.27a} $$

$$ \mathrm{D}_{0+}^{\alpha|\gamma_1,\ldots,\gamma_n} \circ \mathrm{I}_{+}^{\alpha}\big|_{\mathscr{D}'_{0+,\not\delta}} = \mathrm{E}_{\mathscr{D}'_{0+,\not\delta}} \qquad \text{if } 0 < \gamma_1 < \cdots < \gamma_n \le \alpha. \tag{4.27b} $$

The domain $\mathscr{D}'_{0+,\not\delta}$ contains all domains that are commonly used to define Riemann–Liouville operators on the right half axis, as discussed on page 17. In particular, it holds $\mathscr{D}'_{0+,\not\delta} \supseteq X_{FT,+}(\mathbb{R})$, where $X_{FT,+}(\mathbb{R})$ is defined as

$$ \left\{ f \in \mathscr{D}'_{0+}(\mathbb{R}) \cap L^1_{\mathrm{loc}}(\mathbb{R})\,;\ \mathrm{I}_{+}^{\alpha} f \in \mathrm{AC}(\mathbb{R}),\ \mathrm{V}^1(\mathrm{I}_{+}^{\alpha} f) = 0 \right\}. \tag{4.28} $$

Using Lemma 8 it follows that the space X_{FT} from [24, Eq. (3.18)] is characterized by the restrictions $f|_{]0,1[}$ with $f \in X_{FT,+}(\mathbb{R})$. Thus, (4.27b) implies Theorem 3.4 from [24].

Generalized derivatives with Sonine kernels

Generalizations of fractional derivatives and integrals where the kernels Y_α are replaced by pairs of kernels K and L from $L^1_{\mathrm{loc}} \cap \mathscr{D}'_{0+}$ that satisfy the Sonine relations $K * L = Y_1$ were introduced in [18] and recently discussed in [21,29]. The Sonine relations imply

$$\mathrm{D}(K * L) = (\mathrm{D}\, K) * L = K * (\mathrm{D}\, L) = \delta. \tag{4.29}$$

The convolution operator C_K and the composite

$${}_L\mathrm{D}^1_{0+} := \mathrm{C}_L \circ \mathrm{R}^1 \tag{4.30}$$

correspond to the generalized integral and the generalized derivative of Caputo type in [29, Eq. (26), (20)]. Equations (4.29) and (4.7) imply

$$\mathrm{C}_K \circ {}_L\mathrm{D}^1_{0+} = \mathrm{C}_K \circ \mathrm{C}_L \circ \mathrm{R}^1 = \mathrm{R}^1, \tag{4.31a}$$

$${}_L\mathrm{D}^1_{0+} \circ \mathrm{C}_K = \mathrm{C}_L \circ \mathrm{R}^1 \circ \mathrm{C}_K = \mathrm{E} - Y_1 \cdot \mathrm{V}^1(K) \cdot \mathrm{V}\,. \tag{4.31b}$$

From similar considerations as for Equations (4.27) it is found that the Equations (4.31) imply Theorem 1 from [29].

Two counterexamples

The operators of Caputo-Fabrizio [4, Eq.(1)] and Atangana-Baleanu-Caputo [4, Eq. (2)] ${}_{\mathrm{CF}}\,\mathrm{D}^\alpha_{0+}$ and ${}_{\mathrm{ABC}}\,\mathrm{D}^\alpha_{0+}$, can be represented as composites of convolution operators with kernels from $\mathscr{Q}[Y_\mathbb{R}]$ and eliminators. Explicitly, one obtains

$${}_{\mathrm{CF}}\,\mathrm{D}^\alpha_{0+} = \mathrm{C}_{({}_{\mathrm{CF}}K_\alpha)^{*-1}} \circ \mathrm{R}^1 \tag{4.32a}$$

with ${}_{\mathrm{CF}}K_\alpha = (\alpha \cdot \delta + (1-\alpha) \cdot Y_1)/M(\alpha)$, and

$${}_{\mathrm{ABC}}\,\mathrm{D}^\alpha_{0+} = \mathrm{C}_{({}_{\mathrm{ABC}}K_\alpha)^{*-1}} \circ \mathrm{R}^1 \tag{4.32b}$$

with ${}_{\mathrm{ABC}}K_\alpha = (\alpha \cdot \delta + (1-\alpha) \cdot Y_\alpha)/B(\alpha)$ and normalization constants $M(\alpha), B(\alpha) \in \mathbb{R}$. The formulas follow immediately from the form of the derivative operators in Equations (1) and (3) in [4] and the formulas for the corresponding operators, ${}_{\mathrm{CF}}\,\mathrm{J}^\alpha_{0+} = \mathrm{C}_{{}_{\mathrm{CF}}K_\alpha}$ and ${}_{\mathrm{ABC}}\,\mathrm{J}^\alpha_{0+} = \mathrm{C}_{{}_{\mathrm{ABC}}K_\alpha}$, from Equations (5) and (9) in [4]. An application of Equation (4.7) yields the relations

$${}_{\mathrm{CF}}\,\mathrm{D}^\alpha_{0+} \circ {}_{\mathrm{CF}}\,\mathrm{J}^\alpha_{0+} = \mathrm{E} - (\delta + \tfrac{\alpha}{1-\alpha} \cdot Y_{-1})^{*-1} \cdot \mathrm{V}^1, \tag{4.33a}$$

$${}_{\mathrm{ABC}}\,\mathrm{D}^\alpha_{0+} \circ {}_{\mathrm{ABC}}\,\mathrm{J}^\alpha_{0+} = \mathrm{E} - (\delta + \tfrac{\alpha}{1-\alpha} \cdot Y_{-\alpha})^{*-1} \cdot \mathrm{V}^1, \tag{4.33b}$$

that correspond to Equations (6) and (10) from [4]. This shows, once again, that these operators do not satisfy a Fundamental Theorem such as (4.31).

5 Generalized sequential fractional initial value problems

The main purpose of this section is to study the kernels of generalized sequential fractional differential operators. Subsection 5.1 provides an algorithm (Theorem 2) that can be used to simplify linear combinations of sequential derivatives to an eventually simpler normal form. The result is used in Subsection 5.2 to derive a representation formula for kernels (Theorem 3) and a characterization of the maximal injective domains (Theorem 4). The case of sequential derivatives with distinct order is considered in Subsection 5.3 in more detail. This yields a simplified criterion for kernel functions in general (Theorem 5) and a full characterization for important special cases (Theorem 6). Last, in Subsection 5.4, all possible cases with two types are described.

5.1 Simplifying linear combinations of sequential derivatives

Due to annihilation effects evaluating linear combinations of partially defined linear operators can be cumbersome. The theorem to be established below provides an algorithm that reduces sums of composites $\mathrm{C}_K \circ \mathrm{R}^{\Gamma}$ of convolution operators C_K and composite eliminators R^{Γ} to a simple expression. Due to the theorem, the operators to be investigated can be assumed to be given in the form (5.3).

Note, that every finite subset $\Gamma \subseteq \mathbb{R}$ can be represented as $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ with unique $\gamma_1 < \cdots < \gamma_n$ and $n \in \mathbb{N}_0$. Thus, the notations

$$\mathrm{R}^{\Gamma} := \mathrm{R}^{\gamma_1,\dots,\gamma_n}, \qquad \mathrm{P}^{\Gamma} := \mathrm{P}^{\gamma_1,\dots,\gamma_n} \tag{5.1}$$

are well defined with the conventions $\mathrm{R}^{\emptyset} = \mathrm{E}$ and $\mathrm{P}^{\emptyset} = \mathbf{0}$.

Theorem 2 *Every partially defined linear operator C of the form*

$$C = \sum_{k=1}^{n} \mathrm{C}_{U_k} \circ \mathrm{R}^{\Gamma_k} \tag{5.2}$$

with $U_1, \dots, U_n \in \mathscr{D}'_{0+}$, $\Gamma_1, \dots, \Gamma_n \subseteq \mathbb{R}$ finite and $n \in \mathbb{N}_0$, can be written

$$C = \sum_{l=0}^{m} \mathrm{C}_{V_l} \circ \mathrm{R}^{\gamma_1,\dots,\gamma_l} \tag{5.3a}$$

with unique convolution kernels $V_0, \dots, V_m \in \mathscr{D}'_{0+}$, types $\Gamma = \{\gamma_1, \dots, \gamma_m\}$ and a step number $m \in \mathbb{N}_0$, given by

$$\Gamma = \left\{ \gamma \in \mathbb{R}\,; \begin{array}{l} \exists k \in \{1, \dots, n\} : \gamma \in \Gamma_k\,, \\ \forall k \in \{1, \dots, n\} : \gamma \in \Gamma_k \text{ or } \gamma > \sup \Gamma_k \end{array} \right\}, \tag{5.3b}$$

$$V_l = \sum \left(U_k : k = 1, \dots, n \mid \gamma_l = \sup(\Gamma_k \cap \Gamma) \right), \tag{5.3c}$$

$$V_0 = \sum \left(U_k : k = 1, \dots, n \mid \Gamma_k \cap \Gamma = \emptyset \right), \tag{5.3d}$$

for $l = 1, \dots, m$.

Proof *Reducibility to the normal form:* Let $n \in \mathbb{N}_0$. Define the total number of eliminators arising in expression (5.2) as $\Sigma := \sum_{k=1}^{n} \#\Gamma_k$. The reducibility will be proved via of induction over Σ. If $\Sigma = 0$ or $n = 0$ the expression (5.2) obviously has the form (5.3a). Thus, the Lemma holds for $\Sigma = 0$ or $n = 0$.

Assume now, that $\Sigma > 0$ and $n > 0$. The induction hypothesis is, that the statement of the Lemma holds for all expressions of the form (5.2) that have a total number of eliminators $\Sigma' < \Sigma$. Consider the order parameter

$$\gamma_1 := \max\{\, \min \Gamma_k\,;\ k \in \{1, \ldots, n\},\ \Gamma_k \neq \emptyset\,\}\,. \tag{5.4}$$

In the expression (5.2), all operators R^{γ} with $\gamma < \gamma_1$ can be canceled from the right due to (2.14), (3.14) and Corollary 1. A new expression of the form (5.2) emerges with a reduced total number of eliminators $\Sigma' < \Sigma$. Thus, the induction hypothesis applies to the new expression and it thus reduces to the form (5.3a).

If no eliminator can be canceled in this way, then for all $k = 1, \ldots, n$ either $\gamma_1 = \max \Gamma_k$ or $\Gamma_k = \emptyset$. Then, the operator R^{γ_1} can be factored out as

$$C = \sum_{k=1}^{n} \mathrm{C}_{U_k} \circ \mathrm{R}^{\Gamma_k} = \left[\sum_{\substack{k=1 \\ \Gamma_k \neq \emptyset}}^{n} \mathrm{C}_{U_k} \circ \mathrm{R}^{\Gamma_k \setminus \{\gamma_1\}} \right] \circ \mathrm{R}^{\gamma_1} + \sum_{\substack{k=1 \\ \Gamma_k = \emptyset}}^{n} \mathrm{C}_{U_k}\,. \tag{5.5}$$

The expression inside the brackets has the form (5.2) and satisfies $\Sigma' < \Sigma$. Thus, the induction hypothesis applies and the whole expression (5.5) can be written in the form (5.3a) by inserting an expression of the form (5.3a) into the brackets in Equation (5.5) and factoring out all composed eliminators.

Uniqueness of the normal form: According to equations (3.23) and (4.3) it holds

$$\sup\left\{\, \gamma \in \mathbb{R}\,;\ Y_\gamma \notin \operatorname{dom} C \,\right\} = \gamma_m \tag{5.6}$$

and therefore $Y_\gamma \in \operatorname{dom} C$ for all $\gamma \in \mathbb{R}$ with $\gamma \geq \gamma_m$. Further, (3.23) and (4.3) imply

$$\{\gamma_1, \ldots, \gamma_m\} = \left\{\, \gamma \in \mathbb{R}\,;\ Y_\gamma \in \operatorname{dom} C,\ \gamma \leq \gamma_m \,\right\}. \tag{5.7}$$

Thus, the orders $\gamma_1 < \cdots < \gamma_m$ and the number m are uniquely determined by C. For $\gamma \in \mathbb{R}$ one calculates

$$Y_{-\gamma} * C(Y_\gamma) = Y_{-\gamma} * \sum_{l=0}^{m} \mathrm{C}_{U_l} \circ \mathrm{R}^{\gamma_1, \ldots, \gamma_l}(Y_\gamma) = \begin{cases} \sum_{l=0}^{m} U_l & \text{if } \gamma > \gamma_m, \\ \sum_{l'=m-l+1}^{m} U_{l'} & \text{if } \gamma = \gamma_l. \end{cases} \tag{5.8a}$$

From this equations it is clear how to represent the convolution kernels U_l, $l = 0, \ldots, m$ as linear combinations of expressions $Y_{-\gamma} * C(Y_\gamma)$. Therefore, the U_l are uniquely determined by the operator C. ◻

Example 1 Corresponding to the problem from [3] consider

$$D = \sum_{k=1}^{n} \lambda_k \, \mathrm{D}_{0+}^{\alpha_k|\alpha_k-\lceil\alpha_k\rceil+1,\ldots,\alpha_k-1,\alpha_k} \tag{5.9a}$$

with $\lambda_1, \ldots, \lambda_{n-1} \in \mathbb{C}^\times$, $\lambda_n = 1$, $0 < \alpha_1 < \cdots < \alpha_n$ and $n \in \mathbb{N}$. An application of Theorem 2 yields

$$D = \sum_{k=1}^{n^*} \lambda_k \, \mathrm{D}_{0+}^{\alpha_k|} + \sum_{k=n^*+1}^{n} \lambda_k \, \mathrm{D}_{0+}^{\alpha_k|\alpha_n-\lceil\alpha_n-\alpha_{n^*}\rceil+1,\ldots,\alpha_k}, \tag{5.9b}$$

where n^* is the largest k with $\alpha_n - \alpha_k \notin \mathbb{N}$ if it exists and $n^* = n$ otherwise. Thus, the proof of Theorem 2 generalizes Lemma 2 from [3].

5.2 The structure of the kernel

Consider an operator C in the form of Equation (5.3a). Changing to a different normal form makes it easy to obtain certain structural results on its kernel $\ker C$. Using Lemma 5 one calculates directly that

$$C = \sum_{l=0}^{m} \mathrm{C}_{V_l} \circ \mathrm{R}^{\gamma_1,\ldots,\gamma_l} = \mathrm{C}_{W_0} - \sum_{l=1}^{m} \mathrm{C}_{W_l} \circ \mathrm{P}^{\gamma_l} \circ \mathrm{R}^{\gamma_1,\ldots,\gamma_{l-1}} \tag{5.10a}$$

for all $V_0, \ldots, V_m, W_0, \ldots, W_m \in \mathscr{D}'_{0+}$, $\gamma_1 < \cdots < \gamma_m$ and $m \in \mathbb{N}_0$ with

$$W_l = \sum_{l'=l}^{m} V_{l'} \qquad \text{for } l = 0, \ldots, m, \tag{5.10b}$$

$$V_l = W_l - W_{l+1} \qquad \text{for } l = 0, \ldots, m-1, \tag{5.10c}$$

where the convention $W_{m+1} = 0$ applies. Equation (5.10b) is equivalent to Equation (5.10c). The described transformation works as well under the restrictions that the distributions $V_0, \ldots, V_m$ and $W_0, \ldots, W_m$ all belong to $\mathscr{Q}[Y_\mathbb{R}]$, or belong to $\mathscr{A}[Y_\mathbb{R}]$.

For the remainder of this subsection let C be an operator of the form of Equation (5.10) such that the distributions $V_0, \ldots, V_m$, or, equivalently, the distributions $W_0, \ldots, W_m$, belong to the convolution field $\mathscr{Q}[Y_\mathbb{R}]$. Define the number

$$l_0 = l_0(C) := \min\{\, l = 0, \ldots, m\,;\; W_l \neq 0 \,\}\,. \tag{5.11}$$

With these notations, the structure of $\ker C$ is characterized by the following.

Theorem 3 *Every distribution* $K \in \operatorname{dom} C$ *satisfies*

$$K \in \ker C \quad \Leftrightarrow \quad K \in \sum_{l=l_0+1}^{m} K_l \cdot L_l(K) + \left\langle Y_{\gamma_1}, \ldots, Y_{\gamma_{l_0}} \right\rangle, \tag{5.12a}$$

where K_l *and* L_l *are defined as*

$$K_l := \begin{cases} Y_{\gamma_l} * *\dfrac{W_l}{W_{l_0}} & \text{if } l > l_0, \\ Y_{\gamma_l} & \text{if } l \le l_0, \end{cases} \tag{5.12b}$$

$$L_l := \mathrm{V}^{\gamma_l|\gamma_1,\ldots,\gamma_{l-1}}. \tag{5.12c}$$

In particular, the kernel satisfies the inclusions

$$\left\langle Y_{\gamma_1}, \ldots, Y_{\gamma_{l_0}} \right\rangle \subseteq \ker C \subseteq \langle K_1, \ldots, K_m \rangle. \tag{5.13a}$$

Proof The right hand side of (5.10) can be rewritten as

$$\left(\mathrm{C}_{W_{l_0}} - \sum_{l=l_0+1}^{m} \mathrm{C}_{W_l} \circ \mathrm{P}^{\gamma_l} \circ \mathrm{R}^{\gamma_{l_0+1},\ldots,\gamma_{l-1}} \right) \circ \mathrm{R}^{\gamma_1,\ldots,\gamma_{l_0}}. \tag{5.14}$$

Let $K \in \operatorname{dom} C$ and set $\tilde{K} = \mathrm{R}^{\gamma_1,\ldots,\gamma_{l_0}}(K)$. Applying the operator from (5.14) to K and setting the result to zero yields

$$W_{l_0} * \tilde{K} - \sum_{l=l_0+1}^{m} W_l * Y_{\gamma_l} * L_l(\tilde{K}) = 0. \tag{5.15}$$

Solving for $\tilde{K}$ in Eq. (5.15) and noting that the distributions $Y_{\gamma_1}, \ldots, Y_{\gamma_{l_0}}$ belong to $\ker C$, due to (5.14), proves the proposition. □

The following theorem provides the maximal domains for admissible inhomogeneities g of the equation $Cf = g$.

Theorem 4 *The operator* C *from (5.10) restricts to a bijection*

$$C|_{\mathscr{D}'_{\mathbb{Y},\gamma_m}} : \mathscr{D}'_{\mathbb{Y},\gamma_m} \to \mathscr{D}'_{\mathbb{Y},\gamma_m+\mathrm{O}(W_0)}, \quad f \mapsto W_0 * f. \tag{5.16}$$

Proof This is immediate from the formula (4.3) for kernels of sequential derivatives and Proposition 4. □

5.3 Sums of sequential derivatives with distinct orders

In the following, those generalized fractional sequential differential operators are considered that are sums of generalized fractional sequential derivatives with distinct orders. That means, an operator D of the form

$$D := \sum_{k=0}^{n} \lambda_k \, \mathrm{D}_{0+}^{\alpha_k|\gamma_1,\dots,\gamma_{\sigma(k)}} = \mathrm{R}^{\gamma_1,\dots,\gamma_{\sigma(0)}} + \sum_{k=1}^{n} \lambda_k \, \mathrm{D}_{0+}^{\alpha_k|\gamma_1,\dots,\gamma_{\sigma(k)}} \tag{5.17}$$

is considered with parameters $n, m \in \mathbb{N}_0$, $0 = \alpha_0 < \alpha_1 \cdots < \alpha_n$, $1 = \lambda_0, \lambda_1, \dots, \lambda_n \in \mathbb{C}^\times$, $\gamma_1 < \cdots < \gamma_m$ and a function $\sigma : \{0, \dots, n\} \to \{0, \dots, m\}$ such that $m = \max\{\, \sigma(k)\,;\ k = 0, \dots, n\,\}$. In order to exclude trivial cases, $\min\{\, \sigma(k)\,;\ k = 0, \dots, n\,\} = 0$ is assumed as well.

Using Theorem 3 the kernel of the operator D can be expressed in terms of the functions K_l and the composed coefficient operators L_l. The distributions K_l are given by

$$K_l := Y_{\gamma_l} * W_l * (W_0)^{*-1} \qquad \text{for} \quad l = 1, \dots, m, \tag{5.18a}$$

and the distributions W_l are given by

$$W_l = Y_{-\alpha_n} * \sum_{\substack{k=0 \\ \sigma(n-k) \ge l}}^{n} Z_k \qquad \text{for} \quad l = 0, \dots, m, \tag{5.18b}$$

with the notation $Z_k := \mu_k Y_{\beta_k}$ and the parameters $\beta_k := \alpha_n - \alpha_{n-k}$ and $\mu_k := \lambda_{n-k}/\lambda_n$ for $k = 0, \dots, n$.

In the following, elementary properties of the kernel distributions K_l are summarized. That will help to shed more light on the structure of $\ker D$. After an examination of Equation (5.18b), one concludes that

$$K_l \in \begin{cases} Y_{\gamma_l} + \mathscr{D}'_{\not{Y},\gamma_l} & \text{if } l = 1, \dots, \sigma(n), \\ \mathscr{D}'_{\not{Y},\gamma_l} & \text{if } l = \sigma(n) + 1, \dots, n. \end{cases} \tag{5.19}$$

Let $l, l' \in \{1, \dots, m\}$. Equation (5.19) yields the implications

$$l \le l' \quad \Rightarrow \quad K_{l'} \in \operatorname{dom} L_l, \tag{5.20a}$$
$$l < l' \quad \Rightarrow \quad K_{l'} \in \ker L_l, \tag{5.20b}$$

and the equation

$$L_l(K_l) = \begin{cases} 1 & \text{if } l = 1, \dots, \sigma(n), \\ 0 & \text{if } l = \sigma(n) + 1, \dots, m. \end{cases} \tag{5.21}$$

Note, that $\mathrm{O}(K_l) < \mathrm{O}(K_{l'})$ holds for all $l \in \{1, \ldots, \sigma(n)\}$, $l' \in \{1, \ldots, m\}$ with $l < l'$ according to Eq. (5.19). However, if $l \in \{\sigma(n)+1, \ldots, m-1\}$, then $\mathrm{O}(K_l) < \mathrm{O}(K_{l'})$ does not necessarily hold. Nevertheless, the functions $K_1, \ldots, K_l$ are linearly independent. This is because in the expression for W_l from Equation (5.18b), as l increases the number of terms Z_k reduces or $\mathrm{O}(W_l)$ increases.

Theorem 5 *Let $K = \sum_{l=1}^{m} a_l K_l$ with $a_1, \ldots, a_m \in \mathbb{C}$. Then*

1. *It holds $K \in \operatorname{dom} D$ if and only if*

$$\sum_{i=1}^{l-1} a_i \cdot K_i \in \operatorname{dom} L_l \qquad \text{for all } l = 1, \ldots, m. \tag{5.22}$$

2. *It holds $K \in \ker D$ if and only if $K \in \operatorname{dom} D$ and*

$$L_l\left(\sum_{i=1}^{l-1} a_i \cdot K_i\right) = \begin{cases} 0 & \text{if } l \leq \sigma(n), \\ a_l & \text{if } l > \sigma(n), \end{cases} \qquad \text{for all } l = 1, \ldots, m. \tag{5.23}$$

Proof Due to Equation (5.20a), and because $\operatorname{dom} D$ is linear, $K \in \operatorname{dom} D$ is equivalent to (5.22). Inserting $K \in \operatorname{dom} D$ into Equation (5.12a) yields Equation (5.23) after canceling all terms that are zero according to Equation (5.21). □

Corollary 4 *The estimate* $\dim\ker D \leq \sigma(n)$ *holds.*

Corollary 5 *If $l' \in \{1, \ldots, m\}$ is such that $K_l \notin \ker L_{l+1}$ for all $l = 1, \ldots, l'$, then* $\ker D \subseteq \langle K_{l'+1}, \ldots, K_m \rangle$.

Theorem 6 *The kernel of D satisfies* $\dim\ker D = m = n$ *if the conditions $\sigma(n) = m$ and $\beta_{\varphi(l,1)} > \gamma_m - \gamma_l$, $l = 1, \ldots, m-1$ hold, where $\varphi(l, 1) :=$* $\min\{k \in \{1, \ldots, n\}\,;\, \sigma(n-k) < l\}$.

Proof Introducing a term $W_0 * (W_0)^{*-1}$ one calculates

$$K_l = Y_{\gamma_l} * (W_l - W_0) * (W_0)^{*-1} + Y_{\gamma_l}. \tag{5.24}$$

Therefore, $\mathrm{O}_1\left(K_l - Y_{\gamma_l}\right) = \mathrm{O}(W_l - W_0) = \beta_{\varphi(l,1)}$ and the distributions $K_1, \ldots, K_m$ satisfy the condition (5.23) from Theorem 5. □

Example 2 The operator D from Equation (5.9b) satisfies the requirements of Theorem 6. Thus, for linear combinations of Riemann–Liouville operators the cancellation of the dimension of the kernel is already explained by the cancellation effects described in Theorem 3.

Example 3 Let $\tilde{L}$ be the operator from Equation (2.16) in [5] for the special case of constant coefficients $\tilde{p}_0, \ldots, \tilde{p}_n$. For simplicity, assume that the $\tilde{p}_k$ are non-zero. (Within this remark, symbols x from [5] are decorated as $\tilde{x}$.) Recalling Equation (4.25) the operator $\mathrm{D}_+^{-\tilde{\sigma}_0} \circ \tilde{L}$ can be identified with an operator of the form D as in Equation (5.17) above with the definitions $n = \tilde{n} + 1$, $m = \tilde{n}$, $\alpha_1 = -\tilde{\sigma}_0$, $\alpha_k = \tilde{\sigma}_{k-1} - \tilde{\sigma}_0$ for $k = 2, \ldots, n$, $\gamma_l = \tilde{\sigma}_{l-1} + 1$ for $l = 1, \ldots, m$ and $\lambda_1 = 1$, $\lambda_k = \tilde{p}_{\tilde{n}+1-k}$ for $k = 2, \ldots, n-2$, $\lambda_{n-1} = \tilde{p}_{\tilde{n}-1}$, $\lambda_n = \tilde{p}_{\tilde{n}}$.

One verifies, that $\varphi(l, 1) = n + 1 - l$ and thus $\beta_{\varphi(l,1)} = \tilde{\sigma}_{\tilde{n}} - \tilde{\sigma}_l > \tilde{\sigma}_{\tilde{n}-1} - \tilde{\sigma}_l$ for all $l = 1, \ldots, m$. Therefore, Theorem 6 and Theorem 3 imply the existence result [5, Thm. 4] for a trivial inhomogeniety $\tilde{f} = 0$. From $\mathrm{O}\,(W_0) = -\alpha_n$, the relation

$$\gamma_m - \alpha_n = \tilde{\sigma}_{\tilde{n}-1} - \tilde{\sigma}_{\tilde{n}} + 1 + \tilde{\sigma}_0 = 1 - \tilde{\gamma}_n + \tilde{\sigma}_0, \tag{5.25}$$

and Theorem 4 one finds that $\mathrm{D}_+^{-\tilde{\sigma}_0} \circ \tilde{L}$ restricts to a bijection

$$\mathscr{D}'_{\mathbb{Y},\tilde{\sigma}_{\tilde{n}-1}+1} \to \mathscr{D}'_{\mathbb{Y},1-\tilde{\gamma}_{\tilde{n}}+\tilde{\sigma}_0}. \tag{5.26}$$

This extend the result from [5, Thm. 4] to inhomogeneities $\tilde{f} \in \mathscr{D}'_{\mathbb{Y},1-\tilde{\gamma}_{\tilde{n}}}$.

The restriction $\tilde{\gamma}_0 > 1 - \tilde{\gamma}_n$ is required in [5, Thm. 4] in order to ensure that $Y_{\tilde{\sigma}_k+1} = \mathrm{I}_+^{1-\tilde{\gamma}_n}(Y_{\tilde{\sigma}_k+\tilde{\gamma}_n})$ is well defined as a classical Riemann–Liouville integral. The condition becomes superflous in the generalized sequential setting because it is not required that $Y_{\tilde{\sigma}_k+\tilde{\gamma}_n}$ is locally integrable.

5.4 Characterization of the kernel for two primary orders

In the following analysis the representation formula

$$\ast\frac{W_l}{W_0} = \varepsilon(l) \cdot \delta + (-1)^{\varepsilon(l)} \cdot \ast\frac{Z_{\varphi(l,1)} + \cdots + Z_{\varphi(l,\Phi(l))}}{\delta + Z_1 + \cdots + Z_n} \tag{5.27a}$$

is used with the notations

$$\varepsilon(l) := 1_{(l \le \sigma(n))} = \begin{cases} 1 & \text{if } l \le \sigma(n), \\ 0 & \text{if } l > \sigma(n), \end{cases} \tag{5.27b}$$

$$\Phi(l) := \begin{cases} \#\{k \in \{1, \ldots, n\};\ \sigma(n-k) < l\} & \text{if } l \le \sigma(n), \\ \#\{k \in \{1, \ldots, n\};\ \sigma(n-k) \ge l\} & \text{if } l > \sigma(n), \end{cases} \tag{5.27c}$$

$$\varphi(l, i) := i\text{-th element of } \begin{cases} \{k \in \{1, \ldots, n\};\ \sigma(n-k) < l\} & \text{if } l \le \sigma(n), \\ \{k \in \{1, \ldots, n\};\ \sigma(n-k) \ge l\} & \text{if } l > \sigma(n), \end{cases} \tag{5.27d}$$

for $i = 1, \ldots, \Phi(l)$ and $l = 1, \ldots, m$. Further, let

$$\zeta(l) := \max\{i \in \{1, \ldots, \Phi(l)\};\ \forall j < i : Z_{\varphi(l,j+1)} = Z_{\varphi(l,1)} \ast Z_j\} \tag{5.28}$$

and define the notation

$$\delta_{l,l'} := \gamma_{l+l'} - \gamma_l \qquad \text{for } l \in \{1,\dots,m\},\ l' \in \{1,\dots,m-l\}. \tag{5.29}$$

Lemma 9 *Let $U_1,\dots,U_n, V_1,\dots,V_m \in \mathscr{Q}[Y_{\mathbb{R}}] \setminus \{0\}$ with $m \le n$, $m,n \in \mathbb{N}$ and assume, that $0 < \mathrm{O}(U_1) < \dots < \mathrm{O}(U_n)$ and $\mathrm{O}(V_1) < \dots < \mathrm{O}(V_m)$. Then*

$$\ast\frac{\sum_{l=1}^m V_l}{\delta + \sum_{k=1}^n U_k} \in V_1 + \begin{cases} V_{l'+1} - V_1 \ast U_{l'} + \mathscr{D}'_{\mathrm{Y},\mathrm{O}(V_{l'+1}-V_1\ast U_{l'})} & \text{if } l' < m, \\ \qquad\quad - V_1 \ast U_m + \mathscr{D}'_{\mathrm{Y},\mathrm{O}(V_1\ast U_m)} & \text{if } l' = m, \end{cases} \tag{5.30}$$

where l' is the largest k' such that $V_{k+1} - V_1 \ast U_k = 0$ for all $k < k'$.

Proof The assumptions imply $\delta + U_1 + \dots + U_n \neq 0$ and one obtains

$$\begin{aligned} \ast\frac{\sum_{l=1}^m V_l}{\delta + \sum_{k=1}^n U_k} = V_1 + \ast\frac{V_2 - V_1 \ast U_1 + \dots + V_m - V_1 \ast U_{m-1}}{\delta + U_1 + \dots + U_n} \\ - V_1 \ast \quad \ast\frac{U_m + \dots + U_n}{\delta + U_1 + \dots + U_n}. \end{aligned} \tag{5.31}$$

The assumptions and Proposition 4 imply that

$$\mathrm{O}(V_{l+1} - V_1 \ast U_l) < \mathrm{O}\left(V_{l'+1} - V_1 \ast U_{l'}\right) \quad \text{for } l' = l+1,\dots,m-1, \tag{5.32a}$$

$$\mathrm{O}(V_{l+1} - V_1 \ast U_l) < \mathrm{O}(V_1 \ast U_m). \tag{5.32b}$$

With these considerations Equation (5.30) is immediate from (5.31). □

Lemma 10 *For all $l = 1,\dots,m$ one has*

$$\begin{aligned} \ast\frac{W_l}{W_0} &\in \varepsilon(l)\cdot\delta + (-1)^{\varepsilon(l)}\cdot Z_{\varphi(l,1)} + \dots \\ &(-1)^{\varepsilon(l)}\cdot \begin{cases} Z_{\varphi(l,\zeta(l)+1)} - Z_{\varphi(l,1)} \ast Z_{\zeta(l)} + \mathscr{D}'_{\mathrm{Y},\overline{\gamma}_l} & \text{if } \zeta(l) < \Phi(l), \\ \qquad\qquad\quad - Z_{\varphi(l,1)} \ast Z_{\Phi(l)} + \mathscr{D}'_{\mathrm{Y},\overline{\gamma}_l} & \text{if } \zeta(l) = \Phi(l), \end{cases} \end{aligned} \tag{5.33a}$$

with the order parameter

$$\overline{\beta}_l := \begin{cases} \min\{\beta_{\varphi(l,\zeta(l)+1)},\ \beta_{\varphi(l,1)} + \beta_{\zeta(l)}\} & \text{if } \zeta(l) < \Phi(l), \\ \beta_{\varphi(l,1)} + \beta_{\Phi(l)} & \text{if } \zeta(l) = \Phi(l), \end{cases} \tag{5.33b}$$

where β_k was defined just after Eq. (5.18b).

Proof One applies Lemma 9 to Equation (5.27a) with $U_k = Z_k$, $k = 1,\dots,n$ and $V_k = Z_{\varphi(l,k)}$, $k = 1,\dots,\Phi(l)$. □

An application of Lemma 10 yields:

Lemma 11 *Let* $l \in \{1, \dots, m-1\}$. *Then*

$$K_l \in \operatorname{dom} L_{l+1} \quad \Leftrightarrow \quad \beta_{\varphi(l,1)} \geq \delta_{l,1}. \tag{5.34}$$

If $K_l \in \operatorname{dom} L_{l+1}$, *then*

$$L_{l+1}(K_l) = (-1)^{\varepsilon(l)} \cdot \begin{cases} 0 & \text{if } \beta_{\varphi(l,1)} > \delta_{l,1}, \\ \mu_{\varphi(l,1)} & \text{if } \beta_{\varphi(l,1)} = \delta_{l,1}. \end{cases} \tag{5.35}$$

In particular, it holds

$$K_l \in \ker L_{l+1} \quad \Leftrightarrow \quad \beta_{\varphi(l,1)} > \delta_{l,1}. \tag{5.36}$$

Example 4 Combining Theorem 5 and Lemma 11 a complete characterization of ker D can be obtained. The possible cases are given by

$$\ker D = \begin{cases} \{0\} & \text{if } \sigma(n) = 0, \\ \begin{cases} \langle K_1 \rangle & \text{if } \beta_{\varphi(1,1)} > \delta_{1,1}, \\ \left\langle K_1 - \mu_{\varphi(1,1)} \cdot K_2 \right\rangle & \text{if } \beta_{\varphi(1,1)} = \delta_{1,1}, \\ \{0\} & \text{if } \beta_{\varphi(1,1)} < \delta_{1,1}, \end{cases} & \text{if } \sigma(n) = 1, \\ \begin{cases} \langle K_1, K_2 \rangle & \text{if } \beta_{\varphi(1,1)} > \delta_{1,1}, \\ \langle K_2 \rangle & \text{if } \beta_{\varphi(1,1)} \leq \delta_{1,1}, \end{cases} & \text{if } \sigma(n) = 2. \end{cases} \tag{5.37}$$

A minimal example for the first, and trivial, case is the operator

$$D = \mathrm{R}^{\gamma_1,\gamma_2} + \lambda\, \mathrm{D}^{\alpha} \qquad \text{with } 0 < \alpha,\ \gamma_1 < \gamma_2,\ \lambda \in \mathbb{C}^{\times}. \tag{5.38}$$

The cases with $\sigma(n) = 1$ are covered by the minimal examples

$$D = \mathrm{R}^{\gamma_1,\gamma_2} + \lambda_1\, \mathrm{D}_{+}^{\alpha_1} + \lambda_2\, \mathrm{D}_{0+}^{\alpha_2|\gamma_1} \qquad \text{with } \begin{cases} \alpha_2 - \alpha_1 > \gamma_2 - \gamma_1, \\ \alpha_2 - \alpha_1 = \gamma_2 - \gamma_1, \\ \alpha_2 - \alpha_1 < \gamma_2 - \gamma_1, \end{cases} \tag{5.39}$$

where $0 < \alpha_1 < \alpha_2$, $\gamma_1 < \gamma_2$ and $\lambda_1, \lambda_2 \in \mathbb{C}^{\times}$. The cases with $\sigma(n) = 2$ are covered by the examples

$$D = \mathrm{E} + \lambda\, \mathrm{D}_{0+}^{\alpha|\gamma_1,\gamma_2} \qquad \text{with } \begin{cases} \alpha > \gamma_2 - \gamma_1, \\ \alpha \leq \gamma_2 - \gamma_1, \end{cases} \tag{5.40}$$

where $0 < \alpha$, $\gamma_1 < \gamma_2$ and $\lambda \in \mathbb{C}^{\times}$.

Example 5 Linear combinations of a first order time derivative and a generalized Riemann–Liouville fractional derivative were used in [10] as infinitesimal generators for composite fractional time evolutions. Reinterpreting the operator $\mathrm{D}_{0+}^{\alpha,\mu}$ in [10] along the lines of eq. (4.19) as $\mathrm{D}_{0+}^{\alpha;\mu}$ the solution $f(t), t \geq 0$ of the initial value problem (32) in [10] can be represented as the restriction $f := K|_{\mathbb{R}_{0+}}$ of a distribution $K \in \mathscr{D}'_{0+}$, where the distribution K is the solution to the following generalized initial value problem: Define the operator

$$D = \lambda_2\,\mathrm{D}_{0+}^{1|1} + \lambda_1\,\mathrm{D}_{0+}^{\alpha|\alpha+\mu-\alpha\mu} + \mathrm{E}, \tag{5.41a}$$

where $\lambda_1 = (\tau_\alpha)^\alpha, \lambda_2 = \tau_1$ and the parameters $\tau_\alpha, \tau_1 > 0, 0 < \alpha < 1$ and $0 \leq \mu \leq 1$. According to Theorem 2 the equality

$$D = \lambda_2\mathrm{D}_{0+}^{1|1} + \lambda_1\,\mathrm{D}_{0+}^{\alpha|} + \mathrm{E}, \tag{5.41b}$$

holds if and only if $\mu \neq 1$ (corresponding to Equation (34) in [10]). In any case, there exists a unique distribution $K \in \mathscr{D}'_{0+}$ such that

$$K \in \ker D, \qquad\qquad \mathrm{V}^1\,K = 1. \tag{5.41c}$$

The relation $R = S * \check{Y}_1$ between the normalized relaxation function $R = K + \check{Y}_1$ and the time-domain representation of the corresponding normalized susceptibility S was verified using convolutional calculus in [17]. Here, $\check{Y}_1$ denotes the reflection of Y_1. The Laplace-transform of S (in the sense of [33]) defines the susceptibility function $\varepsilon(u)$ in the frequency domain.

Another interpretation of the infinitesimal generator D from [10] was recently given in [11]. It is obtained from observing $\mathrm{D}_{0+}^{\alpha;\mu} = \mathrm{R}^{\mu-\mu\alpha} \circ \mathrm{D}_{0+}^{\alpha|}$ and shifting $\mathrm{R}^{\mu-\mu\alpha}$ from $\mathrm{D}_{0+}^{\alpha|}$ to the infinitesimal generator $\mathrm{D}_{0+}^{1|1}$ of translations in eq. (5.41). The physical motivation for this are relaxation processes that are too fast to be resolved [11]. The modified interpretation leads to a sequential first order derivative

$$\mathrm{R}^{\mu-\mu\alpha} \circ \mathrm{D}_{0+}^{1|1} = \mathrm{R}^{\mu-\mu\alpha} \circ \mathrm{R}^0 \circ \mathrm{D}_+^1 = \mathrm{D}_+^1 \circ \mathrm{R}^\delta \circ \mathrm{R}^1 = \mathrm{D}_{0+}^{1|1,\delta} \tag{5.42}$$

with $\delta = 1+\mu(1-\alpha)$. Instead of D and problem (5.41) the modified fractional initial value problem involves the modified operator

$$\tilde{D} = \lambda_2\,\mathrm{D}_{0+}^{1|1,\delta} + \lambda_1\,\mathrm{D}_{0+}^{\alpha|} + \mathrm{E} \tag{5.43a}$$

where $\lambda_1 = (\tau_\alpha)^\alpha$, $\lambda_2 = \tau_1$, $\delta = 1 + \mu(1-\alpha)$ with the parameters $\tau_\alpha, \tau_1 > 0$, $0 < \alpha < 1$ and $0 < \mu < 1$. Note, that $1 - \delta < 1 - \alpha$. Therefore, Theorem 6 guarantees the existence of a unique distribution $\tilde{K} \in \mathscr{D}'_{0+}$ such that

$$\tilde{K} \in \ker \tilde{D}, \qquad \mathrm{V}^1\,\tilde{K} = 1, \qquad \mathrm{V}^{1,\delta}\,\tilde{K} = -\nu/\lambda_2. \tag{5.43b}$$

with $\nu = (\tau_{\mathrm{ab}})^\delta$. Theorem 3 and Equations (5.18) yield the solution

$$\tilde{K} = \ast\frac{\lambda_2\updelta - \nu Y_{\delta-1}}{\updelta + \lambda_1 Y_{-\alpha} + \lambda_2 Y_{-1}}. \tag{5.44}$$

As above, the relaxation motion $\tilde{R}$ satisfies $\tilde{R} = \tilde{K} + \check{Y}_1 = \tilde{S} * \check{Y}_1$, with the normalized susceptibility $\tilde{S}$ given by the convolution quotient

$$\tilde{S} = \ast\frac{\delta + \lambda_1 Y_{-\alpha} + \nu Y_{\delta-2}}{\delta + \lambda_1 Y_{-\alpha} + \lambda_2 Y_{-1}}. \tag{5.45}$$

The normalized susceptibility function $\varepsilon(u)$ from Equation (3) in [11] coincides with the Laplace-transform of the distribution $\tilde{S}$. Thus, $\tilde{S}$ is the time-domain representation of $\tilde{\varepsilon}(u)$ and it follows that the function $t \mapsto \tilde{K}(t)$ for $t \geq 0$ is the normalized relaxation motion corresponding to $\tilde{\varepsilon}(u)$. As shown in [11,17] the solution (5.45) agrees over a range of 12 decades in time or frequency with a physical relaxation experiment.

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