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SOME BOUNDS FOR ALTERNATING MATHIEU TYPE SERIES

ŽIVORAD TOMOVSKI AND RUDOLF HILFER

(*communicated by A. Čižmešija*)

Abstract. Using recent investigated integral representations for the generalized alternating Mathieu series $\tilde{S}_{\mu}^{(\alpha,\beta)}\left(r;\{a_n\}_{n=1}^{\infty}\right)$ $\left(r,\alpha,\beta,\mu,\ \{a_n\}_{n=1}^{\infty}\in R^{+}\right)$ [9,14,18] with $a_n=n^{\gamma},\ \gamma\in R^{+}$ and Mellin-Laplace type integral transforms for the generalized hypergeometric functions and the Bessel function of first kind, some bounding inequalities for $\tilde{S}_{\mu}^{(\alpha,\beta)}\left(r;\ \{n^{\gamma}\}_{n=1}^{\infty}\right)$ are presented. Namely, it is shown that the series $\tilde{S}_{\mu}^{(\alpha,\beta)}\left(r;\{n^{\gamma}\}_{n=1}^{\infty}\right)$ under some conditions for parameters $\alpha,\ \beta,\ \gamma$ and μ are bounded with constants which do not depend on α,β and γ but only depend on r and μ, i.e.

$$\tilde{S}_{\mu}^{(\alpha,\beta)}\left(r;\{n^{\gamma}\}_{n=1}^{\infty}\right)\leqslant\frac{2}{(1+r^2)^{\mu}}.$$

1. Introduction

The following familiar infinite series

$$S(r)=\sum_{n=1}^{\infty}\frac{2n}{\left(n^2+r^2\right)^2}\qquad\left(r\in R^{+}\right)\tag{1.1}$$

is named after Emile Leonard Mathieu (1835-1890), who investigated it in his 1890 work [7] on elasticity of solid bodies. Bounds for this series are needed for the solution of boundary value problems for the biharmonic equations in a two-dimensional rectangular domain (see [13], p.258, eq. (54)). An alternating version of (1.1)

$$\tilde{S}(r)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2n}{\left(n^2+r^2\right)^2}\qquad\left(r\in R^{+}\right)\tag{1.2}$$

was recently discussed by Pogány *et.al* in [9].

Integral representations of (1.1) and (1.2) are given by (see [5] and [9])

$$S(r)=\frac{1}{r}\int_0^{\infty}\frac{t\sin(rt)}{e^t-1}dt,\tag{1.3}$$

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$$\tilde{S}(r) = \frac{1}{r}\int_0^\infty \frac{t\sin(rt)}{e^t+1}dt \tag{1.4}$$

Several interesting problems and solutions dealing with integral representations and bounds for the following slight generalization of the Mathieu series with a fractional power

$$S_\mu(r) = \sum_{n=1}^\infty \frac{2n}{(n^2+r^2)^\mu} \qquad \left(r \in R^+; \mu > 1\right) \tag{1.5}$$

can be found in the recent works by Diananda [2], Tomovski and Trenčevski [16] and Cerone and Lenard [1]. Motivated essentially by the works of Cerone and Lenard [1] (and Qi [12]) a family of generalized Mathieu series

$$S_\mu^{(\alpha,\beta)}(r;a) = S_\mu^{(\alpha,\beta)}\left(r;\{a_n\}_{n=1}^\infty\right) = \sum_{n=1}^\infty \frac{2a_n^\beta}{(a_n^\alpha+r^2)^\mu} \qquad \left(r,\alpha,\beta,\mu \in R^+\right) \tag{1.6}$$

was defined in [14], where it is tacitly assumed that the positive sequence

$$a = \{a_n\}_{n=1}^\infty = \{a_1, a_2, a_3, ...\} \qquad \left(\lim_{n\to\infty} a_n = \infty\right)$$

is chosen such that the infinite series in definition (1.6) converges, that is, that the following auxiliary series

$$\sum_{n=1}^\infty \frac{1}{a_n^{\mu\alpha-\beta}}$$

is convergent. Comparing the definitions (1.1), (1.5) and (1.6), we see that $S_2(r) = S(r)$ and $S_\mu(r) = S_\mu^{(2,1)}\left(r,\{n\}_{n=1}^\infty\right)$. Furthermore, the special cases $S_2^{(2,1)}\left(r;\{a_n\}_{n=1}^\infty\right)$, $S_\mu(r) = S_\mu^{(2,1)}\left(r;\{n\}_{n=1}^\infty\right)$, $S_\mu^{(2,1)}\left(r;\{n^\gamma\}_{n=1}^\infty\right)$ and $S_\mu^{(\alpha,\alpha/2)}\left(r;\{n\}_{n=1}^\infty\right)$ were investigated by Qi [12]; Diananda [2]; Tomovski [16] and Cerone and Lenard [1].

Let

$$\tilde{S}_\mu^{(\alpha,\beta)}(r;a) = \tilde{S}_\mu^{(\alpha,\beta)}\left(r;\ \{a_n\}_{n=1}^\infty\right) = \sum_{n=1}^\infty (-1)^{n-1}\frac{2a_n^\beta}{(a_n^\alpha+r^2)^\mu} \qquad \left(r,\alpha,\beta,\mu \in R^+\right) \tag{1.7}$$

be an alternating variant of (1.6), where the positive sequence $\{a_n\}_{n=1}^\infty$ satisfied the same conditions of the definition (1.6). In [9, 14, 18] several integral representations of (1.6) and (1.7) in terms of the generalized hypergeometric functions and the Bessel function of first kind were obtained. Here we present some of them:

$$\tilde{S}_\mu^{(\alpha,\beta)}\left(r;\{n^\gamma\}_{n=1}^\infty\right) = \frac{2}{\Gamma(\mu)}\int_0^\infty \frac{x^{\gamma(\mu\alpha-\beta)-1}}{e^x+1}\,{}_1\Psi_1\left[(\mu,1);(\gamma(\mu\alpha-\beta),\gamma\alpha);-r^2x^{\gamma\alpha}\right]dx$$

$$\left(r,\alpha,\beta,\gamma \in R^+,\ \ \gamma(\mu\alpha-\beta) > 1\right); \tag{1.8}$$

$$\tilde{S}_{\mu}^{(\alpha,\beta)}\left(r;\left\{n^{q/\alpha}\right\}_{n=1}^{\infty}\right)=\frac{2}{\Gamma\left(q\left[\mu-\frac{\beta}{\alpha}\right]\right)}$$
$$\times\int_0^{\infty}\frac{x^{q[\mu-\beta/\alpha]-1}}{e^x+1}{}_1F_q\left(\mu;\Delta\left(q;q\left[\mu-\beta/\alpha\right]\right);-r^2\left(\frac{x}{q}\right)^q\right)dx$$
$$\left(r,\alpha,\beta\in R^+,\ \ \mu-\frac{\beta}{\alpha}>q^{-1};\ \ q\in N\right),\tag{1.9}$$

where $\Delta(q;\lambda)$ is the $q-$ tuple $\left(\frac{\lambda}{q},\frac{\lambda+1}{q},...,\frac{\lambda+q-1}{q}\right)$;

$$\tilde{S}_{\mu+1}^{(\alpha,\alpha/2)}\left(r;\left\{n^{2/\alpha}\right\}_{n=1}^{\infty}\right)=\tilde{S}_{\mu+1}^{(2,1)}\left(r;\{n\}_{n=1}^{\infty}\right)=\tilde{S}_{\mu+1}(r)$$
$$=\frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu+1)}\int_0^{\infty}\frac{x^{\mu+\frac{1}{2}}}{e^x+1}J_{\mu-\frac{1}{2}}(rx)\,dx,\quad\left(r,\mu\in R^+\right),\tag{1.10}$$

$$\tilde{S}_{\mu}^{(\alpha,0)}\left(r;\left\{n^{2/\alpha}\right\}_{n=1}^{\infty}\right)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2}{(n^2+r^2)^{\mu}}=\frac{2\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\Gamma(\mu)}\int_0^{\infty}\frac{x^{\mu-\frac{1}{2}}}{e^x+1}J_{\mu-\frac{1}{2}}(rx)\,dx$$
$$\left(r\in R^+,\ \ \mu>\frac{1}{2}\right).\tag{1.11}$$

Here ${}_p\Psi_q$ denotes the Fox-Wright generalization of the hypergeometric ${}_pF_q$ function with p numerator and q denominator parameters (see for example [15, Eq.1.5 (21), p.50])

$${}_p\Psi_q(x)={}_p\Psi_q\left[(a_l,\alpha_l)_{1,p};\ \ (b_j,\beta_j)_{1,q};x\right]=\sum_{k=0}^{\infty}\frac{\prod_{l=1}^{p}\Gamma(a_l+\alpha_l k)}{\prod_{j=1}^{q}\Gamma(b_j+\beta_j k)}\frac{x^k}{k!}\tag{1.12}$$
$$\left(a_l,b_j,\alpha_l,\beta_j\in R;\quad l=1,2,...,p,\ \ j=1,2,...,q;\ \ 1+\sum_{j=1}^{q}\beta_j-\sum_{l=1}^{p}\alpha_l>0\right).$$

The generalized hypergeometric function is defined by

$${}_pF_q\left[(a_l)_{1,p};(b_j)_{1,q};x\right]=\sum_{m=0}^{\infty}\frac{\prod_{l=1}^{p}(a_l)_m}{\prod_{j=1}^{q}(b_j)_m}\frac{x^m}{m!}\tag{1.13}$$

where $(\delta)_m$ is the Pochhammer symbol, defined by

$$(\delta)_0=1,\quad(\delta)_m=\delta(\delta+1)\cdots(\delta+m-1)=\frac{\Gamma(\delta+m)}{\Gamma(\delta)}\quad(m\in N),$$

so that, obviously

$$ {}_p\Psi_q\left[(a_l,1)_{1,p}\,;\ (b_j,1)_{1,q}\,;\,x\right]=\frac{\prod_{l=1}^{p}\Gamma(a_l)}{\prod_{j=1}^{q}\Gamma(b_j)}\,{}_pF_q\left[(a_l)_{1,p}\,;\ (b_j)_{1,q}\,;\,x\right] \tag{1.14}$$

$$\left(a_l>0,\ b_j\notin Z_0^-\right).$$

2. Bounds derivable from the integral representations of $\tilde{S}_{\mu}^{(\alpha,\beta)}\left(r;\{n^{\gamma}\}_{n=1}^{\infty}\right)$

2.1. The Landau estimates (see [6])

$$|J_{\nu}(x)|\leqslant b_L\nu^{-1/3}\ \text{ with }\ b_L=\sqrt[3]{2}\sup_{x\in R^+}\{Ai(x)\}=0.674885\ldots,\ \text{ uniformly in } x, \tag{2.1}$$

$$|J_{\nu}(x)|\leqslant c_Lx^{-1/3}\ \text{ with }\ c_L=\sup_{x\in R^+}\left\{x^{1/3}J_0(x)\right\}=0.78574687\ldots,\ \text{ uniformly in } \nu, \tag{2.2}$$

where $Ai(z)$ denotes the known Airy function, were used in [9] to prove the following bounds:

$$\left|\tilde{S}_{\mu}^{(2,1)}\left(r;\ \{n^{\gamma}\}_{n=1}^{\infty}\right)\right|\leqslant\frac{b_L\tilde{C}_{\mu}(r)\,\Gamma\left(\mu+\frac{1}{2}\right)}{\left(\mu-\frac{3}{2}\right)^{1/3}}\qquad(2\mu-3>0), \tag{2.3}$$

$$\left|\tilde{S}_{\mu}^{(2,1)}\left(r;\ \{n^{\gamma}\}_{n=1}^{\infty}\right)\right|\leqslant\frac{c_L\tilde{C}_{\mu}(r)\,\Gamma\left(\mu+\frac{1}{6}\right)}{\sqrt[3]{r}}\qquad(6\mu+1>0), \tag{2.4}$$

where $\tilde{C}_{\mu}(r)=\frac{\sqrt{\pi}}{(2r)^{\mu-\frac{3}{2}}\Gamma(\mu)}$. Moreover, if $\mu\geqslant\frac{1}{2}\left(1+r^2\right)$, then

$$0<\tilde{S}_{\mu}^{(2,1)}\left(r;\ \{n^{\gamma}\}_{n=1}^{\infty}\right)\leqslant\frac{b_L\tilde{C}_{\mu}(r)\,\Gamma\left(\mu+\frac{1}{2}\right)}{\left(\mu-\frac{3}{2}\right)^{1/3}}=N_b(r,\mu)\qquad(2\mu-3>0), \tag{2.5}$$

$$0<\tilde{S}_{\mu}^{(2,1)}\left(r;\ \{n^{\gamma}\}_{n=1}^{\infty}\right)\leqslant\frac{c_L\tilde{C}_{\mu}(r)\,\Gamma\left(\mu+\frac{1}{6}\right)}{\sqrt[3]{r}}=N_c(r,\mu). \tag{2.6}$$

2.2. Now we shall improve the right sided bounding inequalities (2.5)–(2.6) by showing that $\tilde{S}_{\mu}^{(2,1)}\left(r;\ \{n^{\gamma}\}_{n=1}^{\infty}\right)$ is bounded with the constant

$$M(r,\mu)=\frac{2}{\left(1+r^2\right)^{\mu}} \tag{2.7}$$

under the conditions $\gamma\in R^+,\ \ \mu\geqslant\frac{1}{2}\left(1+r^2\right)$. Let $\varphi(x)=\frac{x^{\gamma}}{(x^{2\gamma}+r^2)^{\mu}}$ with $\mu\geqslant\frac{1}{2}\left(1+r^2\right)$. Since

$$\varphi'(x)=\frac{\gamma x^{\gamma-1}\left(r^2-(2\mu-1)\,x^{2\gamma}\right)}{\left(x^{2\gamma}+r^2\right)^{\mu+1}}<0$$

for all x constrained by the inequality $x > \left(\frac{r^2}{2\mu-1}\right)^{\frac{1}{2\gamma}}$ it follows that $\varphi(x)$ is a decreasing function of x. So we have

$$\tilde{S}_{\mu}^{(2,1)}\left(r;\ \{n^{\gamma}\}_{n=1}^{\infty}\right) = M(r,\mu) - 2\sum_{n=1}^{\infty}\left[\varphi(2n) - \varphi(2n+1)\right] < M(r,\mu) \tag{2.8}$$

when $\left[\left(\frac{r^2}{2\mu-1}\right)^{1/2\gamma}\right] \leqslant 1$. But, this condition holds, since $\mu \geqslant \frac{1}{2}\left(1+r^2\right)$.

Next, it would be of interest to research the efficiency and the sharpness of $M(r,\mu)$. In this goal, the $r-$ domains in which $M(r,\mu)$ is superior to N_b and N_c have to be obtained. We shall prove that $M(r,\mu) \leqslant N_b(r,\mu)$ for all $r \in R^+$ and all $\mu > \frac{3}{2}$.

Let

$$P = P(r,\mu) = \frac{M(r,\mu)}{N_b(r,\mu)}. \tag{2.9}$$

Then

$$P = \frac{(\mu-3/2)^{1/3}}{b_L\sqrt{2\pi}}\frac{\Gamma(\mu)}{\Gamma(\mu+1/2)}\left(\frac{2r}{1+r^2}\right)^{\mu-3/2}\left(1+r^2\right)^{-3/2}. \tag{2.10}$$

We consider the function

$$f(r) = \frac{r^{\mu-3/2}}{(1+r^2)^{\mu}} \qquad \left(r \in R^+,\ \mu > \frac{3}{2}\right) \tag{2.11}$$

It is easy to show that

$$\max_{r\in R^+} f(r) = f\left(\sqrt{\frac{\mu-3/2}{\mu+3/2}}\right) \qquad \left(\mu > \frac{3}{2}\right). \tag{2.12}$$

Using the elementary inequality

$$\left(\frac{2r}{1+r^2}\right)^{\mu-\frac{3}{2}} \leqslant 1 \qquad \left(r \in R^+,\ \mu > \frac{3}{2}\right) \tag{2.13}$$

and Gautschi's inequality (see [4])

$$\frac{\Gamma(\mu)}{\Gamma(\mu+1/2)} \leqslant \frac{1}{\sqrt{\mu-1/4}} \qquad \left(\mu > \frac{3}{2}\right) \tag{2.14}$$

we get

$$\begin{aligned} P &\leqslant \frac{(\mu-3/2)^{1/3}}{b_L\sqrt{2\pi}}\frac{1}{\sqrt{\mu-1/4}}\left(1+\left(\sqrt{\frac{\mu-3/2}{\mu+3/2}}\right)^2\right)^{-3/2} \\ &= \frac{(\mu-3/2)^{1/3}}{b_L\sqrt{2\pi}}\frac{1}{\sqrt{\mu-1/4}}\frac{(\mu+3/2)^{3/2}}{(2\mu)^{3/2}} \end{aligned}$$

$$\begin{aligned} &\leqslant \frac{1}{b_L\sqrt{2\pi}}\frac{(\mu-1/4)^{1/3}}{(\mu-1/4)^{1/2}} = \frac{1}{b_L\sqrt{2\pi}}\frac{1}{(\mu-1/4)^{1/6}} \\ &< \frac{1}{b_L\sqrt{2\pi}}\sqrt[6]{\frac{4}{5}} < 1. \end{aligned} \tag{2.15}$$

The similar comparison involving $M(r,\mu)$ and $N_c(r,\mu)$ we leave to the interested reader which we propose as an open problem.

Open Problem: Does $M(r,\mu) \leqslant N_c(r,\mu)$ for all $r \in R^+$?

2.3. Next, we shall present some elegant bounds for the alternating Mathieu series $\tilde{S}(r), \tilde{S}_\mu^{(\alpha,0)}\left(r;\left\{n^{2/\alpha}\right\}_{n=1}^\infty\right)$, $\tilde{S}_{\mu+1}(r)$, $\tilde{S}_\mu^{(\alpha,\beta)}\left(r;\left\{n^{q/\alpha}\right\}_{n=1}^\infty\right)$, $\tilde{S}_\mu^{(\alpha,\beta)}\left(r;\ \{n^\gamma\}_{n=1}^\infty\right)$ by using their integral representations given above.

2.3.1. Using the well-known formula (see [10, Vol.1, p.446])

$$\int_0^\infty xe^{-x}\sin(xr)\,dx = \frac{2r}{(1+r^2)^2}, \tag{2.16}$$

we get

$$\tilde{S}(r) \leqslant \frac{1}{r}\int_0^\infty xe^{-x}\sin(xr)\,dx = M(r,2). \tag{2.17}$$

2.3.2. In the theory of Bessel functions, it is fairly well-known that (cf; e.g. [3, p.49, Eq. 7.7.3 (16)]

$$\int_0^\infty e^{-st}t^{\lambda-1}J_\nu(\rho t)dt = \left(\frac{\rho}{2s}\right)^\nu s^{-\lambda}\frac{\Gamma(\nu+\lambda)}{\Gamma(\nu+1)}\,{}_2F_1\left[\frac{1}{2}(\nu+\lambda),\ \frac{1}{2}(\nu+\lambda+1);\ \nu+1;\ -\frac{\rho^2}{s^2}\right]$$

$$(\mathrm{Re}(s) > |\mathrm{Im}(\rho)|,\ \mathrm{Re}(\nu+\lambda) > 0). \tag{2.18}$$

Because of

$${}_1F_0(\lambda;\ \ -;\ \ z) = (1-z)^{-\lambda} \qquad (|z|<1;\ \ \lambda \in C) \tag{2.19}$$

the integral formula (2.18) would simplify considerably when $\lambda = \nu+1$ and when $\lambda = \nu+2$, giving us [see also (1.10) and (1.11) above]

$$\int_0^\infty e^{-st}t^\nu J_\nu(\rho t)dt = \frac{(2\rho)^\nu}{\sqrt{\pi}}\frac{\Gamma\left(\nu+\frac{1}{2}\right)}{(s^2+\rho^2)^{\nu+\frac{1}{2}}} \qquad (\mathrm{Re}(s) > |\mathrm{Im}(\rho)|,\ \mathrm{Re}(\nu) > -\frac{1}{2}); \tag{2.20}$$

$$\int_0^\infty e^{-st}t^{\nu+1}J_\nu(\rho t)dt = \frac{2s(2\rho)^\nu}{\sqrt{\pi}}\frac{\Gamma\left(\nu+\frac{3}{2}\right)}{(s^2+\rho^2)^{\nu+\frac{3}{2}}} \qquad (\mathrm{Re}(s) > |\mathrm{Im}(\rho)|,\ \mathrm{Re}(\nu) > -1). \tag{2.21}$$

Using the formulas (2.20), (2.21), and integral representations (1.11) and (1.10), we obtain the following bounds for $\tilde{S}_{\mu}^{(\alpha,0)}\left(r;\left\{n^{2/\alpha}\right\}_{n=1}^{\infty}\right)$ and $\tilde{S}_{\mu+1}(r)$ respectively:

$$\begin{aligned}\tilde{S}_{\mu}^{(\alpha,0)}\left(r;\ \left\{n^{2/\alpha}\right\}_{n=1}^{\infty}\right) &\leqslant \frac{2\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\,\Gamma(\mu)}\int_0^{\infty} e^{-x}x^{\mu-\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx)\,dx \\ &= \frac{2\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\,\Gamma(\mu)}\,\frac{(2r)^{\mu-\frac{1}{2}}\,\Gamma(\mu)}{\sqrt{\pi}\,(1+r^2)^{\mu}} \\ &= M(r,\mu) \qquad \left(r\in R^+,\ \mu>\frac{1}{2}\right)\end{aligned} \tag{2.22}$$

$$\begin{aligned}\tilde{S}_{\mu+1}^{(r)} &\leqslant \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\,\Gamma(\mu+1)}\int_0^{\infty} e^{-x}x^{\mu+\frac{1}{2}}J_{\mu-\frac{1}{2}}(rx)\,dx \\ &= \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}}\,\Gamma(\mu+1)}\,\frac{2\,(2r)^{\mu-\frac{1}{2}}\,\Gamma(\mu+1)}{\sqrt{\pi}\,(1+r^2)^{\mu+1}} = M(r,\mu+1) \quad \left(r,\mu\in R^+\right)\end{aligned} \tag{2.23}$$

2.3.3. In the theory of generalized hypergeometric functions it is known that the following integral formula (see [11], p. 335):

$$\begin{aligned}&\int_0^{\infty} x^{l\alpha-1}e^{-\sigma x}\,{}_pF_q\left((a_p);(b_q);-\omega x^l\right)dx \\ &\qquad = \sigma^{-l\alpha}\Gamma(l\alpha)\,{}_{l+p}F_q\left((a_p),\Delta(l,l\alpha);(b_q);-\frac{l^l\omega}{\sigma^l}\right)\end{aligned} \tag{2.24}$$

holds.

Using the formula (2.24) and integral representation (1.9) we obtain the following bound for $\tilde{S}_{\mu}^{(\alpha,\beta)}\left(r;\left\{n^{q/\alpha}\right\}_{n=1}^{\infty}\right)$:

$$\begin{aligned}&\tilde{S}_{\mu}^{(\alpha,\beta)}\left(r;\left\{n^{q/\alpha}\right\}_{n=1}^{\infty}\right) \\ &\leqslant \frac{2}{\Gamma\left(q\left[\mu-\frac{\beta}{\alpha}\right]\right)}\int_0^{\infty} x^{q\left[\mu-\frac{\beta}{\alpha}\right]-1}e^{-x}\,{}_1F_q\left[\mu;\Delta\left(q;q\left[\mu-\frac{\beta}{\alpha}\right]\right);-r^2\left(\frac{x}{q}\right)^q\right]dx \\ &= \frac{2}{\Gamma\left(q\left[\mu-\frac{\beta}{\alpha}\right]\right)}\Gamma\left(q\left[\mu-\frac{\beta}{\alpha}\right]\right)\,{}_{q+1}F_q\left(\mu,\Delta\left(q,q\left[\mu-\frac{\beta}{\alpha}\right]\right);\Delta\left(q,q\left[\mu-\frac{\beta}{\alpha}\right]\right);-r^2\right) \\ &= 2\,{}_1F_0\left(\mu;-;-r^2\right) = \frac{2}{\left(1+r^2\right)^2},\end{aligned}$$

i.e.

$$\tilde{S}_{\mu}^{(\alpha,\beta)}\left(r;\left\{n^{q/\alpha}\right\}_{n=1}^{\infty}\right) \leqslant M(r,\mu) \quad \left(r,\alpha,\beta\in R^+,\ \mu-\frac{\beta}{\alpha}>q^{-1};\ q\in N\right). \tag{2.25}$$

Specifically for $p=1,\ q=2,\ l=2$, we get from (2.24)

$$\tilde{S}_{\mu}^{(\alpha,\beta)}\left(r;\ \left\{n^{2/\alpha}\right\}_{n=1}^{\infty}\right)$$

$$\leqslant \frac{2}{\Gamma\left(2\left[\mu-\frac{\beta}{\alpha}\right]\right)}\int_{0}^{\infty} x^{2\left[\mu-\frac{\beta}{\alpha}\right]-1}e^{-x}{}_{1}F_{2}\left(\mu;\ \left[\mu-\frac{\beta}{\alpha}\right],\ \left[\mu-\frac{\beta}{\alpha}\right]+\frac{1}{2};-\frac{r^2x^2}{4}\right)dx$$

$$=\frac{2}{\Gamma\left(2\left[\mu-\frac{\beta}{\alpha}\right]\right)}\Gamma\left(2\left[\mu-\frac{\beta}{\alpha}\right]\right)$$

$$\times\,{}_{3}F_{2}\left[\mu,\left[\mu-\frac{\beta}{\alpha}\right],\left[\mu-\frac{\beta}{\alpha}\right]+\frac{1}{2};\left[\mu-\frac{\beta}{\alpha}\right],\left[\mu-\frac{\beta}{\alpha}\right]+\frac{1}{2};-r^2\right]$$

$$=\frac{2}{\left(1+r^2\right)^{\mu}},$$

i.e.

$$\tilde{S}_{\mu}^{(\alpha,\beta)}\left(r;\left\{n^{2/\alpha}\right\}_{n=1}^{\infty}\right)\leqslant M\left(r,\mu\right). \tag{2.26}$$

2.3.4. Now applying the formula (see [11, p.355])

$$\int_{0}^{\infty} x^{\alpha-1}e^{-\sigma x}{}_{p}\Psi_{q}\left[\left(a_p,\alpha_p\right);\left(b_q,B_q\right);-wx^{l}\right]dx$$

$$=\frac{1}{\sigma^{\alpha}}\,{}_{p+1}\Psi_{q}\left[\left(\alpha,r\right),\left(a_p,\alpha_p\right);\left(b_q,\beta_q\right);-\frac{w}{\sigma^{l}}\right]$$

we obtain from the integral representation (1.8),

$$\tilde{S}_{\mu}^{(\alpha,\beta)}\left(r;\left\{n^{\gamma}\right\}_{n=1}^{\infty}\right)$$

$$\leqslant\frac{2}{\Gamma\left(\mu\right)}\int_{0}^{\infty}x^{\gamma(\mu\alpha-\beta)-1}e^{-x}{}_{1}\Psi_{1}\left[\left(\mu,1\right);\left(\gamma\left(\mu\alpha-\beta\right),\gamma\alpha\right);-r^2x^{\gamma\alpha}\right]dx$$

$$=\frac{2}{\Gamma\left(\mu\right)}{}_{2}\Psi_{1}\left[\left(\gamma\left(\mu\alpha-\beta\right),\gamma\alpha\right),\left(\mu,1\right);\left(\gamma\left(\mu\alpha-\beta\right),\gamma\alpha\right);-r^2\right]$$

$$=\frac{2}{\Gamma\left(\mu\right)}\sum_{m=0}^{\infty}\frac{\Gamma\left(\gamma\left(\mu\alpha-\beta\right)+\gamma\alpha m\right)\Gamma\left(\mu+m\right)}{\Gamma\left(\gamma\left(\mu\alpha-\beta\right)+\gamma\alpha m\right)}\frac{\left(-r^2\right)^m}{m!}$$

$$=2{}_{1}F_{0}\left(\mu;-;-r^2\right)=\frac{2}{\left(1+r^2\right)^{\mu}},$$

i.e.

$$\tilde{S}_{\mu}^{(\alpha,\beta)}\left(r;\left\{n^{\gamma}\right\}_{n=1}^{\infty}\right)\leqslant M\left(r,\mu\right)\quad\left(r,\alpha,\beta,\gamma\in R^{+},\ \gamma\left(\mu\alpha-\beta\right)>1\right) \tag{2.27}$$

In Figure 1, we present some numerical results for three alternating series: $\tilde{S}\left(r\right)$, $\tilde{S}_{2}^{(2,2)}\left(r;\ \left\{n\right\}_{n=1}^{\infty}\right)$, $\tilde{S}_{2}^{(2.5,\,2.1)}\left(r;\ \left\{n^{3/2}\right\}_{n=1}^{\infty}\right)$ bounded by the constants $M\left(r,2\right)$ with $0<r<3$.

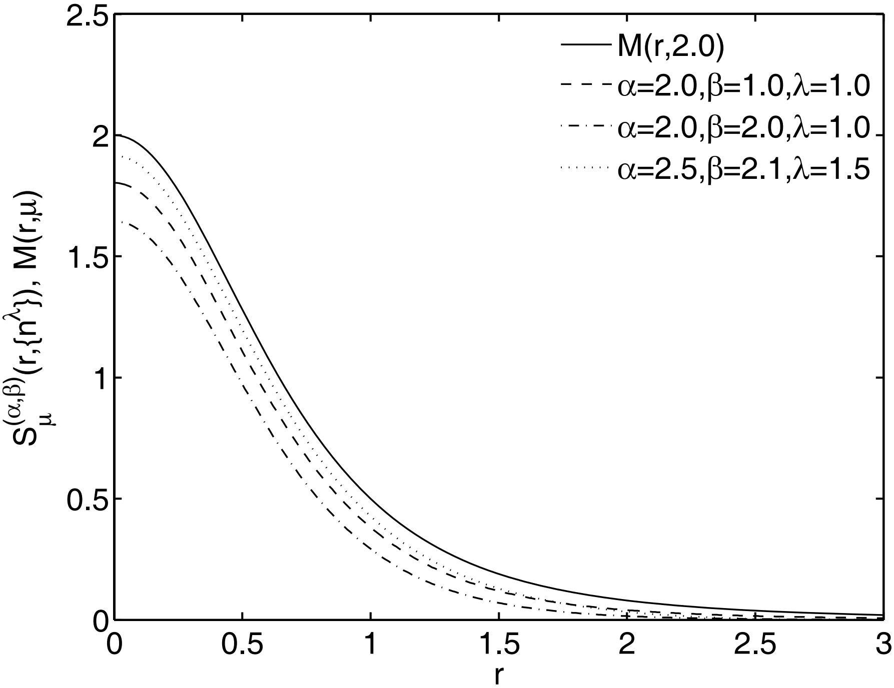

Figure 1. *Alternating Mathieu type series* $\tilde{S}_{\mu}^{(\alpha,\beta)}\left(r;\left\{n^{\lambda}\right\}_{n=1}^{\infty}\right)$ *with* $\alpha=2,\ \beta=1,$ $\lambda=1,\ \mu=2$ *(dashed),* $\alpha=2,\ \beta=2,\ \lambda=1,\ \mu=2$ *(dashdotted) and* $\alpha=2.5,$ $\beta=2.1,\ \lambda=1.5,\ \mu=2$ *(dotted) as functions of* r *with* $0<r<3$ *and their bound* $M(r,2)$ *(solid line).*

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Živorad Tomovski
Institute for Computational Physics
University of Stuttgart
Pfaffenwaldring 27, D-70569
Germany
e-mail: zivorad@icp.uni-stuttgart.de

Faculty of Mathematics and Natural Sciences
Institute of Mathematics
PO.Box.162 Gazi Baba b.b.
1000 Skopje
Macedonia
e-mail: tomovski@iunona.pmf.ukim.edu.mk

Rudolf Hilfer
Institute for Computational Physics
University of Stuttgart
Pfaffenwaldring 27, D-70569
Germany
e-mail: hilfer@icp.uni-stuttgart.de