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Thermodynamic potentials for the infinite range Ising model with strong coupling

R. Hilfer[a,b,*]

[a] ICA-1, Universität Stuttgart, Pfaffenwaldring 27, 70569 Stuttgart, Germany

[b] Institut für Physik, Universität Mainz, 55099 Mainz, Germany

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Abstract

The specific Gibbs free energy has been calculated for the infinite range Ising model with fixed and finite interaction strength. The model shows a temperature driven first-order phase transition that differs from the infinite ranged Ising model with weak coupling. In the temperature-field phase diagram the strong coupling model shows a line of first-order phase transitions that does not end in a critical point.

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1. Introduction

Given the central importance of mean field theories in the modern theory of critical phenomena [1] it is of interest to study the thermodynamics of infinite ranged spin models with fixed and finite interaction strength. Little seems to be known about this class of models. One usually studies the case in which the interaction strength vanishes inversely proportional to the number of spins. Results without this assumption are not available because of divergences in the thermodynamic limit. In Refs. [2–10] the present author has introduced a generalized notion of equilibrium into statistical physics that

* Corresponding author. ICA-1, Universität Stuttgart, Pfaffenwaldring 27, 70569 Stuttgart, Germany. Fax: +49-7111-685-3658.

E-mail address: hilfer@ica1.uni-stuttgart.de (R. Hilfer).

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allows to remove such divergences. An application of these ideas puts the infinite range Ising model with strong coupling into the realm of classical thermodynamics.

Despite the analytical simplicity of the final expressions for the thermodynamic potentials of the strong coupling model their derivations will be given elsewhere [11]. Exact results are presented in this paper for the Gibbs free energy, the specific entropy and specific heat capacity. Other thermodynamic potentials can be obtained as usual from these results, and may then be compared to the results for the weak coupling model.

2. Model

Consider N Ising spins $S_i = \pm 1$, $(i = 1, \ldots, N)$. The model to be studied in this paper is defined by the energy function

$$\mathscr{E}(S_1, \ldots, S_N) = -J_s \sum_{\substack{i=1 \\ i>j}}^{N} \sum_{j=1}^{N} S_i S_j - H \sum_{i=1}^{N} S_i \,, \tag{1}$$

where $H \in \mathbb{R}$ is the external magnetic field, and $J_s > 0$ the ferromagnetic coupling constant. The double sum extends over all $N(N-1)/2$ pairs of spins.

The text book version of the infinite range mean field model (also called weak coupling model) is defined by the energy function

$$\mathscr{E}(S_1, \ldots, S_N) = -\frac{J_w}{N} \sum_{\substack{i=1 \\ i>j}}^{N} \sum_{j=1}^{N} S_i S_j - H \sum_{i=1}^{N} S_i \,, \tag{2}$$

whose two spin coupling strength vanishes with N.

3. Results

The equilibrium specific Gibbs potential $g(T, H)$ for the strong coupling model defined in (1) was evaluated in the canonical ensemble as a function of temperature T and external magnetic field H [11]. It is found as

$$\begin{aligned} g(T,H) = {} & k_B T \left(\frac{1+m}{2}\right) \log\left(\frac{1+m}{2}\right) + k_B T \left(\frac{1-m}{2}\right) \log\left(\frac{1-m}{2}\right) \\ & + \sqrt{\frac{k_B T J_s}{2}(1-m^2)} - \frac{J_s}{2} \,, \end{aligned} \tag{3}$$

where $m = m(T, H)$ is the solution of

$$m = \tanh\left(m \sqrt{\frac{\beta J_s}{2(1-m^2)}} \right) \,. \tag{4}$$

Note that Eq. (4) and hence $g(T, H)$ are independent of H.

Eq. (4) has the following stable solutions:

$$m(T,H)=\begin{cases}\pm 1 & \text{for all } T>0\,,\\ 0 & \text{for all } T>J_s/(4k_B)\,.\end{cases} \tag{5}$$

As a consequence the free energy has two branches intersecting for all H at the same critical temperature

$$T_c=\frac{J_s}{k_B 2\log^2 2}\approx 1.0406845\,\frac{J_s}{k_B}\,. \tag{6}$$

The specific Gibbs free energy becomes

$$g(T,H)=\begin{cases}-\dfrac{J_s}{2} & \text{for } T<T_c,\ H\in\mathbb{R}\,,\\ -\dfrac{J_s}{2}-k_BT\log 2+\sqrt{\dfrac{J_s}{2}\,k_BT} & \text{for } T>T_c,\ H\in\mathbb{R}\,.\end{cases} \tag{7}$$

The specific entropy as a function of temperature

$$s(T,H)=\begin{cases}0 & \text{for } T<T_c,\ H\in\mathbb{R}\,,\\ k_B\log 2-k_B\sqrt{\dfrac{J_s}{8k_BT}} & \text{for } T>T_c,\ H\in\mathbb{R}\end{cases} \tag{8}$$

has a jump discontinuity

$$\Delta s=\lim_{\varepsilon\to 0}\left(s(T_c-\varepsilon,H)-s(T_c+\varepsilon,H)\right)=\frac{k_B}{2}\log 2 \tag{9}$$

at the critical temperature T_c. The latent heat of the transition is therefore

$$\Delta Q=T_c\Delta s=\frac{J_s}{4\log 2}\,. \tag{10}$$

The specific heat capacity is obtained as

$$c_H(T,H)=T\left.\frac{\partial s}{\partial T}\right|_H=\begin{cases}0 & \text{for } T<T_c,\ H\in\mathbb{R}\,,\\ \dfrac{k_B}{4}\sqrt{\dfrac{J_s}{2k_BT}} & \text{for } T>T_c,\ H\in\mathbb{R}\end{cases} \tag{11}$$

and it exhibits a jump of magnitude $(k_B\log 2)/4$ at T_c.

The analogue of Eq. (3) in the weak coupling version of the model reads

$$g(T,H)=k_BT\left(\frac{1+m}{2}\right)\log\left(\frac{1+m}{2}\right)+k_BT\left(\frac{1-m}{2}\right)\log\left(\frac{1-m}{2}\right)$$
$$-\frac{J_w}{2}m^2-Hm\,, \tag{12}$$

where the magnetization per spin $m(T,H)$ is obtained by solving the familiar mean field equation

$$m=\tanh\left(\frac{J_wm+H}{k_BT}\right)\,. \tag{13}$$

It has a line of first-order phase transitions along the temperature axis $H = 0$ that ends in a critical point at $T_c = J_w/k_B$.

4. Discussion

Both models have the same symmetry. They also share the same ground state. Intuitively one expects that at sufficiently high temperatures and zero field both models should exhibit a phase with vanishing macroscopic magnetization. This expectation is indeed confirmed by the exact results. For $T \to \infty$ and $H = 0$ the specific Gibbs potential becomes $g(T, H) \approx -k_B T \log 2$ for both models, and the entropy approaches $k_B \log 2$. Similarly both models have a phase transition into a low temperature phase with nonvanishing magnetization.

Despite these basic similarities there exist also differences. Figs. 1 and 2 show the specific Gibbs potential, and specific entropy for both models for $H = 0$. The solid line corresponds to the strong coupling model, the dashed line represents the familiar weak coupling model at zero magnetic field. While the weak coupling version has a continuous phase transition of order $\frac{4}{3}$ (see [2,3,5,12] for a classification), the strong coupling model shows a strong first order transition.

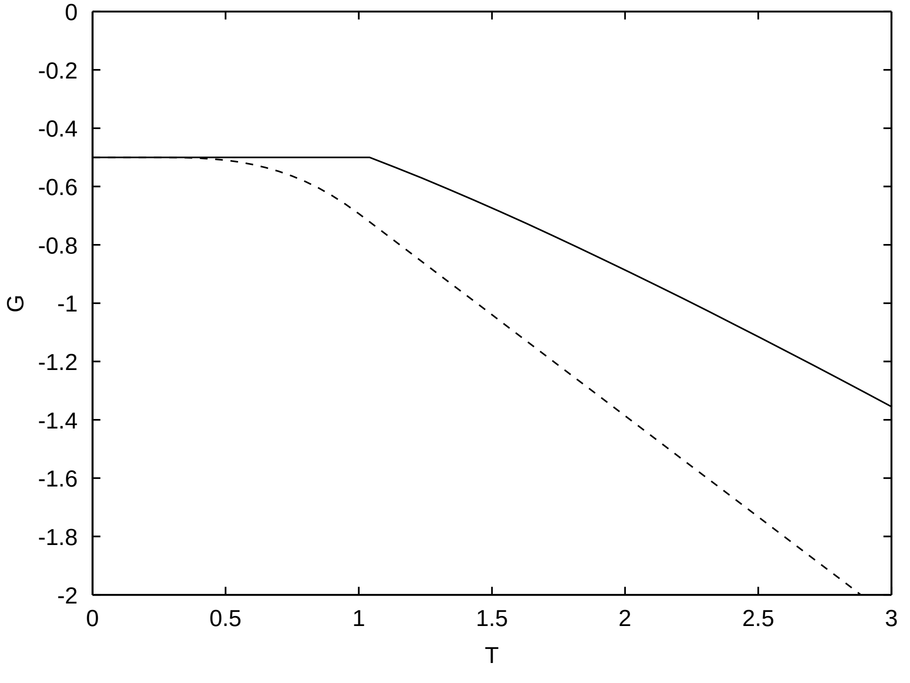

Fig. 1. Specific Gibbs potential $g(T, 0)$ as function of temperature T for the strong coupling model (solid line) and the weak coupling model (dashed line) at vanishing external magnetic field $H = 0$.

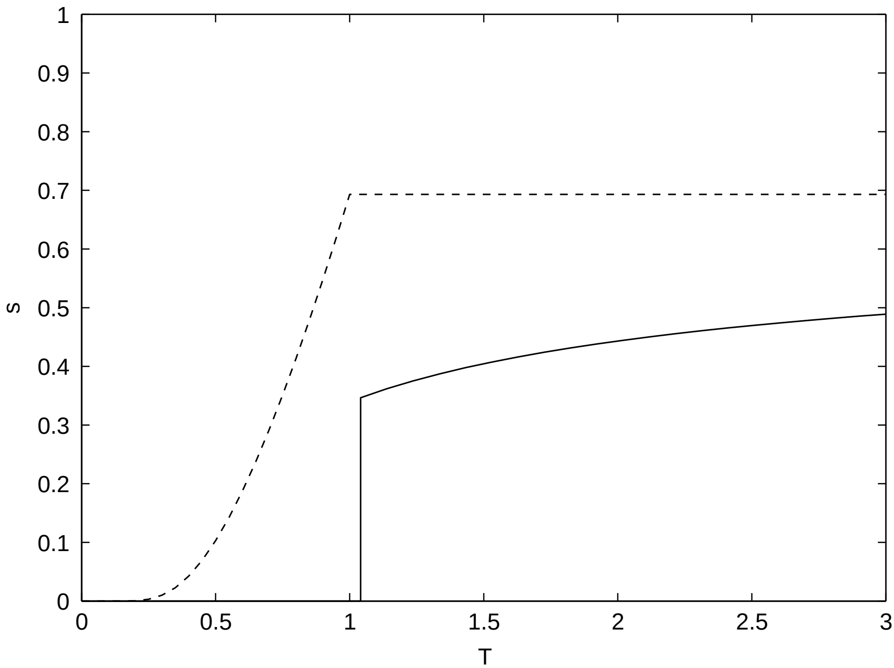

Fig. 2. Specific entropy $s(T, 0)$ as function of temperature T for the strong coupling model (solid line) and the weak coupling model (dashed line) at vanishing external magnetic field $H = 0$.

For the strong coupling model all thermodynamic functions are independent of H. This agrees with expectations because, contrary to the weak model, the field energy becomes negligible compared to the interaction energy in the thermodynamic limit. In the weak coupling model both energies remain of the same magnitude.

In the strong coupling model there appears a phase transition at all values of the external magnetic field. As a consequence the (T, H)-phase diagram for the strong coupling model shows a line of first-order transitions given by the equation $T = T_c$. This line runs from $H = -\infty$ to $H = +\infty$ parallel to the H-axis and does not end in any finite critical point. Such behaviour is reminiscent of a fluid–solid transition.

There are no thermal fluctuations in the low temperature phase of the strong coupling model. The spins are everywhere locked into the groundstate configuration, and the magnetization is $m = \pm 1$. Droplets of opposite orientation have vanishing probability at all $T < T_c$. The low temperature phase has vanishing entropy and heat capacity.

In summary, while the high temperature behaviour of the weak and strong coupling models are very similar their low temperature phases are distinctly different.

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