# Threefold Introduction to Fractional Derivatives

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# Contents

CHAPTER 2

# Threefold Introduction to Fractional Derivatives

*[page 17, §1]*

## 2.1. Historical Introduction to Fractional Derivatives

### 2.1.1. Leibniz

*[17.2.1]* Already at the beginning of calculus one of its founding fathers, namely G.W. Leibniz, investigated fractional derivatives [**72**, **73**]. *[17.2.2]* Differentiation, denoted as $\mathrm{d}^\alpha$ ($\alpha \in \mathbb{N}$), obeys Leibniz' product rule

$$\mathrm{d}^\alpha(fg) = 1\ \mathrm{d}^\alpha f\ \mathrm{d}^0 g + \frac{\alpha}{1}\ \mathrm{d}^{\alpha-1} f\ \mathrm{d}^1 g + \frac{\alpha(\alpha-1)}{1\cdot 2}\ \mathrm{d}^{\alpha-2} f\ \mathrm{d}^2 g + ... \tag{2.1}$$

for integer $\alpha$, and Leibniz was intrigued by the analogy with the binomial theorem

$$\mathrm{p}^\alpha(f+g) = 1\ \mathrm{p}^\alpha f\ \mathrm{p}^0 g + \frac{\alpha}{1}\ \mathrm{p}^{\alpha-1} f\ \mathrm{p}^1 g + \frac{\alpha(\alpha-1)}{1\cdot 2}\ \mathrm{p}^{\alpha-2} f\ \mathrm{p}^2 g + ... \tag{2.2}$$

where he uses the notation $\mathrm{p}^\alpha f$ instead of $f^\alpha$ to emphasize the formal operational analogy.

*[17.3.1]* Moving from integer to noninteger powers $\alpha \in \mathbb{R}$ Leibniz suggests that *"on peut exprimer par une serie infinie une grandeur comme"* $\mathrm{d}^\alpha h$ (with $h = fg$). *[17.3.2]* As his first step he tests the idea of such a generalized differential quantity $\mathrm{d}^\alpha h$ against the rules of his calculus. *[17.3.3]* In his calculus the differential relation $\mathrm{d}h = h\mathrm{d}x$ implies $\mathrm{d}x = \mathrm{d}h/h$ and $\mathrm{d}h/\mathrm{d}x = h$. *[17.3.4]* One has, therefore, also $\mathrm{d}^2 h = h\mathrm{d}x^2$ and generally $\mathrm{d}^\alpha h = h\mathrm{d}x^\alpha$. *[17.3.5]* Regarding $\mathrm{d}^\alpha h = h\mathrm{d}x^\alpha$ with noninteger $\alpha$ as a fractional differential relation subject to the rules of his calculus, however, leads to a paradox. *[17.3.6]* Explicitly, he finds (for $\alpha = 1/2$)

$$\frac{\mathrm{d}^\alpha h}{\mathrm{d}x^\alpha} = \frac{\mathrm{d}^\alpha h}{(\mathrm{d}h/h)^\alpha} \neq h, \tag{2.3}$$

where $\mathrm{d}x = \mathrm{d}h/h$ was used. *[17.3.7]* Many decades had to pass before Leibniz' paradox was fully resolved.

*[page 18, §1]*

### 2.1.2. Euler

*[18.1.1]* Derivatives of noninteger (fractional) order motivated Euler to introduce the Gamma function [**25**]. *[18.1.2]* Euler knew that he needed to generalize (or interpolate, as he calls it) the product $1 \cdot 2 \cdot \ldots \cdot n = n!$ to noninteger values of $n$, and he proposed an integral

$$\prod_{k=1}^{n} k = n! = \int_0^1 (-\log x)^n \, \mathrm{d}x \tag{2.4}$$

for this purpose. *[18.1.3]* In §27-29 of [**25**] he immediately applies this formula to partially resolve Leibniz' paradox, and in §28 he gives the basic fractional derivative (reproduced here in modern notation with $\Gamma(n+1) = n!$)

$$\frac{\mathrm{d}^\alpha x^\beta}{\mathrm{d}x^\alpha} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} x^{\beta-\alpha} \tag{2.5}$$

valid for integer and for noninteger $\alpha, \beta$.

### 2.1.3. Paradoxa and Problems

*[18.2.1]* Generalizing eq. (2.5) to all functions that can be expanded into a power series might seem a natural step, but this "natural" definition of fractional derivatives does not really resolve Leibniz' paradox. *[18.2.2]* Leibniz had implicitly assumed the rule

$$\frac{\mathrm{d}^\alpha \mathrm{e}^{\lambda x}}{\mathrm{d}x^\alpha} = \lambda^\alpha \mathrm{e}^{\lambda x} \tag{2.6}$$

by demanding $\mathrm{d}^\alpha h = h \mathrm{d}x^\alpha$ for integer $\alpha$. *[18.2.3]* One might therefore take eq. (2.6) instead of eq. (2.5) as an equally "natural" starting point (this was later done by Liouville in [**76**, p.3, eq. (1)]), and define fractional derivatives as

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}x^\alpha} = \sum_k c_k \, \lambda_k^\alpha \, \mathrm{e}^{\lambda_k x} \tag{2.7}$$

for functions representable as exponential series $f(x) \sim \sum_k c_k \exp(\lambda_k x)$. *[18.2.4]* Regarding the integral (a Laplace integral)

$$x^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty \mathrm{e}^{-yx} y^{\beta-1} \mathrm{d}y \tag{2.8}$$

as a sum of exponentials, Liouville [**76**, p.7] then applied eq. (2.6) inside the integral to find

$$\frac{\mathrm{d}^\alpha x^{-\beta}}{\mathrm{d}x^\alpha} = \frac{1}{\Gamma(\beta)} \int_0^\infty \mathrm{e}^{-yx} (-y)^\alpha y^{\beta-1} \mathrm{d}y = \frac{(-1)^\alpha \Gamma(\beta+\alpha)}{\Gamma(\beta) \, x^{\beta+\alpha}}, \tag{2.9}$$

*[page 19, §0]* where the last equality follows by substituting $yx = z$ in the integral. *[19.0.1]* If this equation is formally generalized to $-\beta$, disregarding existence of the integral, one finds

$$\frac{\mathrm{d}^\alpha x^\beta}{\mathrm{d}x^\alpha} = \frac{(-1)^\alpha \Gamma(-\beta+\alpha)}{\Gamma(-\beta)}\, x^{\beta-\alpha} \tag{2.10}$$

a formula similar to, but different from eq. (2.5). *[19.0.2]* Although eq. (2.10) agrees with eq. (2.5) for integer $\alpha$ it differs for noninteger $\alpha$. *[19.0.3]* More precisely, if $\alpha = 1/2$ and $\beta = -1/2$, then

$$\frac{\Gamma(3/2)}{\Gamma(0)} x^{-1} = 0 \neq \frac{\mathrm{i}}{x\sqrt{\pi}} = \frac{(-1)^{1/2}\Gamma(1)}{\Gamma(1/2)} x^{-1} \tag{2.11}$$

revealing again an inconsistency between eq. (2.5) and eq. (2.10) (resp. (2.9)).

*[19.1.1]* Another way to see this inconsistency is to expand the exponential function into a power series, and to apply Euler's rule, eq. (2.5), to it. *[19.1.2]* One finds (with obvious notation)

$$\begin{aligned}\left(\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\right)_{(2.5)} \exp(x) &= \left(\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\right)_{(2.5)} \sum_{k=0}^{\infty} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^{k-\alpha}}{\Gamma(k-\alpha+1)} \\ &\neq \left(\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\right)_{(2.6)} \exp(x) = \exp(x)\end{aligned} \tag{2.12}$$

and this shows that Euler's rule (2.5) is inconsistent with the Leibniz/Liouville rule (2.6). *[19.1.3]* Similarly, Liouville found inconsistencies [**75**, p.95/96] when calculating the fractional derivative of $\exp(\lambda x) + \exp(-\lambda x)$ based on the definition (2.7).

*[19.2.1]* A resolution of Leibniz' paradox emerges when eq. (2.5) and (2.6) are compared for $\alpha = -1$, and interpreted as integrals. *[19.2.2]* Such an interpretation was already suggested by Leibniz himself [**73**]. *[19.2.3]* More specifically, one has

$$\frac{\mathrm{d}^{-1}\mathrm{e}^x}{\mathrm{d}x^{-1}} = \mathrm{e}^x = \int_{-\infty}^{x} \mathrm{e}^t \mathrm{d}t \neq \int_0^x \mathrm{e}^t \mathrm{d}t = \mathrm{e}^x - 1 = \frac{\mathrm{d}^{-1}}{\mathrm{d}x^{-1}} \sum_{k=0}^{\infty} \frac{x^k}{k!} \tag{2.13}$$

showing that Euler's fractional derivatives on the right hand side differs from Liouville's and Leibniz' idea on the left. *[19.2.4]* Similarly, eq. (2.5) corresponds to

$$\frac{\mathrm{d}^{-1}x^\beta}{\mathrm{d}x^{-1}} = \frac{x^{\beta+1}}{\beta+1} = \int_0^x y^\beta \mathrm{d}y. \tag{2.14}$$

*[19.2.5]* On the other hand, eq. (2.9) corresponds to

$$\frac{\mathrm{d}^{-1}x^{-\beta}}{\mathrm{d}x^{-1}} = \frac{x^{1-\beta}}{1-\beta} = -\int_x^\infty y^{-\beta}\mathrm{d}y = \int_\infty^x y^{-\beta}\mathrm{d}y. \tag{2.15}$$

*[page 20, §0]* *[20.0.1]* This shows that Euler's and Liouville's definitions differ with respect to their limits of integration.

### 2.1.4. Liouville

*[20.1.1]* It has already been mentioned that Liouville defined fractional derivatives using eq. (2.7) (see [**76**, p.3, eq.(1)]) as

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}x^\alpha} = \sum_k c_k\ \lambda_k^\alpha\ \mathrm{e}^{\lambda_k x} \tag{2.7}$$

for functions representable as a sum of exponentials

$$f(x) \sim \sum_k c_k \exp(\lambda_k x). \tag{2.16}$$

*[20.1.2]* Liouville seems not to have recognized the necessity of limits of integration. *[20.1.3]* From his definition (2.7) he derives numerous integral and series representations. *[20.1.4]* In particular, he finds the fractional integral of order $\alpha > 0$ as

$$\int^\alpha f(x)\mathrm{d}x^\alpha = \frac{1}{(-1)^\alpha \Gamma(\alpha)} \int\limits_0^\infty f(x+y) y^{\alpha-1} \mathrm{d}y \tag{2.17}$$

(see formula [A] on page 8 of [**76**, p.8]). *[20.1.5]* Liouville then gives formula [B] for fractional differentiation on page 10 of [**76**] as

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}x^\alpha} = \frac{1}{(-1)^{n-\alpha}\Gamma(n-\alpha)} \int\limits_0^\infty \frac{\mathrm{d}^n f(x+y)}{\mathrm{d}x^n}\ y^{n-\alpha-1}\mathrm{d}y, \tag{2.18}$$

where $n-1 < \alpha < n$. *[20.1.6]* Liouville restricts the discussion to functions represented by exponential series with $\lambda_k > 0$ so that $f(-\infty) = 0$. *[20.1.7]* Liouville also expands the coefficients $\lambda_k^\alpha$ in (2.7) into binomial series

$$\lambda_k^\alpha = \lim_{h\to 0} \frac{1}{h^\alpha}(1-\mathrm{e}^{-h\lambda_k})^\alpha, \qquad \lambda_k > 0 \tag{2.19a}$$

$$= (-1)^\alpha \lim_{h\to 0} \frac{1}{h^\alpha}(1-\mathrm{e}^{h\lambda_k})^\alpha, \qquad \lambda_k < 0 \tag{2.19b}$$

and inserts the expansion into his defintion (2.7) to arrive at formulae that contain the representation of integer order derivatives as limits of difference quotients (see [**75**, p.106ff]). *[20.1.8]* The results may be written as

$$\frac{\mathrm{d}^\alpha f}{\mathrm{d}x^\alpha} = \lim_{h\to 0} \left\{ \frac{1}{h^\alpha} \sum_{m=0}^\infty \left[ (-1)^m \binom{\alpha}{m} f(x-mh) \right] \right\} \tag{2.20a}$$

$$= (-1)^\alpha \lim_{h\to 0} \left\{ \frac{1}{h^\alpha} \sum_{m=0}^\infty \left[ (-1)^m \binom{\alpha}{m} f(x+mh) \right] \right\}, \tag{2.20b}$$

*[page 21, §0]* where the binomial coefficient $\binom{\alpha}{m}$ is $\Gamma(\alpha-1)\Gamma(m-1)/\Gamma(\alpha+m-1)$. *[21.0.1]* Later, this idea was taken up by Grünwald [**34**], who defined fractional derivatives as limits of generalized difference quotients.

### 2.1.5. Fourier

*[21.1.1]* Fourier [**29**] suggested to define fractional derivatives by generalizing the formula for trigonometric functions,

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\cos(x) = \cos\left(x + \frac{\alpha\pi}{2}\right), \tag{2.21}$$

from $\alpha \in \mathbb{N}$ to $\alpha \in \mathbb{R}$. *[21.1.2]* Again, this is not unique because the generalization

$$\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\cos(x) = (-1)^\alpha \cos\left(x - \frac{\alpha\pi}{2}\right) \tag{2.22}$$

is also possible.

### 2.1.6. Grünwald

*[21.2.1]* Grünwald wanted to free the definition of fractional derivatives from a special form of the function. *[21.2.2]* He emphasized that fractional derivatives are integroderivatives, and established for the first time general fractional derivative operators. *[21.2.3]* His calculus is based on limits of difference quotients. *[21.2.4]* He studies the difference quotients [**34**, p.444]

$$F[u,x,\alpha,h]_f = \sum_{k=0}^{n}(-1)^k\binom{\alpha}{k}\frac{f(x-kh)}{h^\alpha} \tag{2.23}$$

with $n = (x-u)/h$ and calls

$$\mathrm{D}^\alpha[f(x)]_{x=u}^{x=x} = \lim_{h\to 0} F[u,x,\alpha,h]_f \tag{2.24}$$

the *$\alpha$-th differential quotient taken over the straight line from $u$ to $x$* [**34**, p.452]. *[21.2.5]* The title of his work emphasizes the need to introduce limits of integration into the concept of differentiation. *[21.2.6]* His ideas were soon elaborated upon by Letnikov (see [**99**])and applied to differential equations by Most [**89**].

### 2.1.7. Riemann

*[21.3.1]* Riemann, like Grünwald, attempts to define fractional differentiation for general classes of functions. *[21.3.2]* Riemann defines the $n$-th differential quotient of a function $f(x)$ as the coeffcient of $h^n$ in the expansion of $f(x+h)$ into integer

*[page 22, §0]* powers of $h$ [**96**, p.354]. *[22.0.1]* He then generalizes this definition to noninteger powers, and demands that

$$f(x+h) = \sum_{n=-\infty}^{n=\infty} c_{n+\alpha}(\partial_x^{n+\alpha} f)(x)\; h^{n+\alpha} \tag{2.25}$$

holds for $n \in \mathbb{N}, \alpha \in \mathbb{R}$. *[22.0.2]* The factor $c_{n+\alpha}$ is determined such that $\partial^\beta(\partial^\gamma f) = \partial^{\beta+\gamma} f$ holds, and found to be $1/\Gamma(n+\alpha+1)$. *[22.0.3]* Riemann then derives the integral representation [**96**, p.363] for negative $\alpha$

$$\partial^\alpha f = \frac{1}{\Gamma(-\alpha)} \int\limits_k^x (x-t)^{-\alpha-1} f(t)\mathrm{d}t + \sum_{n=1}^{\infty} K_n \frac{x^{-\alpha-n}}{\Gamma(-n-\alpha+1)}, \tag{2.26}$$

where $k, K_n$ are finite constants. *[22.0.4]* He then extends the result to nonnegative $\alpha$ by writing *"für einen Werth von $\alpha$ aber, der $\geq 0$ ist, bezeichnet $\partial^\alpha f$ dasjenige, was aus $\partial^{\alpha-m} f$ (wo $m > \alpha$) durch $m$-malige Differentiation nach $x$ hervorgeht,..."* [**96**, p.341]. *[22.0.5]* The combination of Liouville's and Grünwald's pioneering work with this idea has become the definition of the Riemann-Liouville fractional derivatives (see Section 2.2.2.1 below).

## 2.2. Mathematical Introduction to Fractional Derivatives

*[22.1.1]* The brief historical introduction has shown that fractional derivatives may be defined in numerous ways. *[22.1.2]* A natural and frequently used approach starts from repeated integration and extends it to fractional integrals. *[22.1.3]* Fractional derivatives are then defined either by continuation of fractional integrals to negative order (following Leibniz' ideas [**73**]), or by integer order derivatives of fractional integrals (as suggested by Riemann [**96**]).

### 2.2.1. Fractional Integrals

#### 2.2.1.1. Iterated Integrals

*[22.2.1]* Consider a locally integrable[1] real valued function $f : \mathbb{G} \to \mathbb{R}$ whose domain of definition $\mathbb{G} = [a,b] \subseteq \mathbb{R}$ is an interval with $-\infty \leq a < b \leq \infty$. *[22.2.2]* Integrating

[1]A function $f : \mathbb{G} \to \mathbb{R}$ is called locally integrable if it is integrable on all compact subsets $K \subset \mathbb{G}$ (see eq.(B.9)).

*[page 23, §0]* $n$ times gives the fundamental formula

$$(\mathrm{I}^n_{a+} f)(x) = \int_a^x \int_a^{x_1} \dots \int_a^{x_{n-1}} f(x_n)\, \mathrm{d}x_n \dots \mathrm{d}x_2 \mathrm{d}x_1$$
$$= \frac{1}{(n-1)!} \int_a^x (x-y)^{n-1} f(y)\, \mathrm{d}y, \tag{2.27}$$

where $a < x < b$ and $n \in \mathbb{N}$. *[23.0.1]* This formula may be proved by induction. *[23.0.2]* It reduces $n$-fold integration to a single convolution integral (Faltung). *[23.0.3]* The subscript $a+$ indicates that the integration has $a$ as its lower limit. *[23.0.4]* An analogous formula holds with lower limit $x$ and upper limit $a$. *[23.0.5]* In that case the subscript $a-$ will be used.

### 2.2.1.2. Riemann-Liouville Fractional Integrals

*[23.1.1]* Equation (2.27) for $n$-fold integration can be generalized to noninteger values of $n$ using the relation $(n-1)! = \prod_{k=1}^{n-1} k = \Gamma(n)$ where

$$\Gamma(z) = \int_0^1 (-\log x)^{z-1}\, \mathrm{d}x \tag{2.28}$$

is Euler's $\Gamma$-function defined for all $z \in \mathbb{C}$.

**Definition 2.1** *[23.2.1]* Let $-\infty \le a < x < b \le \infty$. *[23.2.2]* The *Riemann-Liouville fractional integral of order $\alpha > 0$ with lower limit $a$* is defined for locally integrable functions $f : [a, b] \to \mathbb{R}$ as

$$(\mathrm{I}^\alpha_{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-y)^{\alpha-1} f(y)\, \mathrm{d}y \tag{2.29a}$$

for $x > a$. *[23.2.3]* The *Riemann-Liouville fractional integral of order $\alpha > 0$ with upper limit $b$* is defined as

$$(\mathrm{I}^\alpha_{b-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (y-x)^{\alpha-1} f(y)\, \mathrm{d}y \tag{2.29b}$$

for $x < b$. *[23.2.4]* For $\alpha = 0$

$$(\mathrm{I}^0_{a+} f)(x) = (\mathrm{I}^0_{b-} f)(x) = f(x) \tag{2.30}$$

completes the definition. *[23.2.5]* The definition may be generalized to $\alpha \in \mathbb{C}$ with $\mathrm{Re}\,\alpha > 0$.

*[page 24, §1]* *[24.1.1]* Formula (2.29a) appears in [**96**, p.363] with $a > -\infty$ and in [**76**, p.8] with $a = -\infty$. *[24.1.2]* The notation is not standardized. *[24.1.3]* Leibniz, Lagrange and Liouville used the symbol $\int^\alpha$ [**22**, **73**, **76**], Grünwald wrote $\int^\alpha [...\mathrm{d}x^\alpha]_{x=a}^{x=x}$, while Riemann used $\partial_x^{-\alpha}$ [**96**] and Most wrote $\mathrm{d}_a^{-\alpha}/\mathrm{d}x^{-\alpha}$ [**89**]. *[24.1.4]* The notation in (2.29) is that of [**52**, **54**, **98**, **99**]. *[24.1.5]* Modern authors also use $f_\alpha$ [**37**], $\mathrm{I}^\alpha$ [**97**], ${}_aI_x^\alpha$ [**94**], $I_x^\alpha$ [**23**], ${}_aD_x^{-\alpha}$ [**85**, **91**, **102**], or $\mathrm{d}^{-\alpha}/\mathrm{d}(x-a)^{-\alpha}$ [**92**] instead of $\mathrm{I}_{a+}^\alpha$ [2].

*[24.2.1]* The fractional integral operators $\mathrm{I}_{a+}^\alpha, \mathrm{I}_{b-}^\alpha$ are commonly called Riemann-Liouville fractional integrals [**94**, **98**, **99**] although sometimes this name is reserved for the case $a = 0$ [**85**]. *[24.2.2]* Their domain of definition is typically chosen as $D(\mathrm{I}_{a+}^\alpha) = L^1([a,b])$ or $D(\mathrm{I}_{a+}^\alpha) = L_{\mathrm{loc}}^1([a,b])$ [**94**, **98**, **99**]. *[24.2.3]* For the definition of Lebesgue spaces see the Appendix B. *[24.2.4]* If $f \in L^1([a,b])$ then $(\mathrm{I}_{a+}^\alpha f) \in L^1([a,b])$ and $(\mathrm{I}_{a+}^\alpha f)(x)$ is finite for almost all $x$. *[24.2.5]* If $f \in L^p([a,b])$ with $1 \le p \le \infty$ and $\alpha > 1/p$ then $(\mathrm{I}_{a+}^\alpha f)(x)$ is finite for all $x \in [a,b]$. *[24.2.6]* Analogous statements hold for $(\mathrm{I}_{b-}^\alpha f)(x)$ [**98**].

*[24.3.1]* A short table of Riemann-Liouville fractional integrals is given in Appendix 1. *[24.3.2]* For a more extensive list of fractional integrals see [**24**].

### 2.2.1.3. Weyl Fractional Integrals

*[24.4.1]* Examples (2.5) and (2.6) or (A.2) and (A.3) show that Definition 2.1 is well suited for fractional integration of power series, but not for functions defined by Fourier series. *[24.4.2]* In fact, if $f(x)$ is a periodic function with period $2\pi$, and[3]

$$f(x) \sim \sum_{k=-\infty}^{\infty} c_k \mathrm{e}^{\mathrm{i}kx} \tag{2.31}$$

then the Riemann-Liouville fractional $(\mathrm{I}_{a+}^\alpha f)$ will in general not be periodic. *[24.4.3]* For this reason an alternative definition of fractional integrals was investigated by Weyl [**124**].

*[24.5.1]* Functions on the unit circle $\mathbb{G} = \mathbb{R}/2\pi\mathbb{Z}$ correspond to $2\pi$-periodic functions on the real line. *[24.5.2]* Let $f(x)$ be periodic with period $2\pi$ and such that the integral of $f$ over the interval $[0, 2\pi]$ vanishes, so that $c_0 = 0$ in eq. (2.31). *[24.5.3]* Then the integral of $f$ is itself a periodic function, and the constant of integration can be chosen such that the integral over $[0, 2\pi]$ vanishes again. *[24.5.4]* Repeating the integration $n$ times one finds using (2.6) and the integral representation

[2] Some authors [**23**, **26**, **85**, **91**, **92**, **97**] employ the derivative symbol $D$ also for integrals, resp. $I$ for derivatives, to emphasize the similarity between fractional integration and differentiation. If this is done, the choice of Riesz and Feller, namely $I$, seems superior in the sense that fractional derivatives, similar to integrals, are nonlocal operators, while integer derivatives are local operators.

[3] The notation $\sim$ indicates that the sum does not need to converge, and, if it converges, does not need to converge to $f(x)$.

*[page 25, §0]* $c_k = (1/2\pi)\int_0^{2\pi} \mathrm{e}^{-iks} f(s)\mathrm{d}s$ of Fourier coefficients

$$\sum_{k=-\infty}^{\infty} c_k \frac{e^{ikx}}{(ik)^n} = \frac{1}{2\pi}\int\limits_0^{2\pi} f(y) \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \frac{e^{ik(x-y)}}{(ik)^n}\mathrm{d}y \tag{2.32}$$

with $c_0 = 0$. *[25.0.1]* Recall the convolution formula [**132**, p.36]

$$(f * g)(t) = \frac{1}{2\pi}\int\limits_0^{2\pi} f(t-s)g(s)\mathrm{d}s = \sum_{k=-\infty}^{\infty} f_k g_k \mathrm{e}^{ikt} \tag{2.33}$$

for two periodic functions $f(t) \sim \sum_{k=-\infty}^{\infty} f_k \mathrm{e}^{ikt}$ and $g(t) \sim \sum_{k=-\infty}^{\infty} g_k \mathrm{e}^{ikt}$. *[25.0.2]* Using eq. (2.33) and generalizing (2.32) to noninteger $n$ suggests the following definition. [**94**, **99**].

**Definition 2.2** *[25.1.1]* Let $f \in L^p(\mathbb{R}/2\pi\mathbb{Z}), 1 \le p < \infty$ be periodic with period $2\pi$ and such that its integral over a period vanishes. *[25.1.2]* The *Weyl fractional integral of order $\alpha$* is defined as

$$(\mathrm{I}^{\alpha}_{\pm} f)(x) = (\Psi^{\alpha}_{\pm} * f)(x) = \frac{1}{2\pi}\int\limits_0^{2\pi} \Psi^{\alpha}_{\pm}(x-y)f(y)\mathrm{d}y, \tag{2.34}$$

where

$$\Psi^{\alpha}_{\pm}(x) = \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \frac{e^{ikx}}{(\pm \mathrm{i}k)^{\alpha}} \tag{2.35}$$

for $0 < \alpha < 1$.

*[25.2.1]* It can be shown that the series for $\Psi^{\alpha}_{\pm}(x)$ converges and that the Weyl definition coincides with the Riemann-Liouville definition [**133**]

$$(\mathrm{I}^{\alpha}_{+} f)(x) = \frac{1}{\Gamma(\alpha)}\int\limits_{-\infty}^{x} (x-y)^{\alpha-1} f(y)\,\mathrm{d}y, \tag{2.36a}$$

respectively

$$(\mathrm{I}^{\alpha}_{-} f)(x) = \frac{1}{\Gamma(\alpha)}\int\limits_{x}^{\infty} (y-x)^{\alpha-1} f(y)\,\mathrm{d}y \tag{2.36b}$$

for $2\pi$ periodic functions whose integral over a period vanishes. *[25.2.2]* This is eq. (2.29) with $a = -\infty$ resp. $b = \infty$. *[25.2.3]* For this reason the Riemann-Liouville *[page 26, §0]* fractional integrals with limits $\pm\infty$, $\mathrm{I}^{\alpha}_{+} f = \mathrm{I}^{\alpha}_{(-\infty)+} f$ and $\mathrm{I}^{\alpha}_{-} f = \mathrm{I}^{\alpha}_{\infty-} f$, are often called *Weyl fractional integrals* [**24, 85, 94, 99**].

*[26.1.1]* The Weyl fractional integral may be rewritten as a convolution

$$(\mathrm{I}^{\alpha}_{\pm} f)(x) = (K^{\alpha}_{\pm} * f)(x), \tag{2.37}$$

where the convolution product for functions on $\mathbb{R}$ is defined as[4]

$$(K * f)(x) := \int\limits_{-\infty}^{\infty} K(x-y)f(y)\mathrm{d}y \tag{2.38}$$

and the convolution kernels are defined as

$$K^{\alpha}_{\pm}(x) := \Theta(\pm x)\frac{(\pm x)^{\alpha-1}}{\Gamma(\alpha)} \tag{2.39}$$

for $\alpha > 0$. *[26.1.3]* Here

$$\Theta(x) = \begin{cases} 1 & , x > 0 \\ \\ 0 & , x \le 0 \end{cases} \tag{2.40}$$

is the Heaviside unit step function, and $x^{\alpha} = \exp \alpha \log x$ with the convention that $\log x$ is real for $x > 0$. *[26.1.4]* For $\alpha = 0$ the kernel

$$K^{0}_{+}(x) = K^{0}_{-}(x) = \delta(x) \tag{2.41}$$

is the Dirac $\delta$-function defined in (C.2) in Appendix 1. *[26.1.5]* Note that $K^{\alpha}_{\pm} \in L^{1}_{\mathrm{loc}}(\mathbb{R})$ for $\alpha > 0$.

### 2.2.1.4. Riesz Fractional Integrals

*[26.2.1]* Riemann-Liouville and Weyl fractional integrals have upper or lower limits of integration, and are sometimes called left-sided resp. right-sided integrals. *[26.2.2]* A more symmetric definition was advanced in [**97**].

**Definition 2.3** *[26.3.1]* Let $f \in L^{1}_{\mathrm{loc}}(\mathbb{R})$ be locally integrable. *[26.3.2]* The *Riesz fractional integral* or *Riesz potential* of order $\alpha > 0$ is defined as the linear combination [**99**]

$$\begin{aligned}(\mathrm{I}^{\alpha} f)(x) &= \frac{(\mathrm{I}^{\alpha}_{+} f)(x) + (\mathrm{I}^{\alpha}_{-} f)(x)}{2\cos(\alpha\pi/2)} \\ &= \frac{1}{2\Gamma(\alpha)\cos(\alpha\pi/2)} \int\limits_{-\infty}^{\infty} \frac{f(y)}{|x-y|^{1-\alpha}}\mathrm{d}y \end{aligned} \tag{2.42}$$

[4] If $K, f \in L^{1}(\mathbb{R})$ then $(K * f)(t)$ exists for almost all $t \in \mathbb{R}$ and $f \in L^{1}(\mathbb{R})$. *[26.1.2]* If $K \in L^{p}(\mathbb{R})$, $f \in L^{q}(\mathbb{R})$ with $1 < p, q < \infty$ and $1/p + 1/q = 1$ then $K * f \in C_0(\mathbb{R})$, the space of continuous functions vanishing at infinity.

*[page 27, §0]* of right- and left-sided Weyl fractional integrals. *[27.0.1]* The *conjugate Riesz potential* is defined by

$$(\widetilde{\mathrm{I}^\alpha} f)(x) = \frac{(\mathrm{I}_+^\alpha f)(x) - (\mathrm{I}_-^\alpha f)(x)}{2\sin(\alpha\pi/2)}$$
$$= \frac{1}{2\Gamma(\alpha)\sin(\alpha\pi/2)} \int\limits_{-\infty}^{\infty} \frac{\mathrm{sgn}(x-y)f(y)}{|x-y|^{1-\alpha}}\mathrm{d}y. \tag{2.43}$$

*[27.0.2]* Of course, $\alpha \neq 2k+1, k \in \mathbb{Z}$ in (2.42) and $\alpha \neq 2k, k \in \mathbb{Z}$ in (2.43). *[27.0.3]* The definition is again completed with

$$(\mathrm{I}^0 f)(x) = (\widetilde{\mathrm{I}^0} f)(x) = f(x) \tag{2.44}$$

for $\alpha = 0$.

*[27.1.1]* Riesz fractional integration may be written as a convolution

$$(\mathrm{I}^\alpha f)(x) = (K^\alpha * f)(x) \tag{2.45a}$$
$$(\widetilde{\mathrm{I}^\alpha} f)(x) = (\widetilde{K}^\alpha * f)(x) \tag{2.45b}$$

with the (one-dimensional) Riesz kernels

$$K^\alpha(x) = \frac{K_+^\alpha(x) + K_-^\alpha(x)}{2\cos(\alpha\pi/2)} = \frac{|x|^{\alpha-1}}{2\cos(\alpha\pi/2)\Gamma(\alpha)} \tag{2.46}$$

for $\alpha \neq 2k+1, k \in \mathbb{Z}$, and

$$\widetilde{K}^\alpha(x) = \frac{K_+^\alpha(x) - K_-^\alpha(x)}{2\sin(\alpha\pi/2)} = \frac{|x|^{\alpha-1}\,\mathrm{sgn}(x)}{2\sin(\alpha\pi/2)\Gamma(\alpha)} \tag{2.47}$$

for $\alpha \neq 2k, k \in \mathbb{Z}$. *[27.1.2]* Subsequently, Feller introduced the generalized Riesz-Feller kernels **[26]**

$$K^{\alpha,\beta}(x) = \frac{|x|^{\alpha-1}\sin\left[\alpha\left(\pi/2 + \beta\,\mathrm{sgn}\,x\right)\right]}{2\sin(\alpha\pi/2)\Gamma(\alpha)} \tag{2.48}$$

with parameter $\beta \in \mathbb{R}$. *[27.1.3]* The corresponding generalized *Riesz-Feller fractional integral of order* $\alpha$ *and type* $\beta$ is defined as

$$(\mathrm{I}^{\alpha,\beta} f)(x) = (K^{\alpha,\beta} * f)(x). \tag{2.49}$$

*[27.1.4]* This formula interpolates continuously from the Weyl integral $\mathrm{I}_-^\alpha = \mathrm{I}^{\alpha,-\pi/2}$ for $\beta = -\pi/2$ through the Riesz integral $\mathrm{I}^\alpha = \mathrm{I}^{\alpha,0}$ for $\beta = 0$ to the Weyl integral $\mathrm{I}_+^\alpha = \mathrm{I}^{\alpha,\pi/2}$ for $\beta = \pi/2$. *[27.1.5]* Due to their symmetry Riesz-Feller fractional integrals are readily generalized to higher dimensions.

*[page 28, §1]*

### 2.2.1.5. Fractional Integrals of Distributions

*[28.1.1]* Fractional integration can be extended to distributions using the convolution formula (2.37) above. *[28.1.2]* Distributions are generalized functions [**31**,**105**]. *[28.1.3]* They are defined as linear functionals on a space $X$ of conveniently chosen "test functions". *[28.1.4]* For every locally integrable function $f \in L^1_{\text{loc}}(\mathbb{R})$ there exists a distribution $F_f : X \to \mathbb{C}$ defined by

$$F_f(\varphi) = \langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\, \mathrm{d}x, \tag{2.50}$$

where $\varphi \in X$ is test function from a suitable space $X$ of test functions. *[28.1.5]* By abuse of notation one often writes $f$ for the associated distribution $F_f$. *[28.1.6]* Distributions that correspond to functions via (2.50) are called *regular distributions*. *[28.1.7]* Examples for regular distributions are the convolution kernels $K^\alpha_\pm \in L^1_{\text{loc}}(\mathbb{R})$ defined in (2.39). *[28.1.8]* They are locally integrable functions on $\mathbb{R}$ when $\alpha > 0$. *[28.1.9]* Distributions that are not regular are sometimes called *singular*. *[28.1.10]* An important example for a singular distribution is the Dirac $\delta$-function. *[28.1.11]* It is defined as $\delta : X \to \mathbb{C}$

$$\int \delta(x)\varphi(x)\mathrm{d}x = \varphi(0) \tag{2.51}$$

for every test function $\varphi \in X$. *[28.1.12]* The test function space $X$ is usually chosen as a subspace of $C^\infty(\mathbb{R})$, the space of infinitely differentiable functions. *[28.1.13]* A brief introduction to distributions is given in Appendix 1.

*[28.2.1]* In order to generalize (2.37) to distributions one must define the convolution of two distributions. *[28.2.2]* To do so one multiplies eq. (2.38) on both sides with a smooth test function $\varphi \in C^\infty_\mathrm{c}(\mathbb{R})$ of compact support. *[28.2.3]* Integrating gives

$$\begin{aligned}
\langle K * f, \varphi \rangle &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x-y)f(y)\varphi(x)\mathrm{d}y\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x)f(y)\varphi(x+y)\mathrm{d}y\mathrm{d}x \\
&= \langle K(x), \langle f(y), \varphi(x+y)\rangle\rangle,
\end{aligned} \tag{2.52}$$

where the notation $\langle f(y), \varphi(x+y)\rangle$ means that the functional $F_f$ is applied to the function $\varphi(x+\cdot)$ for fixed $x$. *[28.2.4]* Explicitly, for fixed $x$

$$F_f(\varphi_x) = \langle f(y), \varphi_x(y)\rangle = \langle f(y), \varphi(x+y)\rangle = \int_{-\infty}^{\infty} f(y)\varphi(x+y)\mathrm{d}x, \tag{2.53}$$

*[page 29, §0]* where $\varphi_x(\cdot) = \varphi(x+\cdot)$. *[29.0.1]* Equation (2.52) can be used as a definition for the convolution of distributions provided that the right hand side has meaning. *[29.0.2]* This is not always the case as the counterexample $K = f = 1$ shows. *[29.0.3]* In general the convolution product is not associative (see eq. (2.113)). *[29.0.4]* However, associative and commutative convolution algebras exist [**21**]. *[29.0.5]* Equation (2.52) is always meaningful when $\operatorname{supp} K$ or $\operatorname{supp} f$ is compact [**63**]. *[29.0.6]* Another case is when $K$ and $f$ have support in $\mathbb{R}_+$. *[29.0.7]* This will be assumed in the following.

**Definition 2.4** *[29.1.1]* Let $f$ be a distribution $f \in C_0^\infty(\mathbb{R})'$ with $\operatorname{supp} f \subset \mathbb{R}_+$. *[29.1.2]* Then its fractional integral is the distribution $\mathrm{I}_{0+}^\alpha f$ defined as

$$\langle \mathrm{I}_{0+}^\alpha f, \varphi\rangle = \langle \mathrm{I}_{+}^\alpha f, \varphi\rangle = \langle K_+^\alpha * f, \varphi\rangle \tag{2.54}$$

for $\operatorname{Re}\alpha > 0$. *[29.1.3]* It has support in $\mathbb{R}_+$.

*[29.2.1]* If $f \in C_0^\infty(\mathbb{R})'$ with $\operatorname{supp} f \subset \mathbb{R}_+$ then also $\mathrm{I}_{0+}^\alpha f \in C_0^\infty(\mathbb{R})'$ with $\operatorname{supp} \mathrm{I}_{0+}^\alpha f \subset \mathbb{R}_+$.

#### 2.2.1.6. Integral Transforms

*[29.3.1]* The Fourier transformation is defined as

$$\mathcal{F}\{f\}(k) = \int\limits_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}kx} f(x)\,\mathrm{d}x \tag{2.55}$$

for functions $f \in L^1(\mathbb{R})$. *[29.3.2]* Then

$$\mathcal{F}\left\{\mathrm{I}_\pm^\alpha f\right\}(k) = (\pm\mathrm{i}k)^{-\alpha}\mathcal{F}\{f\}(k) \tag{2.56}$$

holds for $0 < \alpha < 1$ by virtue of the convolution theorem. *[29.3.3]* The equation cannot be extended directly to $\alpha \geq 1$ because the Fourier integral on the left hand side may not exist. *[29.3.4]* Consider e.g. $\alpha = 1$ and $f \in C_\mathrm{c}^\infty(\mathbb{R})$. *[29.3.5]* Then $(\mathrm{I}_+^1 f)(x) \to$const as $x \to \infty$ and $\mathcal{F}\left\{\mathrm{I}_+^1 f\right\}$ does not exist [**94**]. *[29.3.6]* Equation (2.56) can be extended to all $\alpha$ with $\operatorname{Re}\alpha > 0$ for functions in the so called Lizorkin space [**99**, p.148] defined as the space of functions $f \in \mathcal{S}(\mathbb{R})$ such that $(\mathrm{D}^m\,\mathcal{F}\{f\})(0) = 0$ for all $m \in \mathbb{N}_0$.

*[29.4.1]* For the Riesz potentials one has

$$\mathcal{F}\{\mathrm{I}^\alpha f\}(k) = |k|^{-\alpha}\mathcal{F}\{f\}(k) \tag{2.57a}$$

$$\mathcal{F}\left\{\widetilde{\mathrm{I}^\alpha} f\right\}(k) = (-\mathrm{i}\operatorname{sgn} k)|k|^{-\alpha}\mathcal{F}\{f\}(k) \tag{2.57b}$$

for functions in Lizorkin space.

*[page 30, §1]* *[30.1.1]* The Laplace transform is defined as

$$\mathcal{L}\{f\}(u) = \int\limits_0^\infty \mathrm{e}^{-ux} f(x)\,\mathrm{d}x \tag{2.58}$$

for locally integrable functions $f : \mathbb{R}_+ \to \mathbb{C}$. *[30.1.2]* Now

$$\mathcal{L}\left\{\mathrm{I}^\alpha_{0+} f\right\}(u) = u^{-\alpha}\mathcal{L}\{f\}(u) \tag{2.59}$$

by the convolution theorem for Laplace transforms. *[30.1.3]* The Laplace transform of $\mathrm{I}^\alpha_{0-} f$ leads to a more complicated operator.

#### 2.2.1.7. Fractional Integration by Parts

*[30.2.1]* If $f(x) \in L^p([a,b]), g \in L^q([a,b])$ with $1/p+1/q \le 1+\alpha$, $p,q \ge 1$ and $p \neq 1$, $q \neq 1$ for $1/p + 1/q = 1 + \alpha$ then the formula

$$\int\limits_a^b f(x)(\mathrm{I}^\alpha_{a+} g)(x)\mathrm{d}x = \int\limits_a^b g(x)(\mathrm{I}^\alpha_{b-} f)(x)\mathrm{d}x \tag{2.60}$$

holds. *[30.2.2]* The formula is known as fractional integration by parts [**99**]. *[30.2.3]* For $f(x) \in L^p(\mathbb{R}), g \in L^q(\mathbb{R})$ with $p > 1, q > 1$ and $1/p + 1/q = 1 + \alpha$ the analogous formula

$$\int\limits_{-\infty}^\infty f(x)(\mathrm{I}^\alpha_{+} g)(x)\mathrm{d}x = \int\limits_{-\infty}^\infty g(x)(\mathrm{I}^\alpha_{-} f)(x)\mathrm{d}x \tag{2.61}$$

holds for Weyl fractional integrals.

*[30.3.1]* These formulae provide a second method of generalizing fractional integration to distributions. *[30.3.2]* Equation (2.60) may be read as

$$\langle \mathrm{I}^\alpha_{a+} f, \varphi\rangle = \langle f, \mathrm{I}^\alpha_{b-}\varphi\rangle \tag{2.62}$$

for a distribution $f$ and a test function $\varphi$. *[30.3.3]* It shows that right- and left-sided fractional integrals are adjoint operators. *[30.3.4]* The formula may be viewed as a definition of the fractional integral $\mathrm{I}^\alpha_{a+} f$ of a distribution provided that the operator $\mathrm{I}^\alpha_{b-}$ maps the test function space into itself.

#### 2.2.1.8. Hardy-Littlewood Theorem

*[30.4.1]* The mapping properties of convolutions can be studied with the help of Youngs inequality. Let $p, q, r$ obey $1 \le p, q, r \le \infty$ and $1/p + 1/q = 1 + 1/r$. *[30.4.2]* If $K \in L^p(\mathbb{R})$ and $f \in L^q(\mathbb{R})$ then $K * f \in L^r(\mathbb{R})$ and Youngs inequality $\|K * f\|_r \le \|K\|_p\,\|f\|_q$ holds. *[30.4.3]* It follows that $\|K * f\|_q \le C\|f\|_p$ if

*[page 31, §0]* $1 \leq p \leq q \leq \infty$ and $K \in L^r(\mathbb{R})$ with $1/r = 1 + (1/q) - (1/p)$. *[31.0.1]* The Hardy-Littlewood theorem states that these estimates remain valid for $K^\alpha_\pm$ although these kernels do not belong to any $L^p(\mathbb{R})$-space [**37**,**38**]. *[31.0.2]* The theorem was generalized to higher dimensions by Sobolev in 1938, and is also known as the Hardy-Littlewood-Sobolev inequality (see [**37**, **38**, **63**, **113**]).

**Theorem 2.5** *[31.1.1] Let $0 < \alpha < 1$, $1 < p < 1/\alpha$, $-\infty \leq a < b \leq \infty$. [31.1.2] Then $\mathrm{I}^\alpha_{a+}, \mathrm{I}^\alpha_{b-}$ are bounded linear operators from $L^p([a,b])$ to $L^q([a,b])$ with $1/q = (1/p) - \alpha$, i.e. there exists a constant $C(p,q)$ independent of $f$ such that $\| \mathrm{I}^\alpha_{a+} f\|_q \leq C\|f\|_p$.*

### 2.2.1.9. Additivity

*[31.2.1]* The basic composition law for fractional integrals follows from

$$\begin{aligned}
(K^\alpha_+ * K^\beta_+)(x) &= \int_0^x K^\alpha_+(x-y) K^\beta_+(y)\,\mathrm{d}y = \int_0^x \frac{(x-y)^{\alpha-1}}{\Gamma(\alpha)} \frac{y^{\beta-1}}{\Gamma(\beta)} \mathrm{d}y \\
&= \frac{x^{\alpha-1}}{\Gamma(\alpha)} \frac{x^{\beta-1}}{\Gamma(\beta)} \int_0^1 (1-z)^{\alpha-1} z^{\beta-1} x \mathrm{d}z \\
&= \frac{x^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} = K^{\alpha+\beta}_+(x),
\end{aligned} \tag{2.63}$$

where Euler's Beta-function

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} = \int_0^1 (1-z)^{\alpha-1} z^{\beta-1} \mathrm{d}z = \mathrm{B}(\alpha, \beta) \tag{2.64}$$

was used. *[31.2.2]* This implies the semigroup law for exponents

$$\mathrm{I}^\alpha_{a+}\, \mathrm{I}^\beta_{a+} = \mathrm{I}^{\alpha+\beta}_{a+}, \tag{2.65}$$

also called additivity law. *[31.2.3]* It holds for Riemann-Liouville, Weyl and Riesz-Feller fractional integrals of functions.

## 2.2.2. Fractional Derivatives

### 2.2.2.1. Riemann-Liouville Fractional Derivatives

*[31.3.1]* Riemann [**96**, p.341] suggested to define fractional derivatives as integer order derivatives of fractional integrals.

**Definition 2.6** *[31.4.1]* Let $-\infty \leq a < x < b \leq \infty$. *[31.4.2]* The *Riemann-Liouville fractional derivative of order* $0 < \alpha < 1$ *with lower limit a (resp. upper limit b)* is defined for *[page 32, §0]* functions such that $f \in L^1([a,b])$ and $f * K^{1-\alpha} \in W^{1,1}([a,b])$ as

$$(\mathrm{D}^{\alpha}_{a\pm} f)(x) = \pm\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{I}^{1-\alpha}_{a\pm} f)(x) \tag{2.66}$$

and $(\mathrm{D}^0_{a\pm} f)(x) = f(x)$ for $\alpha = 0$. *[32.0.1]* For $\alpha > 1$ the definition is extended for functions $f \in L^1([a,b])$ with $f * K^{n-\alpha} \in W^{n,1}([a,b])$ as

$$(\mathrm{D}^{\alpha}_{a\pm} f)(x) = (\pm 1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\mathrm{I}^{n-\alpha}_{a\pm} f)(x), \tag{2.67}$$

where[5] $n = [\mathrm{Re}\,\alpha] + 1$ is smallest integer larger than $\alpha$.

*[32.1.1]* Here $W^{k,p}(G) = \{f \in L^p(G) : \mathrm{D}^k f \in L^p(G)\}$ denotes a Sobolev space defined in (B.17). *[32.1.2]* For $k = p = 1$ the space $W^{1,1}([a,b]) = \mathrm{AC}^0([a,b])$ coincides with the space of absolutely continuous functions.

*[32.2.1]* The notation for fractional derivatives is not standardized[6]. *[32.2.2]* Leibniz and Euler used $\mathrm{d}^\alpha$ **[25,72,73]** Riemann wrote $\partial^\alpha_x$ **[96]**, Liouville preferred $\mathrm{d}^\alpha/\mathrm{d}x^\alpha$ **[76]**, Grünwald used $\{\mathrm{d}^\alpha f/\mathrm{d}x^\alpha\}^{x=x}_{x=a}$ or $\mathrm{D}^\alpha[f]^{x=x}_{x=a}$ **[34]**, Marchaud wrote $\mathrm{D}^{(\alpha)}_a$, and Hardy-Littlewood used an index $f^\alpha$ **[37]**. *[32.2.3]* The notation in (2.67) follows **[52,54,98,99]**. Modern authors also use $I^{-\alpha}$ **[97]**, $I^{-\alpha}_x$ **[23]**, ${}_aD^\alpha_x$ **[85,94,102]**, $\mathrm{d}^\alpha/\mathrm{d}x^\alpha$ **[102,129]**, $\mathrm{d}^\alpha/\mathrm{d}(x-a)^\alpha$ **[92]** instead of $\mathrm{D}^\alpha_{a+}$.

*[32.3.1]* Let $f(x)$ be absolutely continuous on the finite interval $[a,b]$. *[32.3.2]* Then, its derivative $f'$ exists almost everywhere on $[a,b]$ with $f' \in L^1([a,b])$, and the function $f$ can be written as

$$f(x) = \int_a^x f'(y)\mathrm{d}y + f(a) = (\mathrm{I}^1_{a+} f')(x) + f(a). \tag{2.68}$$

Substituting this into $\mathrm{I}^\alpha_{a+} f$ gives

$$(\mathrm{I}^\alpha_{a+} f)(x) = (\mathrm{I}^1_{a+}\,\mathrm{I}^\alpha_{a+} f')(x) + \frac{f(a)}{\Gamma(\alpha+1)}(x-a)^\alpha, \tag{2.69}$$

where commutativity of $\mathrm{I}^1_{a+}$ and $\mathrm{I}^\alpha_{a+}$ was used. *[32.3.3]* It follows that

$$(\mathrm{D}\,\mathrm{I}^\alpha_{a+} f)(x) - (\mathrm{I}^\alpha_{a+}\,\mathrm{D}\, f)(x) = \frac{f(a)}{\Gamma(\alpha)}(x-a)^{\alpha-1} \tag{2.70}$$

for $0 < \alpha < 1$. *[32.3.4]* Above, the notations

$$(\mathrm{D}\, f)(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = f'(x) \tag{2.71}$$

were used for the first order derivative.

[5] $[x]$ is the largest integer smaller than $x$.
[6] see footnote 2

*[page 33, §1]* *[33.1.1]* This observation suggests to introduce a modified Riemann-Liouville fractional derivative through

$$(\widetilde{\mathrm{D}}^{\alpha}_{a+} f)(x) := \mathrm{I}^{n-\alpha}_{a+} f^{(n)}(x) = \frac{1}{\Gamma(n-\alpha)} \int\limits_a^x \frac{f^{(n)}(y)}{(x-y)^{\alpha-n+1}} \mathrm{d}y, \tag{2.72}$$

where $n = [\operatorname{Re}\alpha] + 1$. *[33.1.2]* Note, that $f$ must be at least $n$-times differentiable. *[33.1.3]* Formula (2.72) is due to Liouville [**76**, p.10] (see eq. (2.18) above), but nowadays sometimes named after Caputo [**17**].

*[33.2.1]* The relation between (2.72) and (2.67) is given by

**Theorem 2.7** *[33.2.2] For $f \in \mathrm{AC}^{n-1}([a,b])$ with $n = [\operatorname{Re}\alpha] + 1$ the Riemann-Liouville fractional derivative $(\mathrm{D}^{\alpha}_{a+} f)(x)$ exists almost everywhere for $\operatorname{Re}\alpha \geq 0$. [33.2.3] It can be written as*

$$(\mathrm{D}^{\alpha}_{a+} f)(x) = (\widetilde{\mathrm{D}}^{\alpha}_{a+} f)(x) + \sum_{k=0}^{n-1} \frac{(x-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}(a) \tag{2.73}$$

*in terms of the Liouville(-Caputo) derivative defined in* (2.72).

*[33.3.1]* The Riemann-Liouville fractional derivative is the left inverse of Riemann-Liouville fractional integrals. *[33.3.2]* More specifically, [**99**, p.44]

**Theorem 2.8** *[33.3.3] Let $f \in L^1([a,b])$. [33.3.4] Then*

$$\mathrm{D}^{\alpha}_{a+} \mathrm{I}^{\alpha}_{a+} f(x) = f(x) \tag{2.74}$$

*holds for all $\alpha$ with $\operatorname{Re}\alpha \geq 0$.*

*[33.4.1]* For the right inverses of fractional integrals one finds

**Theorem 2.9** *[33.4.2] Let $f \in L^1([a,b])$ and $\operatorname{Re}\alpha > 0$. [33.4.3] If in addition $\mathrm{I}^{n-\alpha}_{a+} f \in \mathrm{AC}^n([a,b])$ where $n = [\operatorname{Re}\alpha] + 1$ then*

$$\mathrm{I}^{\alpha}_{a+} \mathrm{D}^{\alpha}_{a+} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(x-a)^{\alpha-k-1}}{\Gamma(\alpha-k)} \left(\mathrm{D}^{n-k-1} \mathrm{I}^{n-\alpha}_{a+} f\right)(a) \tag{2.75}$$

*holds. [33.4.4] For $0 < \operatorname{Re}\alpha < 1$ this becomes*

$$\mathrm{I}^{\alpha}_{a+} \mathrm{D}^{\alpha}_{a+} f(x) = f(x) - \frac{(\mathrm{I}^{1-\alpha}_{a+} f)(a)}{\Gamma(\alpha)} (x-a)^{\alpha-1}. \tag{2.76}$$

*[33.5.1]* The last theorem implies that for $f \in L^1([a,b])$ and $\operatorname{Re}\alpha > 0$ with $n = [\operatorname{Re}\alpha] + 1$ the equality

$$\mathrm{I}^{\alpha}_{a+} \mathrm{D}^{\alpha}_{a+} f(x) = f(x) \tag{2.77}$$

*[page 34, §0]* holds only if

$$\mathrm{I}_{a+}^{n-\alpha} f \in \mathrm{AC}^n([a,b]) \tag{2.78a}$$

and

$$\left(\mathrm{D}^k \, \mathrm{I}_{a+}^{n-\alpha} f\right)(a) = 0 \tag{2.78b}$$

for all $k = 0, 1, 2, ..., n-1$. *[34.0.1]* Note that the existence of $g(x) = \mathrm{D}_{a+}^{\alpha} f(x)$ in eq. (2.77) does not imply that $f(x)$ can be written as $(\mathrm{I}_{a+}^{\alpha} g)(x)$ for some integrable function $g$ [**99**]. *[34.0.2]* This holds only if both conditions (2.78) are satisfied. *[34.0.3]* As an example where one of them fails, consider the function $f(x) = (x-a)^{\alpha-1}$ for $0 < \alpha < 1$. *[34.0.4]* Then $\mathrm{D}_{a+}^{\alpha}(x-a)^{\alpha-1} = 0$ exists. *[34.0.5]* Now $\mathrm{D}^0 \, \mathrm{I}_{a+}^{1-\alpha}(x-a)^{\alpha-1} \neq 0$ so that (2.78b) fails. *[34.0.6]* There does not exist an integrable $g$ such that $\mathrm{I}_{a+}^{\alpha} g = (x-a)^{\alpha-1}$. *[34.0.7]* In fact, $g$ corresponds to the $\delta$-distribution $\delta(x-a)$.

#### 2.2.2.2. General Types of Fractional Derivatives

*[34.1.1]* Riemann-Liouville fractional derivatives have been generalized in [**52**, p.433] to fractional derivatives of different types.

**Definition 2.10** *[34.2.1]* The *generalized Riemann-Liouville fractional derivative of order* $0 < \alpha < 1$ *and type* $0 \leq \beta \leq 1$ *with lower (resp. upper) limit* $a$ is defined as

$$(\mathrm{D}_{a\pm}^{\alpha,\beta} f)(x) = \left(\pm \, \mathrm{I}_{a\pm}^{\beta(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \left(\mathrm{I}_{a\pm}^{(1-\beta)(1-\alpha)} f\right)\right)(x) \tag{2.79}$$

for functions such that the expression on the right hand side exists.

*[34.3.1]* The type $\beta$ of a fractional derivative allows to interpolate continuously from $\mathrm{D}_{a\pm}^{\alpha} = \mathrm{D}_{a\pm}^{\alpha,0}$ to $\widetilde{\mathrm{D}}_{a\pm}^{\alpha} = \mathrm{D}_{a\pm}^{\alpha,1}$. *[34.3.2]* A relation between fractional derivatives of the same order but different types was given in [**52**, p.434].

#### 2.2.2.3. Marchaud-Hadamard Fractional Derivatives

*[34.4.1]* Marchaud's approach [**78**] is based on Hadamards finite parts of divergent integrals [**36**]. *[34.4.2]* The strategy is to define fractional derivatives as analytic continuation of fractional integrals to negative orders. [see [**99**, p.225]]

**Definition 2.11** *[34.5.1]* Let $-\infty < a < b < \infty$ and $0 < \alpha < 1$. *[34.5.2]* The *Marchaud fractional derivative of order* $\alpha$ *with lower limit* $a$ is defined as

$$(\mathrm{M}_{a+}^{\alpha} f)(x) = \frac{f(x)}{\Gamma(1-\alpha)(x-a)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^x \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} \mathrm{d}y \tag{2.80}$$

*[page 35, §0]* and the *Marchaud fractional derivative of order $\alpha$ with upper limit $b$* is defined as

$$(\mathrm{M}^{\alpha}_{b-} f)(x) = \frac{f(x)}{\Gamma(1-\alpha)(b-x)^{\alpha}} + \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_{x}^{b} \frac{f(x)-f(y)}{(x-y)^{\alpha+1}} \mathrm{d}y. \tag{2.81}$$

*[35.0.1]* For $a = -\infty$ (resp. $b = \infty$) the definition is

$$(\mathrm{M}^{\alpha}_{\pm} f)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_{0}^{\infty} \frac{f(x)-f(x \mp y)}{y^{\alpha+1}} \mathrm{d}y. \tag{2.82}$$

*[35.0.2]* The definition is completed with $\mathrm{M}^0 f = f$ for all variants.

*[35.1.1]* The idea of Marchaud's method is to extend the Riemann-Liouville integral from $\alpha > 0$ to $\alpha < 0$, and to define

$$(\mathrm{I}^{-\alpha}_{+} f)(x) = \frac{1}{\Gamma(-\alpha)} \int\limits_{0}^{\infty} y^{-\alpha-1} f(x-y)\, \mathrm{d}y, \tag{2.83}$$

where $\alpha > 0$. *[35.1.2]* However, this is not possible because the integral in (2.83) diverges. *[35.1.3]* The idea is to subtract the divergent part of the integral,

$$\int_{\varepsilon}^{\infty} y^{-\alpha-1} f(x) \mathrm{d}y = \frac{f(x)}{\alpha \varepsilon^{\alpha}} \tag{2.84}$$

obtained by setting $f(x-y) \approx f(x)$ for $y \approx 0$. *[35.1.4]* Subtracting (2.83) from (2.84) for $0 < \alpha < 1$ suggests the definition

$$(\mathrm{M}^{\alpha}_{+} f)(x) = \lim_{\varepsilon \to 0+} \frac{1}{\Gamma(-\alpha)} \int\limits_{\varepsilon}^{\infty} \frac{f(x)-f(x-y)}{y^{\alpha+1}}\, \mathrm{d}y \tag{2.85}$$

*[35.1.5]* Formal integration by parts leads to $(\mathrm{I}^{1-\alpha}_{+} f')(x)$, showing that this definition contains the Riemann-Liouville definition.

*[35.2.1]* The definition may be extended to $\alpha > 1$ in two ways. *[35.2.2]* The first consists in applying (2.85) to the $n$-th derivative $\mathrm{d}^n f/\mathrm{d}x^n$ for $n < \alpha < n+1$. *[35.2.3]* The second possibility is to regard $f(x-y) - f(x)$ as a first order difference, and to generalize to $n$-th order differences. *[35.2.4]* The $n$-th order difference is

$$(\Delta^n_y f)(x) = (\mathbf{1} - \mathsf{T}_y)^n f(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-ky), \tag{2.86}$$

where $(\mathbf{1}\, f)(x) = f(x)$ is the identity operator and

$$(\mathsf{T}_h f)(x) = f(x-h) \tag{2.87}$$

*[page 36, §0]* is the translation operator. *[36.0.1]* The Marchaud fractional derivative can then be extended to $0 < \alpha < n$ through [**94, 98**]

$$(\mathrm{M}_{+}^{\alpha} f)(x) = \lim_{\varepsilon \to 0+} \frac{1}{C_{\alpha,n}} \int\limits_{\varepsilon}^{\infty} \frac{\Delta_y^n f(x)}{y^{\alpha+1}} \,\mathrm{d}y, \tag{2.88}$$

where

$$C_{\alpha,n} = \int\limits_{0}^{\infty} \frac{(1-\mathrm{e}^{-y})^n}{y^{\alpha+1}} \mathrm{d}y, \tag{2.89}$$

where the limit may be taken in the sense of pointwise or norm convergence.

*[36.1.1]* The Marchaud derivatives $\mathrm{M}_{\pm}^{\alpha}$ are defined for a wider class of functions than Weyl derivatives $\mathrm{D}_{\pm}^{\alpha}$. *[36.1.2]* As an example consider the function $f(x) =$const.

*[36.2.1]* Let $f$ be such that there exists a function $g \in L^1([a,b])$ with $f = \mathrm{I}_{a+}^{\alpha}\, g$. *[36.2.2]* Then the Riemann-Liouville derivative and the Marchaud derivative coincide almost everywhere, i.e. $(\mathrm{M}_{a+}^{\alpha} f)(x) = (\mathrm{D}_{a+}^{\alpha} f)(x)$ for almost all $x$ [**99**, p.228].

### 2.2.2.4. Weyl Fractional Derivatives

*[36.3.1]* There are two kinds of Weyl fractional derivatives for periodic functions. *[36.3.2]* The *Weyl-Liouville fractional derivative* is defined as [**99**, p.351], [**94**]

$$(\mathrm{D}_{\pm}^{\alpha} f)(x) = \pm \frac{\mathrm{d}}{\mathrm{d}x} (\mathrm{I}_{\pm}^{1-\alpha} f)(x) \tag{2.90}$$

for $0 < \alpha < 1$ where the Weyl integral $\pm\, \mathrm{I}_{\pm}^{\alpha} f$ was defined in (2.34). *[36.3.3]* The *Weyl-Marchaud fractional derivative* is defined as [**99**, p.352], [**94**]

$$(\mathrm{W}_{\pm}^{\alpha} f)(x) = \frac{1}{2\pi} \int\limits_{0}^{2\pi} [f(x-y) - f(x)]\, (\mathrm{D}^1\, \Psi_{\pm}^{1-\alpha})(y) \mathrm{d}y \tag{2.91}$$

for $0 < \alpha < 1$ where $\Psi_{\pm}(x)$ is defined in eq. (2.35). *[36.3.4]* The Weyl derivatives are defined for periodic functions of with zero mean in $C^{\beta}(\mathbb{R}/2\pi\mathbb{Z})$ where $\beta > \alpha$. *[36.3.5]* In this space $(\mathrm{D}_{\pm}^{\alpha} f)(x) = (\mathrm{W}_{\pm}^{\alpha} f)(x)$, i.e. the Weyl-Liouville and Weyl-Marchaud form coincide [**99**]. *[36.3.6]* As for fractional integrals, it can be shown that the Weyl-Liouville derivative $(0 < \alpha < 1)$

$$(\mathrm{D}_{+}^{\alpha} f)(x) = \frac{1}{\Gamma(1-\alpha)} \int\limits_{-\infty}^{x} \frac{f(y)}{(x-y)^{\alpha}} \mathrm{d}y \tag{2.92}$$

coincides with the Riemann-Liouville derivative with lower limit $-\infty$. *[36.3.7]* In addition one has the equivalence $\mathrm{D}_{+}^{\alpha} f = \mathrm{W}_{+}^{\alpha} f$ with the Marchaud-Hadamard fractional derivative in a suitable sense [**99**, p.357].

*[page 37, §1]*

#### 2.2.2.5. Riesz Fractional Derivatives

*[37.1.1]* To define the Riesz fractional derivative as integer derivatives of Riesz potentials consider the Fourier transforms

$$\mathcal{F}\left\{\mathrm{D}\,\mathrm{I}^{1-\alpha} f\right\}(k) = (\mathrm{i}k)|k|^{\alpha-1}\mathcal{F}\{f\}(k) = (\mathrm{i}\,\mathrm{sgn}\,k)|k|^{\alpha}\mathcal{F}\{f\}(k) \tag{2.93}$$

$$\mathcal{F}\left\{\mathrm{D}\,\widetilde{\mathrm{I}^{1-\alpha}} f\right\}(k) = (\mathrm{i}k)(-\mathrm{i}\,\mathrm{sgn}\,k)|k|^{\alpha-1}\mathcal{F}\{f\}(k) = |k|^{\alpha}\mathcal{F}\{f\}(k) \tag{2.94}$$

for $0 < \alpha < 1$. *[37.1.2]* Comparing this to eq. (2.57) suggests to consider

$$\frac{\mathrm{d}}{\mathrm{d}x}(\widetilde{\mathrm{I}^{1-\alpha}} f)(x) = \lim_{h\to 0}\frac{1}{h}\left[(\widetilde{\mathrm{I}^{1-\alpha}} f)(x+h) - (\widetilde{\mathrm{I}^{1-\alpha}} f)(x)\right] \tag{2.95}$$

as a candidate for the Riesz fractional derivative.

*[37.2.1]* Following **[94]** the *strong Riesz fractional derivative of order* $\alpha$ $\mathrm{R}^{\alpha} f$ of a function $f \in L^p(\mathbb{R})$, $1 \le p < \infty$, is defined through the limit

$$\lim_{h\to 0}\left\|\frac{1}{h}(f * K_h^{1-\alpha}) - \mathrm{R}^{\alpha} f\right\|_p = 0, \tag{2.96}$$

whenever it exists. *[37.2.2]* The convolution kernel defined as

$$K_h^{1-\alpha} = \frac{1}{2\Gamma(1-\alpha)\sin(\alpha\pi/2)}\left[\frac{\mathrm{sgn}\,(x+h)}{|x+h|^{\alpha}} - \frac{\mathrm{sgn}\,x}{|x|^{\alpha}}\right] \tag{2.97}$$

is obtained from eq. (2.95). *[37.2.3]* Indeed, this definition is equivalent to eq. (2.94). *[37.2.4]* A function $f \in L^p(\mathbb{R})$ where $1 \le p \le 2$ has a strong Riesz derivative of order $\alpha$ if and only if there exsists a function $g \in L^p(\mathbb{R})$ such that $|k|^{\alpha}\mathcal{F}\{f\}(k) = \mathcal{F}\{g\}(k)$. *[37.2.5]* Then $\mathrm{R}^{\alpha} f = g$.

#### 2.2.2.6. Grünwald-Letnikov Fractional Derivatives

*[37.3.1]* The basic idea of the Grünwald approach is to generalize finite difference quotients to noninteger order, and then take the limit to obtain a differential quotient. *[37.3.2]* The first order derivative is the limit

$$\frac{\mathrm{d}}{\mathrm{d}x}f(x) = (\mathrm{D}\,f)(x) = \lim_{h\to 0}\frac{f(x) - f(x-h)}{h} = \lim_{h\to 0}\frac{[\mathbf{1} - \mathsf{T}(h)]}{h}f(x) \tag{2.98}$$

of a difference quotient. *[37.3.3]* In the last equality $(\mathbf{1}\,f)(x) = f(x)$ is the identity operator, and

$$[\mathsf{T}(h)f](x) = f(x-h) \tag{2.99}$$

is the translation operator. *[37.3.4]* Repeated application of $\mathsf{T}$ gives

$$[\mathsf{T}(h)^n f](x) = f(x-nh), \tag{2.100}$$

*[page 38, §0]* where $n \in \mathbb{N}$. *[38.0.1]* The second order derivative can then be written as

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) = (\mathrm{D}^2\, f)(x) = \lim_{h\to 0} \frac{f(x) - 2f(x-h) + f(x-2h)}{h^2}$$
$$= \lim_{h\to 0} \left\{ \frac{[\mathbf{1} - \mathsf{T}(h)]}{h} \right\}^2 f(x), \tag{2.101}$$

and the $n$-th derivative

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x) = (\mathrm{D}^n\, f)(x) = \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-kh)$$
$$= \lim_{h\to 0} \left\{ \frac{[\mathbf{1} - \mathsf{T}(h)]}{h} \right\}^n f(x), \tag{2.102}$$

which exhibits the similarity with the binomial formula. *[38.0.2]* The generalization to noninteger $n$ gives rise to fractional difference quotients defined through

$$(\Delta_h^\alpha f)(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x-kh) \tag{2.103}$$

for $\alpha > 0$. *[38.0.3]* These are generally divergent for $\alpha < 0$. *[38.0.4]* For example, if $f(x) = 1$, then

$$\sum_{k=0}^{N} (-1)^k \binom{\alpha}{k} = \frac{1}{\Gamma(1-\alpha)} \frac{\Gamma(N+1-\alpha)}{\Gamma(N+1)} \tag{2.104}$$

diverges as $N \to \infty$ if $\alpha < 0$. *[38.0.5]* Fractional difference quotients were studied in [**68**]. Note that fractional differences obey [**99**]

$$(\Delta_h^\alpha(\Delta_h^\beta f))(x) = (\Delta_h^{\alpha+\beta} f)(x). \tag{2.105}$$

**Definition 2.12** *[38.1.1]* The *Grünwald-Letnikov fractional derivative of order* $\alpha > 0$ is defined as the limit

$$(\mathrm{G}_\pm^\alpha\, f)(x) = \lim_{h\to 0^+} \frac{1}{h^\alpha} (\Delta_{\pm h}^\alpha f)(x) \tag{2.106}$$

of fractional difference quotients whenever the limit exists. *[38.1.2]* The Grünwald Letnikov fractional derivative is called *pointwise* or *strong* depending on whether the limit is taken pointwise or in the norm of a suitable Banach space.

*[38.2.1]* For a definition of Banach spaces and their norms see e.g. [**128**].

*[38.3.1]* The Grünwald-Letnikov fractional derivative has been studied for periodic functions in $L^p(\mathbb{R}/2\pi\mathbb{Z})$ with $1 \le p < \infty$ in [**94, 99**]. *[38.3.2]* It has the following properties.

*[page 39, §1]*

**Theorem 2.13** *[39.1.1] Let $f \in L^p(\mathbb{R}/2\pi\mathbb{Z})$, $1 \le p < \infty$ and $\alpha > 0$. [39.1.2] Then the following statements are equivalent:*

*(1)* $\mathrm{G}_+^\alpha f \in L^p(\mathbb{R}/2\pi\mathbb{Z})$

*(2) [39.1.3] There exists a function $g \in L^p(\mathbb{R}/2\pi\mathbb{Z})$ such that $(\mathrm{i}k)^\alpha \mathcal{F}\{f(x)\}(k) = \mathcal{F}\{g(x)\}(k)$ where $k \in \mathbb{Z}$.*

*(3) [39.1.4] There exists a function $g \in L^p(\mathbb{R}/2\pi\mathbb{Z})$ such that $f(x) - \mathcal{F}\{f(x)\}(0) = (\mathrm{I}_+^\alpha g)(x)$ holds for almost all $x$.*

**Theorem 2.14** *[39.2.1] Let $f \in L^p(\mathbb{R}/2\pi\mathbb{Z})$, $1 \le p < \infty$ and $\alpha, \beta > 0$. [39.2.2] Then:*

*(1)* $\mathrm{G}_+^\alpha f \in L^p(\mathbb{R}/2\pi\mathbb{Z})$ *implies* $\mathrm{G}_+^\beta f \in L^p(\mathbb{R}/2\pi\mathbb{Z})$ *for every* $0 < \beta < \alpha$.

*(2)* $\mathrm{G}_+^\alpha \mathrm{G}_+^\beta f = \mathrm{G}_+^{\alpha+\beta} f$

*(3)* $\mathrm{G}_+^\alpha (\mathrm{I}_+^\alpha f) = f(x) - \mathcal{F}\{f\}(0)$

### 2.2.2.7. Fractional Derivatives of Distributions

*[39.3.1]* The basic idea for defining fractional differentiation of distributions is to extend the definition of fractional integration (2.54) to negative $\alpha$. *[39.3.2]* However, for $\operatorname{Re}\alpha < 0$ the distribution $K_+^\alpha$ becomes singular because $x^{\alpha-1}$ is not locally integrable in this case. *[39.3.3]* The extension of $K_+^\alpha$ to $\operatorname{Re}\alpha < 0$ requires regularization [**31**, **63**, **128**]. *[39.3.4]* It turns out that the regularization exists and is essentially unique as long as $(-\alpha) \notin \mathbb{N}_0$.

**Definition 2.15** *[39.4.1]* Let $f$ be a distribution $f \in C_0^\infty(\mathbb{R})'$ with $\operatorname{supp} f \subset \mathbb{R}_+$. *[39.4.2]* Then the *fractional derivative of order $\alpha$ with lower limit* 0 is the distribution $\mathrm{D}_{0+}^\alpha f$ defined as

$$\langle \mathrm{D}_{0+}^\alpha f, \varphi \rangle = \langle \mathrm{D}_+^\alpha f, \varphi \rangle = \langle K_+^{-\alpha} * f, \varphi \rangle, \tag{2.107}$$

where $\alpha \in \mathbb{C}$ and

$$K_+^\alpha(x) = \begin{cases} \Theta(x)\dfrac{x^{\alpha-1}}{\Gamma(\alpha)} & ,\operatorname{Re}\alpha > 0 \\ \dfrac{\mathrm{d}^N}{\mathrm{d}x^N}\left[\Theta(x)\dfrac{x^{\alpha+N-1}}{\Gamma(\alpha+N)}\right] & ,\operatorname{Re}\alpha + N > 0, N \in \mathbb{N} \end{cases} \tag{2.108}$$

is the kernel distribution. *[39.4.3]* For $\alpha = 0$ one finds $K_+^0(x) = (\mathrm{d}/\mathrm{d}x)\Theta(x) = \delta(x)$ and $\mathrm{D}_{0+}^0 = \mathbf{1}$ as the identity operator. *[39.4.4]* For the $\alpha = -k, k \in \mathbb{N}$ one finds

$$K_+^{-k}(x) = \delta^{(k)}(x), \tag{2.109}$$

where $\delta^{(k)}$ is the $k$-th derivative of the $\delta$ distribution.

*[page 40, §1]* *[40.1.1]* The kernel distribution in (2.108) is

$$K_+^{-\alpha}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left[\Theta(x)\frac{x^{-\alpha}}{\Gamma(1-\alpha)}\right] = \frac{\mathrm{d}}{\mathrm{d}x}K_+^{1-\alpha}(x) \tag{2.110}$$

for $0 < \alpha < 1$. *[40.1.2]* Its regularized action is

$$\left\langle K_+^{-\alpha}(x), \varphi(x)\right\rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}x}K_+^{1-\alpha}(x), \varphi(x)\right\rangle = -\left\langle K_+^{1-\alpha}(x), \varphi(x)'\right\rangle \tag{2.111a}$$

$$= -\frac{1}{\Gamma(1-\alpha)}\lim_{\varepsilon\to 0}\int_{\varepsilon}^{\infty} x^{-\alpha}\varphi(x)'\mathrm{d}x \tag{2.111b}$$

$$= -\lim_{\varepsilon\to 0}\left\{\left.\frac{\varphi(x)+C}{\Gamma(\alpha)x^{\alpha}}\right|_{\varepsilon}^{\infty} - \int_{\varepsilon}^{\infty}\frac{\varphi(x)+C}{\Gamma(-\alpha)x^{1+\alpha}}\mathrm{d}x\right\} \tag{2.111c}$$

$$= \int_0^{\infty}\frac{\varphi(x)-\varphi(0)}{\Gamma(-\alpha)x^{1+\alpha}}\mathrm{d}x, \tag{2.111d}$$

where $\varphi(\infty) < \infty$ was assumed in the last step and the arbitrary constant was chosen as $C = -\varphi(0)$. *[40.1.3]* This choice regularizes the divergent first term in (2.111c). *[40.1.4]* If this rule is used for the distributional convolution

$$(K_+^{-\alpha} * f)(x) = \frac{1}{\Gamma(-\alpha)}\int_0^{\infty}\frac{f(x)-f(x-y)}{y^{\alpha+1}}\mathrm{d}y = (\mathrm{M}_+^{\alpha} f)(x) \tag{2.112}$$

then the Marchaud-Hadamard form is recovered with $0 < \alpha < 1$.

*[40.2.1]* It is now possible to show that the convolution of distributions is in general not associative. *[40.2.2]* A counterexample is

$$(1 * \delta') * \Theta = 1' * \Theta = 0 * \Theta = 0 \neq 1 = 1 * \delta = 1 * \Theta' = 1 * (\delta' * \Theta), \tag{2.113}$$

where $\Theta$ is the Heaviside step function.

*[40.3.1]* $\mathrm{D}_{0+}^{\alpha} f$ has support in $\mathbb{R}_+$. *[40.3.2]* The distributions in $f \in C_0^{\infty}(\mathbb{R})'$ with $\operatorname{supp} f \subset \mathbb{R}_+$ form a convolution algebra [**21**] and one finds [**31**, **99**]

**Theorem 2.16** *[40.3.3] If $f \in C_0^{\infty}(\mathbb{R})'$ with $\operatorname{supp} f \subset \mathbb{R}_+$ then also $\mathrm{I}_{0+}^{\alpha} f \in C_0^{\infty}(\mathbb{R})'$ with $\mathrm{I}_{0+}^{\alpha} \operatorname{supp} f \subset \mathbb{R}_+$. [40.3.4] Moreover, for all $\alpha, \beta \in \mathbb{C}$*

$$\mathrm{D}_{0+}^{\alpha}\,\mathrm{D}_{0+}^{\beta} f = \mathrm{D}_{0+}^{\alpha+\beta} f \tag{2.114}$$

*with $\mathrm{D}_{0+}^{\alpha} f = \mathrm{I}_{0+}^{-\alpha} f$ for $\operatorname{Re}\alpha < 0$. [40.3.5] For each $f \in C_0^{\infty}(\mathbb{R})'$ with $\operatorname{supp} f \subset \mathbb{R}_+$ there exists a unique distribution $g \in C_0^{\infty}(\mathbb{R})'$ with $\operatorname{supp} g \subset \mathbb{R}_+$ such that $f = \mathrm{I}_{0+}^{\alpha} g$.*

*[page 41, §1]* *[41.1.1]* Note that

$$\mathrm{D}_{0+}^{\alpha} f = \mathrm{D}_{0+}^{\alpha}(\mathbf{1}\, f) = (K_{+}^{-\alpha} * K_{+}^{0}) * f = (\mathrm{D}_{0+}^{\alpha}\, \delta) * f = \delta^{(\alpha)} * f \tag{2.115}$$

for all $\alpha \in \mathbb{C}$.

*[41.2.1]* Also, the differentiation rule

$$\mathrm{D}_{0+}^{\alpha}\, K_{+}^{\beta} = K_{+}^{\beta-\alpha} \tag{2.116}$$

holds for all $\alpha, \beta \in \mathbb{C}$. *[41.2.2]* It contains

$$\mathrm{D}\, K_{+}^{\beta} = K_{+}^{\beta-1} \tag{2.117}$$

for all $\beta \in \mathbb{C}$ as a special case.

#### 2.2.2.8. Fractional Derivatives at Their Lower Limit

*[41.3.1]* All fractional derivatives defined above are nonlocal operators. *[41.3.2]* A local fractional derivative operator was introduced in [**40**, **41**, **52**].

**Definition 2.17** *[41.4.1]* For $-\infty < a < \infty$ the *Riemann-Liouville fractional derivative of order* $0 < \alpha < 1$ *at the lower limit* $a$ is defined by

$$\left.\frac{\mathrm{d}^{\alpha} f}{\mathrm{d}x^{\alpha}}\right|_{x=a} = f^{(\alpha)}(a) = \lim_{x \to a\pm} (\mathrm{D}_{a\pm}^{\alpha}\, f)(x), \tag{2.118}$$

whenever the two limits exist and are equal. *[41.4.2]* If $f^{(\alpha)}(a)$ exists the function $f$ is called fractionally differentiable at the limit $a$.

*[41.5.1]* These operators are useful for the analysis of singularities. *[41.5.2]* They were applied in [**40**–**42**, **44**, **52**] to the analysis of singularities in the theory of critical phenomena and to the generalization of Ehrenfests classification of phase transitions. *[41.5.3]* There is a close relationship to the theory of regularly varying functions [**107**] as evidenced by the following result [**52**].

**Theorem 2.18** *[41.6.1] Let the function* $f : [0, \infty[ \to \mathbb{R}$ *be monotonously increasing with* $f(x) \geq 0$ *and* $f(0) = 0$*, and such that* $(\mathrm{D}_{0+}^{\alpha,\lambda}\, f)(x)$ *with* $0 < \alpha < 1$ *and* $0 \leq \lambda \leq 1$ *is also monotonously increasing on a neighbourhood* $[0, \delta]$ *for small* $\delta > 0$*. [41.6.2] Let* $0 \leq \beta < \lambda(1-\alpha) + \alpha$*, let* $C \geq 0$ *be a constant and* $\Lambda(x)$ *a slowly varying function for* $x \to 0$*. [41.6.3] Then*

$$\lim_{x \to 0} \frac{f(x)}{x^{\beta} \Lambda(x)} = C \tag{2.119}$$

*holds if and only if*

$$\lim_{x \to 0} \frac{(\mathrm{D}_{0+}^{\alpha,\lambda}\, f)(x)}{x^{\beta-\alpha} \Lambda(x)} = C \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} \tag{2.120}$$

*holds.*

*[page 42, §1]*
*[42.1.1]* A function $f$ is called slowly varying at infinity if $\lim_{x\to\infty} f(bx)/f(x) = 1$ for all $b > 0$. *[42.1.2]* A function $f(x)$ is called slowly varying at $a \in \mathbb{R}$ if $f(1/(x-a))$ is slowly varying at infinity.

#### 2.2.2.9. Fractional Powers of Operators

*[42.2.1]* The spectral decomposition of selfadjoint operators is a familiar mathematical tool from quantum mechanics [**116**]. *[42.2.2]* Let $A$ denote a selfadjoint operator with domain $D(A)$ and spectral family $E_\lambda$ on a Hilbert space $X$ with scalar product $(\cdot,\cdot)$. *[42.2.3]* Then

$$(Au, v) = \int\limits_{\sigma(A)} \lambda \mathrm{d}(E_\lambda u, v) \tag{2.121}$$

holds for all $u, v \in D(A)$. *[42.2.4]* Here $\sigma(A)$ is the spectrum of $A$. *[42.2.5]* It is then straightforward to define the fractional power $A^\alpha u$ by

$$(A^\alpha u, u) = \int\limits_{\sigma(A)} \lambda^\alpha \mathrm{d}(E_\lambda u, u) \tag{2.122}$$

on the domain

$$D(A^\alpha) = \{u \in X : \int\limits_{\sigma(A)} \lambda^\alpha \mathrm{d}(E_\lambda u, u) < \infty\}. \tag{2.123}$$

*[42.2.6]* Similarly, for any measurable function $g : \sigma(A) \to \mathbb{C}$ the operator $g(A)$ is defined with an integrand $g(\lambda)$ in eq. (2.122). *[42.2.7]* This yields an operator calculus that allows to perform calculations with functions instead of operators.

*[42.3.1]* Fractional powers of the Laplacian as the generator of the diffusion semigroup were introduced by Bochner [**13**] and Feller [**26**] based on Riesz' fractional potentials. *[42.3.2]* The fractional diffusion equation

$$\frac{\partial f}{\partial t} = -(-\Delta)^{\alpha/2} f \tag{2.124}$$

was related by Feller to the Levy stable laws [**74**] using one dimensional fractional integrals $\mathrm{I}^{-\alpha,\beta}$ of order $-\alpha$ and type $\beta$ [**26**][7]. *[42.3.3]* For $\alpha = 2$ eq. (2.124) reduces to the diffusion equation. *[42.3.4]* This type of fractional diffusion will be referred to as *fractional diffusion of Bochner-Levy type* (see Section 2.3.4 for more discussion). *[42.3.5]* Later, these ideas were extended to fractional powers of closed[8] semigroup generators [**4**,**5**,**69**,**70**]. *[42.3.6]* If $(-\mathrm{A})$ is the infinitesimal generator of a

[7]Fellers motivation to introduce the type $\beta$ was this relation.

[8] An operator $A : B \to B$ on a Banach space $B$ is called *closed* if the set of pairs $(x, Ax)$ with $x \in D(A)$ is closed in $B \times B$.

*[page 43, §0]* semigroup $T(t)$ (see Section 2.3.3.2 for definitions of $T(t)$ and A) on a Banach space $B$ then its fractional power is defined as

$$(-\mathrm{A})^\alpha f = \lim_{\varepsilon\to 0+} \frac{1}{-\Gamma(-\alpha)} \int_\varepsilon^\infty t^{-\alpha-1}[\mathbf{1} - T(t)]f\mathrm{d}t \tag{2.125}$$

for every $f \in B$ for which the limit exists in the norm of $B$ [**93**,**120**,**121**,**123**]. *[43.0.1]* This aproach is clearly inspired by the Marchaud form (2.82). Alternatively, one may use the Grünwald approach to define fractional powers of semigroup generators [**99**, **122**].

### 2.2.2.10. Pseudodifferential Operators

*[43.1.1]* The calculus of pseudodifferential operators represents another generalization of the operator calculus in Hilbert spaces. *[43.1.2]* It has its roots in Hadamard's ideas [**36**], Riesz potentials [**97**], Feller's suggestion [**26**] and Calderon-Zygmund singular integrals [**16**]. *[43.1.3]* Later it was generalized and became a tool for treating elliptic partial differential operators with nonconstant coefficients.

**Definition 2.19** *[43.2.1]* A (Kohn-Nirenberg) *pseudodifferential operator of order* $\alpha \in \mathbb{R}$ $\sigma(x,\mathrm{D}) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$ is defined as

$$\sigma(x,\mathrm{D})f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}xk}\sigma(x,k)\mathcal{F}\{f\}(k)\mathrm{d}k \tag{2.126}$$

and the function $\sigma(x,k)$ is called its *symbol.* *[43.2.2]* The symbol is in the Kohn-Nirenberg symbol class $S^\alpha$ if it is in $C^\infty(\mathbb{R}^{2d})$, and there exists a compact set $K \subset \mathbb{R}^d$ such that $\operatorname{supp}\sigma \subset K \times \mathbb{R}^d$, and for any pair of multiindices $\beta,\gamma$ there is a constant $C_{\beta,\gamma}$ such that

$$\mathrm{D}_k^\beta\,\mathrm{D}_x^\gamma\,\sigma(x,k) \le C_{\beta,\gamma}(1+|k|)^{\alpha-|\beta|}. \tag{2.127}$$

*[43.2.3]* The Hörmander symbol class $S^\alpha_{\rho,\delta}$ is obtained by replacing the exponent $\alpha - |\beta|$ on the right hand side with $\alpha - \rho|\beta| + \delta|\gamma|$ where $0 \le \rho,\delta \le 1$.

*[43.3.1]* Pseudodifferential operators provide a unified approach to differential and integral or convolution operators that are "nearly" translation invariant. *[43.3.2]* They have a close relation with Weyl quantization in physics [**28**, **116**]. However, they will not be discussed further because the traditional symbol classes do not contain the usual fractional derivative operators. *[43.3.3]* Fractional Riesz derivatives are not pseudodifferential operators in the sense above. *[43.3.4]* Their symbols do not fall into any of the standard Kohn-Nirenberg or Hörmander symbol classes due to lack of differentiability at the origin.

*[page 44, §1]*

### 2.2.3. Eigenfunctions

*[44.1.1]* The eigenfunctions of Riemann-Liouville fractional derivatives are defined as the solutions of the fractional differential equation

$$(\mathrm{D}^{\alpha}_{0+} f)(x) = \lambda f(x), \tag{2.128}$$

where $\lambda$ is the eigenvalue. *[44.1.2]* They are readily identifed using eq. (A.6) as

$$f(x) = x^{1-\alpha}\mathrm{E}_{\alpha,\alpha}(\lambda x^{\alpha}), \tag{2.129}$$

where

$$\mathrm{E}_{\alpha,\beta} = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k + \beta)} \tag{2.130}$$

is the generalized Mittag-Leffler function [**125**, **126**]. *[44.1.3]* More generally the eigenvalue equation for fractional derivatives of order $\alpha$ and type $\beta$ reads

$$(\mathrm{D}^{\alpha,\beta}_{0+} f)(x) = \lambda f(x), \tag{2.131}$$

and it is solved by [**54**, eq.124]

$$f(x) = x^{(1-\beta)(1-\alpha)}\mathrm{E}_{\alpha,\alpha+\beta(1-\alpha)}(\lambda x^{\alpha}), \tag{2.132}$$

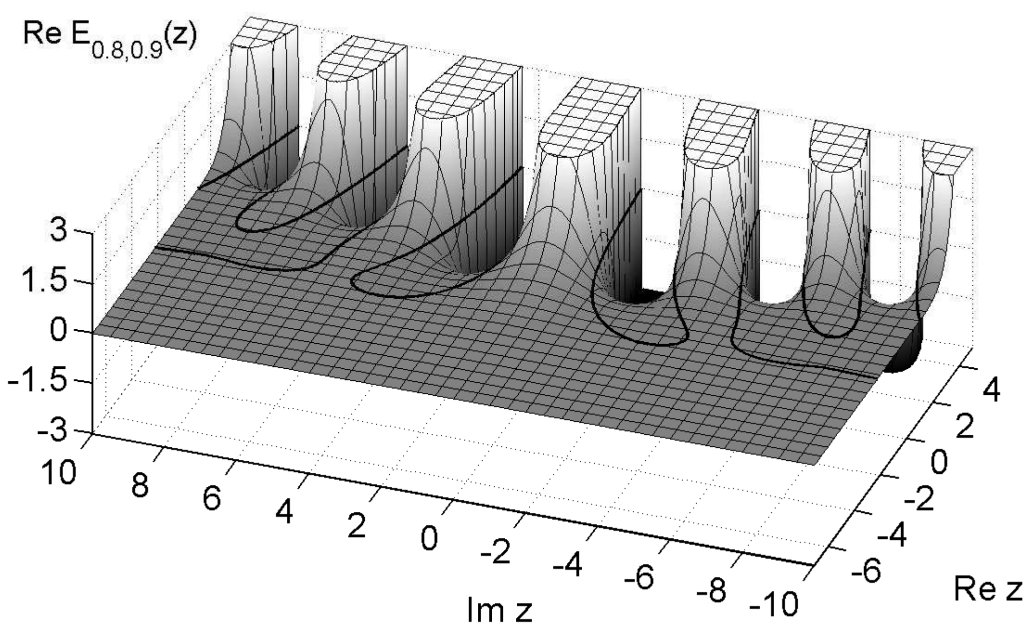


**Figure 2.1**. Truncated real part of the generalized Mittag-Leffler function $-3 \le \mathrm{Re}\,\mathrm{E}_{0.8,0.9}(z) \le 3$ for $z \in \mathbb{C}$ with $-7 \le \mathrm{Re}\,z \le 5$ and $-10 \le \mathrm{Im}\,z \le 10$. The solid line is defined by $\mathrm{Re}\,\mathrm{E}_{0.8,0.9}(z) = 0$.

*[page 45, §0]* where the case $\beta = 0$ corresponds to (2.128). *[45.0.1]* A second important special case is the equation

$$(\mathrm{D}^{\alpha,1}_{0+} f)(x) = \lambda f(x), \tag{2.133}$$

with $\mathrm{D}^{\alpha,1}_{0+} = \widetilde{\mathrm{D}}^{\alpha}_{0+}$. *[45.0.2]* In this case the eigenfunction

$$f(x) = \mathrm{E}_\alpha(\lambda x^\alpha), \tag{2.134}$$

where $\mathrm{E}_\alpha(x) = \mathrm{E}_{\alpha,1}(x)$ is the Mittag-Leffler function [**86**]. *[45.0.3]* The Mittag-Leffler function plays a central role in fractional calculus. *[45.0.4]* It has only recently been calculated numerically in the full complex plane [**62, 108**]. *[45.0.5]* Figure 2.1 and 2.2 illustrate $\mathrm{E}_{0.8,0.9}(z)$ for a rectangular region in the complex plane (see [**108**]).

*[45.1.1]* The solid line in Figure 2.1 is the line $\mathrm{Re}\,\mathrm{E}_{0.8,0.9}(z) = 0$, in Figure 2.2 it is $\mathrm{Im}\,\mathrm{E}_{0.8,0.9}(z) = 0$.

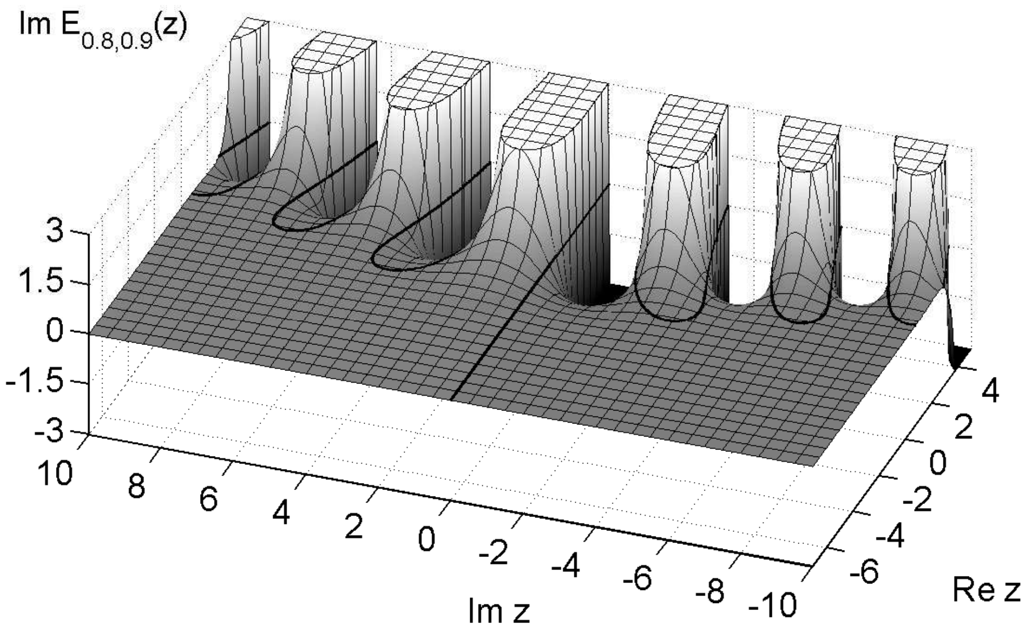


**Figure 2.2**. Same as Fig. 2.1 for the imaginary part of $\mathrm{E}_{0.8,0.9}(z)$. The solid line is $\mathrm{Im}\,\mathrm{E}_{0.8,0.9}(z) = 0$.

*[45.2.1]* Note, that some authors are avoiding the operator $\mathrm{D}^{\alpha,1}_{0+}$ in fractional differential equations (see e.g. [**7, 82, 84, 101, 111, 112**] or chapters in this volume). *[45.2.2]* In their notation the eigenvalue equation (2.133) becomes (c.f. [**112**, eq.(22)])

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x) = \lambda\, \mathrm{D}^{1-\alpha}_{0+} f(x) \tag{2.135}$$

containing two derivative operators instead of one.

*[page 46, §1]*

## 2.3. Physical Introduction to Fractional Derivatives

### 2.3.1. Basic Questions

*[46.1.1]* An introduction to fractional derivatives would be incomplete without an introduction to applications. *[46.1.2]* In the past fractional calculus has been used predominantly as a convenient calculational tool [**26**, **76**, **89**]. *[46.1.3]* A well known example is Riesz' interpolation method for solving the wave equation [**20**]. *[46.1.4]* In recent times, however, fractional differential equations appear as "generalizations" of more or less fundamental equations of physics [**3**, **12**, **18**, **23**, **43**, **46**, **52**, **54–56**, **58**, **60**, **90**, **91**, **102**, **104**, **119**, **129**]. *[46.1.5]* The idea is that physical phenomena can be described by fractional differential equations. *[46.1.6]* This practice raises at least two fundamental questions:

(1) *[46.1.7]* Are mathematical models with fractional derivatives consistent with the fundamental laws and fundamental symmetries of nature ?
(2) *[46.1.8]* How can the fractional order $\alpha$ of differentiation be observed or how does a fractional derivative emerge from concrete models ?

*[46.1.9]* Both questions will be addressed here. *[46.1.10]* The answer to the first question is provided by the theory of fractional time evolutions [**43**, **47**], the answer to the second question by anomalous subdiffusion [**46**, **60**].

### 2.3.2. Fractional Space

*[46.2.1]* Fractional derivatives are nonlocal operators. *[46.2.2]* Nevertheless, numerous authors have proposed fractional differential equations involving fractional spatial derivatives. *[46.2.3]* Particularly popular are fractional powers of the Laplace operator due to the well known work of Riesz, Feller and Bochner [**13**, **27**, **97**]. The nonlocality of fractional spatial derivatives raises serious (largely) unresolved physical problems.

*[46.3.1]* As an illustration of the problem with spatial fractional derivatives consider the one dimensional potential equation for functions $f \in C^2(\mathbb{R})$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} f(x) = 0, \quad x \in \mathbb{G} \tag{2.136}$$

on the open interval $\mathbb{G} = ]a, b[$ with boundary conditions $f(a) = 0, f(b) = 0$ with $a < b$. A solution of this boundary value problem is $f(x) = 0$ with $x \in \mathbb{G}$. *[46.3.2]* This trivial solution remains unchanged as long as the boundary values $f(a) = f(b) = 0$ remain unperturbed. *[46.3.3]* All functions $f \in C^2(\mathbb{R})$ that vanish on $[a, b]$ are solutions of the boundary value problem. *[46.3.4]* In particular, the boundary *[page 47, §0]* specification

$$f(x) = 0, \quad \text{for } x \in \mathbb{R} \setminus \mathbb{G} \tag{2.137}$$

and the perturbed boundary specification

$$f(x) = g(x), \quad \text{for } x \in \mathbb{R} \setminus \mathbb{G} \tag{2.138}$$

with $g \geq 0$ and $\operatorname{supp} g \cap [a, b] = \emptyset$ have the same trivial solution $f = 0$ in $\mathbb{G}$. *[47.0.1]* The reason is that $\mathrm{d}^2/\mathrm{d}x^2$ is a local operator.

*[47.1.1]* Consider now a fractional generalization of (2.136) that arises for example as the stationary limit of (Bochner-Levy) fractional diffusion equations with a fractional Laplace operator [**13**]. *[47.1.2]* Such a onedimensional fractional Laplace equation reads

$$\mathrm{R}^\alpha f(x) = 0, \tag{2.139}$$

where $\mathrm{R}^\alpha$ is a Riesz fractional derivative of order $0 < \alpha < 1$. *[47.1.3]* For the boundary specification (2.137) it has the same trivial solution $f(x) = 0$ for all $x \in \mathbb{G}$. *[47.1.4]* But this solution no longer applies for the perturbed boundary specification (2.138). *[47.1.5]* In fact, assuming (2.138) for $x \in \mathbb{R} \setminus \mathbb{G}$ and $f(x) = 0$ for $x \in \mathbb{G}$ now yields $(\mathrm{R}^\alpha f)(x) \neq 0$ for all $x \in \mathbb{G}$. *[47.1.6]* The exterior $\mathbb{R} \setminus \mathbb{G}$ of the domain $\mathbb{G}$ cannot be isolated from the interior of $\mathbb{G}$ using classical boundary conditions. *[47.1.7]* The reason is that $\mathrm{R}^\alpha$ is a nonlocal operator.

*[47.2.1]* Locality in space is a basic and firmly established principle of physics (see e.g. [**35, 115**]). *[47.2.2]* Of course, one could argue that relativistic effects are negligible, and that fractional spatial derivatives might arise as an approximate phenomenological model describing an underlying physical reality that obeys spatial locality. *[47.2.3]* However, spatial fractional derivatives imply not only action at a distance. *[47.2.4]* As seen above, they imply also that the exterior domain cannot be decoupled from the interior by conventional walls or boundary conditions. *[47.2.5]* This has far reaching consequences for theory and experiment. *[47.2.6]* In theory it invalidates all arguments based on surface to volume ratios becoming negligible in the large volume limit. *[47.2.7]* This includes many concepts and results in thermodynamics and statistical physics that depend on the lower dimensionality of the boundary. *[47.2.8]* Experimentally it becomes difficult to isolate a system from its environment. *[47.2.9]* Fractional diffusion would never come to rest inside a vessel with thin rigid walls unless the equilibrium concentration prevails also outside the vessel. *[47.2.10]* A fractionally viscous fluid at rest inside a container with thin rigid walls would have to start to move when the same fluid starts flowing outside the vessel. *[47.2.11]* It seems therefore difficult to reconcile nonlocality in space with theory and experiment.

*[page 48, §1]*

### 2.3.3. Fractional Time

#### 2.3.3.1. Basic Questions

*[48.1.1]* Nonlocality in time, unlike space, does not violate basic principles of physics, as long as it respects causality [**43**, **47**–**49**, **54**]. *[48.1.2]* In fact, causal nonlocality in time is a common nonequilibrium phenomenon known as history dependence, hysteresis and memory.

*[48.2.1]* Theoretical physics postulates time translation invariance as a fundamental symmetry of nature. *[48.2.2]* As a consequence energy conservation is fundamental, and the infinitesimal generator of time translations is a first order time derivative. *[48.2.3]* Replacing integer order time derivatives with fractional time derivatives raises at least three basic questions:

(1) *[48.2.4]* What replaces time translations as the physical time evolution ?
(2) *[48.2.5]* Is the nonlocality of fractional time derivatives consistent with the laws of nature ?
(3) *[48.2.6]* Is the asymmetry of fractional time derivatives consistent with the laws of nature ?

*[48.2.7]* These questions as well as ergodicity breaking, stationarity, long time limits and temporal coarse grainig were discussed first within ergodic theory [**47**–**49**] and later from a general perspective in [**54**].

*[48.3.1]* The third question requires special remarks because irreversibility is a longstanding and controversial subject [**71**]. *[48.3.2]* The problem of irreversibility may be formulated briefly in two ways.

**Definition 2.20** (The normal irreversibility problem) *[48.4.1]* Assume that time is reversible. Explain how and why time irreversible equations arise in physics.

**Definition 2.21** (The reversed irreversibility problem) *[48.5.1]* Assume that time is irreversible. Explain how and why time reversible equations arise in physics.

*[48.6.1]* While the normal problem has occupied physicists and mathematicians for more than a century, the reversed problem was apparently first formulated in [**59**]. *[48.6.2]* Surprisingly, the reversed irreversibility problem has a clear and quantitiative solution within the theory of fractional time. *[48.6.3]* The solution is based on the simple postulate that every time evolution of a physical system is irreversible. *[48.6.4]* It is not possible to repeat an experiment in the past [**59**]. *[48.6.5]* This empiricial fact seems to reflect a fundamental law of nature that rivals the law of energy conservation.

*[48.7.1]* The mathematical concepts corresponding to irreversible time evolutions are operator semigroups and abstract Cauchy problems [**15**, **93**]. *[48.7.2]* The following brief introduction to fractional time evolutions (sections 2.3.3.2–2.3.3.8) is in large parts identical to the brief exposition in [**59**]. *[48.7.3]* For more details see [**54**].

*[page 49, §1]*

#### 2.3.3.2. Time Evolution

*[49.1.1]* A physical time evolution $\{T(\Delta t) : 0 \leq \Delta t < \infty\}$ is defined as a one-parameter family (with time parameter $\Delta t$) of bounded linear time evolution operators $T(\Delta t)$ on a Banach space $B$. *[49.1.2]* The parameter $\Delta t$ represents time durations. The one-parameter family fulfills the conditions

$$[T(\Delta t_1)T(\Delta t_2)f](t_0) = [T(\Delta t_1 + \Delta t_2)f](t_0) \tag{2.140}$$

$$[T(0)f](t_0) = f(t_0) \tag{2.141}$$

for all $\Delta t_1, \Delta t_2 \geq 0$, $t_0 \in \mathbb{R}$ and $f \in B$. *[49.1.3]* The elements $f \in B$ represent time dependent physical observables, i.e. functions on the time axis $\mathbb{R}$. *[49.1.4]* Note that the argument $\Delta t \geq 0$ of $T(\Delta t)$ has the meaning of a time duration, while $t \in \mathbb{R}$ in $f(t)$ means a time instant. *[49.1.5]* Equations (2.140) and (2.141) define a semigroup. *[49.1.6]* The inverse elements $T(-\Delta t)$ are absent. *[49.1.7]* This reflects the fundamental difference between past and future.

*[49.2.1]* The linear operator A defined as

$$\mathrm{A}f = \operatorname*{s-lim}_{\Delta t \to 0+} \frac{T(\Delta t)f - f}{\Delta t} \tag{2.142}$$

with domain

$$D(\mathrm{A}) = \left\{ f \in B : \operatorname*{s-lim}_{\Delta t \to 0+} \frac{T(\Delta t)f - f}{\Delta t} \text{ exists} \right\} \tag{2.143}$$

is called the infinitesimal generator of the semigroup. *[49.2.2]* Here $\operatorname{s-lim} f = g$ is the strong limit and means $\lim \|f - g\| = 0$ in the norm of $B$ as usual.

#### 2.3.3.3. Continuity

*[49.3.1]* Physical time evolution is continuous. *[49.3.2]* This requirement is represented mathematically by the assumption that

$$\operatorname*{s-lim}_{\Delta t \to 0} T(\Delta t)f = f \tag{2.144}$$

holds for all $f \in B$, where s-lim is again the strong limit. *[49.3.3]* Semigroups of operators satisfying this condition are called strongly continuous or $C^0$-semigroups [**15, 93**]. *[49.3.4]* Strong continuity is weaker than uniform continuity and has become recognized as an important continuity concept that covers most applications [**2**].

#### 2.3.3.4. Homogeneity

*[49.4.1]* Homogeneity of time means two different requirements: *[49.4.2]* Firstly, it requires that observations are independent of a particular instant or position in

*[page 50, §0]* time. *[50.0.1]* Secondly, it requires arbitrary divisibility of time durations and self-consistency for the transition between time scales.

*[50.1.1]* Independence of physical processes from their position on the time axis requires that physical experiments are reproducible if they are *ceteris paribus* shifted in time. *[50.1.2]* The first requirement, that the start of an experiment can be shifted, is expressed mathematically as the requirement of invariance under time translations. *[50.1.3]* As a consequence one demands commutativity of the time evolution with time translations in the form

$$[\mathsf{T}(\tau)T(\Delta t)f](t_0) = [T(\Delta t)\mathsf{T}(\tau)f](t_0) = [T(\Delta t)f](t_0 - \tau) \tag{2.145}$$

for all $\Delta t \geq 0$ und $t_0, \tau \in \mathbb{R}$. *[50.1.4]* Here the translation operator $\mathsf{T}(t)$ is defined by

$$\mathsf{T}(\tau)f(t_0) = f(t_0 - \tau). \tag{2.146}$$

*[50.1.5]* Note that $\tau \in \mathbb{R}$ is a time shift, not a duration. *[50.1.6]* It can also be negative. Physical experiments in the past have the same outcome as in the present or in the future. *[50.1.7]* Outcomes of past experiments can be studied in the present with the help of documents (e.g. a video recording), irrespective of the fact that the experiment cannot be repeated in the past.

*[50.2.1]* The second requirement of homogeneity is homogeneous divisibility. *[50.2.2]* The semigroup property (2.140) implies that for $\Delta t > 0$

$$T(\Delta t)...T(\Delta t) = [T(\Delta t)]^n = T(n\Delta t) \tag{2.147}$$

holds. *[50.2.3]* Homogeneous divisibility of a physical time evolution requires that there exist rescaling factors $D_n$ for $\Delta t$ such that with $\Delta t = \overline{\Delta t}/D_n$ the limit

$$\lim_{n\to\infty} T(n\overline{\Delta t}/D_n) = \overline{T}(\overline{\Delta t}) \tag{2.148}$$

exists und defines a time evolution $\overline{T}(\overline{\Delta t})$. *[50.2.4]* The limit $n \to \infty$ corresponds to two simultaneous limits $n \to \infty, \Delta t \to 0$, and it corresponds to the passage from a microscopic time scale $\Delta t$ to a macroscopic time scale $\overline{\Delta t}$.

#### 2.3.3.5. Causality

*[50.3.1]* Causality of the physical time evolution requires that the values of the image function $g(t) = (T(\Delta t)f)(t)$ depend only upon values $f(s)$ of the original function with time instants $s < t$.

#### 2.3.3.6. Fractional Time Evolution

*[50.4.1]* The requirement (2.145) of homogeneity implies that the operators $T(\Delta t)$ are convolution operators [**114, 128**]. Let $T$ be a bounded linear operator on $L^1(\mathbb{R})$ that commutes with time translations, i.e. that fulfills eq. (2.145). *[50.4.2]* Then there

*[page 51, §0]* exists a finite Borel measure $\mu$ such that

$$(Tf)(s) = (\mu * f)(s) = \int f(s-x)\mu(\mathrm{d}x) \tag{2.149}$$

holds [**128**], [**114**, p.26]. *[51.0.1]* Applying this theorem to physical time evolution operators $T(\Delta t)$ yields a convolution semigroup $\mu_{\Delta t}$ of measures $T(\Delta t)f(t) = (\mu_{\Delta t} * f)(t)$

$$\mu_{\Delta t_1} * \mu_{\Delta t_2} = \mu_{\Delta t_1 + \Delta t_2} \tag{2.150}$$

with $\Delta t_1, \Delta t_2 \geq 0$. *[51.0.2]* For $\Delta t = 0$ the measure $\mu_0$ is the Dirac-measure concentrated at 0.

*[51.1.1]* The requirement of causality implies that the support $\operatorname{supp}\mu_{\Delta t} \subset \mathbb{R}_+ = [0,\infty)$ of the semigroup is contained in the positive half axis.

*[51.2.1]* The convolution semigroups with support in the positive half axis $[0,\infty)$ can be characterized completely by Bernstein functions [**10**]. *[51.2.2]* An arbitrarily often differentiable function $b : (0,\infty) \to \mathbb{R}$ with continuous extension to $[0,\infty)$ is called Bernstein function if for all $x \in (0,\infty)$

$$b(x) \geq 0 \tag{2.151}$$

$$(-1)^n \frac{\mathrm{d}^n b(x)}{\mathrm{d}x^n} \leq 0 \tag{2.152}$$

holds for all $n \in \mathbb{N}$. *[51.2.3]* Bernstein functions are positive, monotonously increasing and concave.

*[51.3.1]* The characterization is given by the following theorem [**10**, p.68]. *[51.3.2]* There exists a one-to-one mapping between the convolution semigroups $\{\mu_t : t \geq 0\}$ with support on $[0,\infty)$ and the set of Bernstein functions $b : (0,\infty) \to \mathbb{R}$ [**10**]. *[51.3.3]* This mapping is given by

$$\int_0^\infty e^{-ux}\mu_{\Delta t}(\mathrm{d}x) = e^{-\Delta t b(u)} \tag{2.153}$$

with $\Delta t > 0$ and $u > 0$.

*[51.4.1]* The requirement of homogeneous divisibility further restricts the set of admissible Bernstein functions. *[51.4.2]* It leaves only those measures $\mu$ that can appear as limits

$$\lim_{n\to\infty,\Delta t\to 0} \underbrace{\mu_{\Delta t} * \ldots * \mu_{\Delta t}}_{n \text{ factors}} = \lim_{n\to\infty} \mu_{n\overline{\Delta t}/D_n} = \overline{\mu}_{\overline{\Delta t}}. \tag{2.154}$$

*[51.4.3]* Such limit measures $\overline{\mu}$ exist if and only if $b(x) = x^\alpha$ with $0 < \alpha \leq 1$ and $D_n \sim n^{1/\alpha}$ holds [**11**, **32**, **54**].

*[51.5.1]* The remaining measures define the class of fractional time evolutions $T_\alpha(\Delta t)$ that depend only on one parameter, the fractional order $\alpha$.

*[page 52, §1]* *[52.1.1]* These remaining fractional measures have a density and they can be written as [**43**, **47**–**49**, **54**]

$$T_\alpha(\Delta t)f(t_0) = \int_0^\infty f(t_0 - s)h_\alpha\left(\frac{s}{\Delta t}\right)\frac{\mathrm{d}s}{\Delta t}, \tag{2.155}$$

where $\Delta t \geq 0$ and $0 < \alpha \leq 1$. *[52.1.2]* The density functions $h_\alpha(x)$ are the one-sided stable probability densities [**43**, **47**–**49**, **54**]. *[52.1.3]* They have a Mellin transform [**45**, **103**, **131**]

$$\mathcal{M}\left\{h_\alpha(x)\right\}(s) = \int_0^\infty x^{s-1}h_\alpha(x)\mathrm{d}x = \frac{1}{\alpha}\frac{\Gamma((1-s)/\alpha)}{\Gamma(1-s)} \tag{2.156}$$

allowing to identify

$$h_\alpha(x) = \frac{1}{\alpha x}H^{10}_{11}\left(\frac{1}{x}\,\middle|\,\begin{matrix}(0,1)\\(0,1/\alpha)\end{matrix}\right) \tag{2.157}$$

in terms of $H$-functions [**30**, **45**, **95**, **103**].

#### 2.3.3.7. Infinitesimal Generator

*[52.2.1]* The infinitesimal generators of the fractional semigroups $T_\alpha(\Delta t)$

$$\mathrm{A}_\alpha f(t) = -(\mathrm{M}^\alpha_+ f)(t) = -\frac{1}{\Gamma(-\alpha)}\int_0^\infty \frac{f(t-s)-f(t)}{s^{\alpha+1}}\,\mathrm{d}s \tag{2.158}$$

are fractional time derivatives of Marchaud-Hadamard type [**51**, **98**]. *[52.2.2]* This fundamental and general result provides the basis for generalizing physical equations of motion by replacing the integer order time derivative with a fractional time derivative as the generator of time evolution [**43**, **54**].

*[52.3.1]* For $\alpha = 1$ one finds $h_1(x) = \delta(x-1)$ from eq. (2.158), and the fractional semigroup $T_{\alpha=1}(\Delta t)$ reduces to the conventional translation semigroup $T_1(\Delta t)f(t_0) = f(t_0 - \Delta t)$. *[52.3.2]* The special case $\alpha = 1$ occurs more frequently in the limit (2.154) than the cases $\alpha < 1$ in the sense that it has a larger domain of attraction. *[52.3.3]* The fact that the semigroup $T_1(\Delta t)$ can often be extended to a group on all of $\mathbb{R}$ provides an explanation for the seemingly fundamental reversibility of mechanical laws and equations. *[52.3.4]* This solves the "reversed irreversibility problem".

#### 2.3.3.8. Remarks

*[52.4.1]* Homogeneous divisibility formalizes the fact that a verbal statement in the present tense presupposes always a certain time scale for the duration of an

*[page 53, §0]* instant. *[53.0.1]* In this sense the present should not be thought of as a point, but as a short time interval [**48**, **54**, **59**].

*[53.1.1]* Fractional time evolutions seem to be related to the subjective human experience of time. *[53.1.2]* In physics the time duration is measured by comparison with a periodic reference (clock) process. *[53.1.3]* Contrary to this, the subjective human experience of time amounts to the comparison with an hour glass, i.e. with a nonperiodic reference. *[53.1.4]* It seems that a time duration is experienced as "long" if it is comparable to the time interval that has passed since birth. *[53.1.5]* This phenomenon seems to be reflected in fractional stationary states defined as solutions of the stationarity condition $T_\alpha(\Delta t)f(t) = f(t)$. Fractional stationarity requires a generalization of concepts such as "stationarity" or "equilibrium". *[53.1.6]* This outlook could be of interest for nonequilibrium and biological systems [**43**, **47**–**49**, **54**].

*[53.2.1]* Finally, also the special case $\alpha \to 0$ challenges philosophical remarks [**59**]. *[53.2.2]* In the limit $\alpha \to 0$ the time evolution operator degenerates into the identity. *[53.2.3]* This could be expressed verbally by saying that for $\alpha = 0$ "becoming" and "being" coincide. *[53.2.4]* In this sense the paradoxical limit $\alpha \to 0$ is reminiscent of the eternity concept known from philosophy.

### 2.3.4. Identification of $\alpha$ from Models

*[53.3.1]* Consider now the second basic question of Section 2.3.1: How can the fractional order $\alpha$ be observed in experiment or identified from concrete models. *[53.3.2]* To the best knowledge of this author there exist two examples where this is possible. *[53.3.3]* Both are related to diffusion processes. *[53.3.4]* There does not seem to exist an example of a rigorous identification of $\alpha$ from Hamiltonian models, although it has been suggested that such a relation might exist (see [**129**]).

#### 2.3.4.1. Bochner-Levy Fractional Diffusion

*[53.4.1]* The term fractional diffusion can refer either to diffusion with a fractional Laplace operator or to diffusion equations with a fractional time derivative. *[53.4.2]* Fractional diffusion (or Fokker-Planck) equations with a fractional Laplacian may be called *Bochner-Levy diffusion*. *[53.4.3]* The identification of the fractional order $\alpha$ in Bochner-Levy diffusion equations has been known for more than five decades [**13**, **14**, **26**]. *[53.4.4]* For a lucid account see also [**27**]. *[53.4.5]* The fractional order $\alpha$ in this case is the index of the underlying stable process [**13**, **27**]. *[53.4.6]* With few exceptions [**77**] these developments in the nation of mathematics did, for many years, not find much attention or application in the nation of physics although eminent mathematical physicists such as Mark Kac were thoroughly familiar

*[page 54, §0]* with Bochner-Levy diffusion [**65**][9]. *[54.0.1]* A possible reason might be the unresolved problem of locality discussed above. *[54.0.2]* Bochner himself writes *"Whether this (equation) might have physical interpretation, is not known to us"* [**13**, p.370].

#### 2.3.4.2. Montroll-Weiss Fractional Diffusion

*[54.1.1]* Diffusion equations with a fractional time derivative will be called *Montroll-Weiss diffusion* although fractional time derivatives do not appear in the original paper [**87**] and the connection was not discovered until 30 years later [**46**,**60**]. *[54.1.2]* As shown in Section 2.3.3, the locality problem does not arise. *[54.1.3]* Montroll-Weiss diffusion is expected to be consistent with all fundamental laws of physics. *[54.1.4]* The fact that the relation between Montroll-Weiss theory and fractional time derivatives was first established in [**46**, **60**] seems to be widely unknown at present, perhaps because this fact is never mentioned in widely read reviews [**82**] and popular introductions to the subject [**112**][10].

*[54.2.1]* There exist several versions of diffusion equations with fractional time derivatives, and they differ physically or mathematically from each other [**54**, **82**, **104**, **127**, **130**]. *[54.2.2]* Of interest here will be the fractional diffusion equation for $f : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$

$$\mathrm{D}_{0+}^{\alpha,1} f(\mathbf{r},t) = C\,\Delta f(\mathbf{r},t) \tag{2.159}$$

with a fractional time derivative of order $\alpha$ and type 1. *[54.2.3]* The Laplace operator is $\Delta$ and the fractional diffusion constant is $C$. *[54.2.4]* The function $f(\mathbf{r},t)$ is assumed to obey the initial condition $f(\mathbf{r},0+) = f_0\delta(\mathbf{r})$. *[54.2.5]* Equation (2.159) was introduced in integral form in [**104**], but the connection with [**87**] was not given.

*[54.3.1]* An alternative to eq. (2.159), introduced in [**53**, **54**], is

$$\mathrm{D}_{0+}^{\alpha,0} f(\mathbf{r},t) = C\,\Delta f(\mathbf{r},t) \tag{2.160}$$

with a Riemann-Liouville fractional time derivative $\mathrm{D}_{0+}^{\alpha}$ of type 0. *[54.3.2]* This equation does not describe diffusion of Montroll-Weiss type [**53**]. *[54.3.3]* It has therefore been called "inconsistent" in [**81**, p.3566]. As emphasized in [**53**] the choice of $\mathrm{D}_{0+}^{\alpha}$ in (2.159) is physically and mathematically consistent, but corresponds to a modified initial condition, namely $\mathrm{I}_{0+}^{1-\alpha} f(\mathbf{r},0+) = f_0\delta(\mathbf{r})$. *[54.3.4]* Similarly, fractional diffusion equations with time derivative $\mathrm{D}_{0+}^{\alpha,\beta}$ of order $\alpha$ and type $\beta$ have been investigated in [**54**]. *[54.3.5]* For $\alpha = 1$ they all reduce to the diffusion equation.

*[54.4.1]* Before discussing how $\alpha$ arises from an underlying continuous time random walk it is of interest to give an overall comparison of ordinary diffusion with

[9] Also, Herrmann Weyl, who pioneered fractional as well as functional calculus and worked on the foundations of physics, seems not to have applied fractional derivatives to problems in physics.

[10] Note that, contrary to [**112**, p.51], fractional derivatives are never mentioned in [**6**].

*[page 55, §0]* $\alpha = 1$ and fractional diffusion of the form (2.159) with $\alpha \neq 1$. *[55.0.1]* This is conveniently done using the following table published in [**46**]. *[55.0.2]* The first column gives the results for $\alpha = 1$, the second for $0 < \alpha < 1$ and the third for the limit $\alpha \to 0$. *[55.0.3]* The first row compares the infinitesimal generators of time evolution $\mathrm{A}_\alpha$. *[55.0.4]* The second row gives the fundamental solution $f(\mathbf{k}, u)$ in Fourier-Laplace space. *[55.0.5]* The third row gives $f(\mathbf{k}, t)$ and the fourth $f(\mathbf{r}, t)$. *[55.0.6]* In the fifth and sixth row the asymptotic behaviour is collected for $r^2/t^\alpha \to 0$ and $r^2/t^\alpha \to \infty$.

**Table 2.1.** Table from [**46**].

| | $\alpha = 1$ | $0 < \alpha < 1$ | $\alpha \to 0$ |
|---|---|---|---|
| $\mathrm{A}_\alpha$ | $\dfrac{\mathrm{d}}{\mathrm{d}t}$ | $\widetilde{\mathrm{D}}^{\alpha}_{0+}$ | $\to \mathbf{1}$ |
| $f(\mathbf{k}, u)$ | $\dfrac{f_0}{u + C\mathbf{k}^2}$ | $\dfrac{f_0 u^{\alpha-1}}{u^\alpha + C\mathbf{k}^2}$ | $\to \dfrac{f_0}{u(1 + C\mathbf{k}^2)}$ |
| $f(\mathbf{k}, t)$ | $f_0 \mathrm{e}^{-Ct\mathbf{k}^2}$ | $f_0 \mathrm{E}_\alpha\left(-Ct\mathbf{k}^2\right)$ | $\to \dfrac{f_0}{1 + C\mathbf{k}^2}$ |
| $f(\mathbf{r}, t)$ | $\dfrac{f_0 \mathrm{e}^{-\mathbf{r}^2/4Ct}}{(4\pi Ct)^{-d/2}}$ | $\dfrac{f_0}{(\mathbf{r}^2\pi)^{d/2}} H^d_\alpha\left(\dfrac{\mathbf{r}^2}{4Ct^\alpha}\right)$ | $\dfrac{f_0 \lvert\mathbf{r}\rvert^{1-\frac{d}{2}}}{\sqrt{C(2\pi)^d}} K_{\frac{d-2}{d}}\left(\dfrac{\lvert\mathbf{r}\rvert}{\sqrt{C}}\right)$ |
| $\dfrac{\mathbf{r}^2}{t^\alpha} \to 0$ | $t^{-d/2}$ | $\dfrac{\lvert\mathbf{r}\rvert^{2-d}}{t^\alpha}$ | $\lvert\mathbf{r}\rvert^{(2/d)-(d/2)}$ |
| $\dfrac{\mathbf{r}^2}{t^\alpha} \to \infty$ | $\exp\left[-\dfrac{\mathbf{r}^2}{4Ct}\right]$ | $\exp\left[-c_\alpha\left(\dfrac{\mathbf{r}^2}{4Ct^\alpha}\right)^{\frac{1}{2-\alpha}}\right]$ | $\exp\left(-\dfrac{\lvert\mathbf{r}\rvert}{\sqrt{C}}\right)$ |

*[55.1.1]* In the table $\mathrm{E}_{\alpha,\beta}(x)$ denotes the generalized Mittag-Leffler function from eq. (2.130), $K_\nu(x)$ is the modified Bessel function [**1**], $d > 2$, $c_\alpha = (2-\alpha)\alpha^{\alpha/(2-\alpha)}$ and the shorthand

$$H^d_\alpha(x) = H^{20}_{12}\left(x \,\middle|\, \begin{matrix} (1, \alpha) \\ (d/2, 1), (1, 1) \end{matrix}\right) \tag{2.161}$$

was used for the $H$-function $H^{20}_{12}$. *[55.1.2]* For information on $H$-functions see [**30**, **54**, **79**, **95**].

*[page 56, §1]* *[56.1.1]* The results in the table show that the normal diffusion ($\alpha = 1$) is slowed down for $0 < \alpha < 1$ and comes to a complete halt for $\alpha \to 0$. *[56.1.2]* For more discussion of the solution see [**46**].

### 2.3.4.3. Continuous Time Random Walks

*[56.2.1]* The fractional diffusion equation (2.159) can be related rigorously to the microscopic model of Montroll-Weiss continuous time random walks (CTRW's) [**64**, **87**] in the same way as ordinary diffusion is related to random walks [**27**]. *[56.2.2]* The fractional order $\alpha$ can be identified and has a physical meaning related to waiting times in the Montroll-Weiss model. *[56.2.3]* The relation between fractional time derivatives and CTRW's was first exposed in [**46**, **60**]. *[56.2.4]* The relation was established in two steps. First, it was shown in [**60**] that Montroll-Weiss continuous time random walks with a Mittag-Leffler waiting time density are rigorously equivalent to a fractional master equation. *[56.2.5]* Then, in [**46**] this underlying random walk model was connected to the fractional diffusion equation (2.159) in the usual asymptotic sense [**109**] of long times and large distances[11]. *[56.2.6]* For additional results see also [**50**, **53**, **54**, **57**]

*[56.3.1]* The basic integral equation for separable continuous time random walks describes a random walker in continuous time without correlation between its spatial and temporal behaviour. *[56.3.2]* It reads [**39**, **64**, **87**, **88**, **118**]

$$f(\mathbf{r},t) = \delta_{\mathbf{r},\mathbf{0}}\Phi(t) + \int_0^t \psi(t-t') \sum_{\mathbf{r}'} \lambda(\mathbf{r}-\mathbf{r}') f(\mathbf{r}',t')\mathrm{d}t', \tag{2.162}$$

where $f(\mathbf{r},t)$ denotes the probability density to find the walker at position $\mathbf{r} \in \mathbb{R}^d$ after time $t$ if it started from $\mathbf{r} = \mathbf{0}$ at time $t = 0$. *[56.3.3]* The function $\lambda(\mathbf{r})$ is the probability for a displacement by $\mathbf{r}$ in each step, and $\psi(t)$ gives the probability density of waiting time intervals between steps. *[56.3.4]* The transition probabilities obey $\sum_{\mathbf{r}} \lambda(\mathbf{r}) = 1$, and $\Phi(t) = 1 - \int_0^t \psi(t')\mathrm{d}t'$ is the survival probability at the initial site.

*[56.4.1]* The fractional master equation introduced in [**60**] with inital condition $f(\mathbf{r},0) = \delta_{\mathbf{r},\mathbf{0}}$ reads

$$\mathrm{D}^{\alpha,1}_{0+} f(\mathbf{r},t) = \sum_{\mathbf{r}'} w(\mathbf{r}-\mathbf{r}') f(\mathbf{r}',t) \tag{2.163}$$

with fractional transition rates $w(\mathbf{r})$ obeying $\sum_{\mathbf{r}} w(\mathbf{r}) = 0$. *[56.4.2]* Note, that eq. (2.162) contains a free function $\psi(t)$ that has no counterpart in eq. (2.163). *[56.4.3]* The rigorous relation between eq. (2.162) and eq. (2.163), first established in [**60**], is given by the relation

$$\lambda(\mathbf{k}) = 1 + \tau^{\alpha} w(\mathbf{k}) \tag{2.164}$$

[11] This is emphasized in eqs. (1.8) and (2.1) in [**46**] that are, of course, asymptotic.

*[page 57, §0]* for the Fourier transformed transition rates $w(\mathbf{r})$ and probabilities $\lambda(\mathbf{r})$, and the choice

$$\psi(t) = \frac{t^{\alpha-1}}{\tau^\alpha}\mathrm{E}_{\alpha,\alpha}\left(-\left(\frac{t}{\tau}\right)^\alpha\right) \tag{2.165}$$

for the waiting time density, where $\tau > 0$ is a characteristic time constant. *[57.0.1]* With $\mathrm{E}_{\alpha,\alpha}(0) = 1$ it follows that

$$\psi(t) \sim t^{\alpha-1} \tag{2.166}$$

for $t \to 0$. *[57.0.2]* From $\mathrm{E}_{\alpha,\alpha}(x) \sim x^{-2}$ for $x \to \infty$ one finds

$$\psi(t) \sim t^{-\alpha-1} \tag{2.167}$$

for $t \to \infty$. *[57.0.3]* For $\alpha = 1$ the waiting time density becomes the exponential distribution, and for $\alpha \to 0$ it approaches $1/t$.

*[57.1.1]* It had been observed already in the early 1970's that continuous time random walks are equivalent to generalized master equations [**9**,**66**]. *[57.1.2]* Similarly, the Fourier-Laplace formula

$$f(\mathbf{k}, u) = u^{\alpha-1}/(u^\alpha + C\mathbf{k}^2) \tag{2.168}$$

for the solution of CTRW's with algbraic tails of the form (2.167) was well known (see [**117**, eq.(21), p.402] [**110**, eq.(23), p.505] [**67**, eq.(29), p.3083]). *[57.1.3]* Comparison with row 2 of the table makes the connection between the fractional diffusion equation (2.159) and the CTRW-equation (2.162) evident. *[57.1.4]* However, this connection with fractional calculus was not made before the appearance of [**46**, **60**]. *[57.1.5]* In particular, there is no mention of fractional derivatives or fractional calculus in [**6**].

*[57.2.1]* The rigorous relation between fractional diffusion and CTRW's, established in [**46**, **60**] and elaborated in [**50**, **53**, **54**, **57**], has become a fruitful starting point for subsequent investigations, particularly into fractional Fokker-Planck equations with drift [**19**, **33**, **51**, **61**, **80–83**, **100**, **111**, **112**, **130**].

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APPENDIX A

*[page 59, §1]*

# Tables

*[59.1.1]* Let $\alpha \in \mathbb{C}$, $x > a$

| $f(x)$ | $(\mathrm{I}^{\alpha}_{a+} f)(x)$ | |
|---|---|---|
| $f(\lambda x)$ | $\lambda^{-\alpha}(\mathrm{I}^{\alpha}_{\lambda a+} f)(\lambda x), \lambda > 0$ | (A.1) |
| $(x-a)^{\beta}$ | $\dfrac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}(x-a)^{\alpha+\beta}$<br>$\operatorname{Re}\beta > 0$ | (A.2) |
| $\mathrm{e}^{\lambda x}$ | $\mathrm{e}^{\lambda a}(x-a)^{\alpha}\mathrm{E}_{1,\alpha+1}(\lambda(x-a))$<br>$\lambda \in \mathbb{R}$ | (A.3) |
| $(x-a)^{\beta-1}\ \mathrm{e}^{\lambda x}$ | $\dfrac{\Gamma(\beta)\mathrm{e}^{\lambda a}}{\Gamma(\alpha+\beta)}(x-a)^{\alpha+\beta-1}\ {}_1F_1(\beta;\alpha+\beta;\lambda(x-a))$<br>$\operatorname{Re}\beta > 0$ | (A.4) |
| $(x-a)^{\beta-1}\ \log(x-a)$ | $\dfrac{\Gamma(\beta)(x-a)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}[\psi(\beta)-\psi(\alpha+\beta)+\log(x-a)]$ | (A.5) |
| $(x-a)^{\beta-1}\mathrm{E}_{\gamma,\beta}((x-a)^{\gamma})$ | $(x-a)^{\alpha+\beta-1}\mathrm{E}_{\gamma,\alpha+\beta}((x-a)^{\gamma})$<br>$\operatorname{Re}\beta > 0, \operatorname{Re}\gamma > 0$ | (A.6) |

**Table A.1.** Some fractional integrals

APPENDIX B

*[page 61, §1]*

# Function Spaces

*[61.1.1]* The set $\mathbb{G}$ denotes an interval, a domain in $\mathbb{R}^d$ or a measure space $(\mathbb{G}, \mathcal{A}, \mu)$ **[8]** depending on the context. *[61.1.2]* $\mathbb{K}$ stands for $\mathbb{R}$ or $\mathbb{C}$. $\gamma = (\gamma_1, ..., \gamma_d) \in \mathbb{N}_0^d$ is a multiindex and $|\gamma| = \sum_{i=1}^d \gamma_i$. *[61.1.3]* For the definition of Hilbert and Banach spaces the reader may consult e.g. **[128]**. *[61.1.4]* The following notation is used for various spaces of continuous functions:

$$C^0(\mathbb{G}) := \{f : \mathbb{G} \to \mathbb{K} | f \text{ is continuous}\} \tag{B.1}$$

$$C^k(\mathbb{G}) := \{f \in C^0(\mathbb{G}) | f \text{ is } k\text{-times continuously differentiable}\} \tag{B.2}$$

$$C^k_0(\mathbb{G}) := \{f \in C^k(\mathbb{G}) | f \text{ vanishes at the boundary } \partial\mathbb{G}\} \tag{B.3}$$

$$C^k_{\mathrm{b}}(\mathbb{G}) := \{f \in C^k(\mathbb{G}) | f \text{ is bounded}\} \tag{B.4}$$

$$C^k_{\mathrm{c}}(\mathbb{G}) := \{f \in C^k(\mathbb{G}) | f \text{ has compact support}\} \tag{B.5}$$

$$C^k_{\mathrm{ub}}(\mathbb{G}) := \{f \in C^k(\mathbb{G}) | f \text{ is bounded and uniformly continuous}\} \tag{B.6}$$

$$\mathrm{AC}^k([a,b]) := \{f \in C^k([a,b]) | f^{(k)} \text{ is absolutely continuous}\} \tag{B.7}$$

*[61.1.5]* For compact $\mathbb{G}$ the norm on these spaces is

$$\|f\|_\infty := \sup_{x\in\mathbb{G}} |f(x)|. \tag{B.8}$$

*[61.1.6]* The Lebesgue spaces over $(\mathbb{G}, \mathcal{A}, \mu)$ are defined as

$$L^p_{\mathrm{loc}}(\mathbb{G}, \mu) := \{f : \mathbb{G} \to \mathbb{K} \mid f^p \text{ is integrable on every compact } K \subset \mathbb{G}\} \tag{B.9}$$

$$L^p(\mathbb{G}, \mu) := \{f : \mathbb{G} \to \mathbb{K} \mid f^p \text{ is integrable}\} \tag{B.10}$$

with norm

$$\|f\|_p := \left(\int_{\mathbb{G}} |f(s)|^p \mathrm{d}\mu(s)\right)^{1/p}. \tag{B.11}$$

*[page 62, §0]* *[62.0.1]* For $p = \infty$

$$L^\infty(\mathbb{G},\mu) := \{f : \mathbb{G} \to \mathbb{K} \mid f \text{ is measurable and } \|f\|_\infty < \infty\} \tag{B.12}$$

where

$$\|f\|_\infty := \sup\{|z| : z \in f_{\text{ess}}(\mathbb{G})\} \tag{B.13}$$

and

$$f_{\text{ess}}(\mathbb{G}) := \{z \in \mathbb{C} : \mu(\{x \in \mathbb{G} : |f(x) - z| < \varepsilon\}) \neq 0 \text{ for all } \varepsilon > 0\} \tag{B.14}$$

is the *essential range of* $f$.

*[62.1.1]* The Hölder spaces $C^\alpha(\mathbb{G})$ with $0 < \alpha < 1$ are defined as

$$C^\alpha(\mathbb{G}) := \{f : \mathbb{G} \to \mathbb{K} | \exists c \geq 0 \text{ s.t. } |f(x) - f(y)| \leq c|x - y|^\alpha, \forall x, y \in \mathbb{G}\} \tag{B.15}$$

with norm

$$\|f\|_\alpha := \|f\|_\infty + c_\alpha \tag{B.16}$$

where $c_\alpha$ is the smallest constant $c$ in (B.15). *[62.1.2]* For $\alpha > 1$ the Hölder space $C^\alpha(\mathbb{G})$ contains only the constant functions and therefore $\alpha$ is chosen as $0 < \alpha < 1$. *[62.1.3]* The spaces $C^{k,\alpha}(\mathbb{G})$, $k \in \mathbb{N}$, consist of those functions $f \in C^k(\mathbb{G})$ whose partial derivatives of order $k$ all belong to $C^\alpha(\mathbb{G})$.

*[62.2.1]* The Sobolev spaces are defined by

$$W^{k,p}(\mathbb{G}) = \left\{ f \in L^p(\mathbb{G}) : \begin{array}{c} f \text{ is } k\text{-times differentiable in the} \\ \text{sense of distributions and } \mathrm{D}^\gamma f \in \\ L^p(\mathbb{G}) \text{ for all } \gamma \in \mathbb{N}_0^d \text{ with } |\gamma| \leq k \end{array} \right\} \tag{B.17}$$

where the derivative $\mathrm{D}^\gamma = \partial_1^{\gamma_1} ... \partial_d^{\gamma_d}$ with multiindex $\gamma = (\gamma_1, ..., \gamma_d) \in \mathbb{N}_0^d$ is understood in the sense of distributions. *[62.2.2]* A distribution $f$ is in $W^{k,p}(\mathbb{G})$ if and only if for each $\gamma \in \mathbb{N}_0^d$ with $|\gamma| \leq k$ there exists $f_\gamma \in L^p(\mathbb{G})$ such that

$$\int\limits_{\mathbb{G}} \phi f_\gamma \mathrm{d}x = (-1)^{|\gamma|} \int\limits_{\mathbb{G}} (\mathrm{D}^\gamma \phi) f \mathrm{d}x \tag{B.18}$$

for all test functions $\phi$. *[62.2.3]* In the special case $d = 1$ one has $f \in W^{k,p}(\mathbb{G})$ if and only if $f \in C^{k-1}(\mathbb{G})$, $f^{(k-1)} \in \mathrm{AC}(\mathbb{G})$, and $f^{(j)} \in L^p(\mathbb{G})$ for $j = 0, 1, ..., k$. *[62.2.4]* The Sobolev spaces are equipped with the norm

$$\|f\|_{W^{k,p}(\mathbb{G})} = \sum_{|\gamma| \leq m} \| \mathrm{D}^\gamma f\|_p \tag{B.19}$$

(see **[2]**). *[62.2.5]* A function is called *rapidly decreasing* if it is infinitely many times differentiable, i.e. $f \in C^\infty(\mathbb{R}^d)$ and

$$\lim_{|x| \to \infty} |x|^n \, \mathrm{D}^\gamma f(x) = 0 \tag{B.20}$$

*[page 63, §0]* for all $n \in \mathbb{N}$ and $\gamma \in \mathbb{N}^d$. *[63.0.1]* The test function space

$$\mathcal{S}(\mathbb{R}^d) := \{f \in C^\infty(\mathbb{R}^d) | f \text{ is rapidly decreasing}\} \tag{B.21}$$

is called *Schwartz space*.

APPENDIX C

*[page 65, §1]*

# Distributions

*[65.1.1]* Distributions are generalized functions [**31**]. *[65.1.2]* They were invented to overcome the differentiability requirements for functions in analysis and mathematical physics [**63, 105**]. *[65.1.3]* Distribution theory has also a physical origin. *[65.1.4]* A physical observable $f$ can never be measured at a point $x \in \mathbb{R}^d$ because every measurement apparatus averages over a small volume around $x$ [**115**]. *[65.1.5]* This "smearing out" can be modelled as an integration with smooth "test functions" having compact support.

*[65.2.1]* Let $X$ denote the space of admissible test functions. *[65.2.2]* Commonly used test function spaces are $C^\infty(\mathbb{R}^d)$, the space of infinitely often differentiable functions, $C_c^\infty(\mathbb{R}^d)$, the space of smooth functions with compact support (see (B.5)), $C_0^\infty(\mathbb{R}^d)$, the space of smooth functions vanishing at infinity (see (B.3)), or the so called Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of smooth functions decreasing rapidly at infinity (see (B.21)).

*[65.3.1]* A distribution $F : X \to \mathbb{K}$ is a linear and continuous mapping that maps $\varphi \in X$ to a real ($\mathbb{K} = \mathbb{R}$) or complex ($\mathbb{K} = \mathbb{C}$) number [1]. *[65.3.2]* There exists a canonical correspondence between functions and distributions. *[65.3.3]* More precisely, for every locally integrable function $f \in L^1_{\text{loc}}(\mathbb{R}^d)$ there exists a distribution $F_f = \langle f, .\rangle$ (often also denoted with the same symbol $f$) defined by

$$F_f(\varphi) = \langle f, \varphi\rangle = \int\limits_{\mathbb{R}^d} f(x)\varphi(x)\, \mathrm{d}x \tag{C.1}$$

for every test function $\varphi \in X$. *[65.3.4]* Distributions that can be written in this way are called *regular distributions*. *[65.3.5]* Distributions that are not regular are sometimes called *singular*. *[65.3.6]* The mapping $f \to \langle f, .\rangle$ that assigns to a locally integrable $f$ its associated distribution is injective and continuous. *[65.3.7]* The set of distributions is again a vector space, namely the dual space of the vector space of test functions, and it is denoted as $X'$ where $X$ is the test function space.

[1] For vector valued distributions see [**106**]

*[page 66, §1]* *[66.1.1]* Important examples for singular distributions are the Dirac $\delta$-function and its derivatives. *[66.1.2]* They are defined by the rules

$$\int \delta(x)\varphi(x)\mathrm{d}x = \varphi(0) \tag{C.2}$$

$$\int \delta^{(n)}(x)\varphi(x) = (-1)^n \left.\frac{\mathrm{d}^n\varphi}{\mathrm{d}x^n}\right|_{x=0} \tag{C.3}$$

for every test function $\varphi \in X$ and $n \in \mathbb{N}$. *[66.1.3]* Clearly, $\delta(x)$ is not a function, because if it were a function, then $\int \delta(x)\varphi(x)\mathrm{d}x = 0$ would have to hold. *[66.1.4]* Another example for a singular distribution is the finite part or principal value $\mathcal{P}\{1/x\}$ of $1/x$. *[66.1.5]* It is defined by

$$\left\langle \mathcal{P}\left\{\frac{1}{x}\right\}, \varphi \right\rangle = \lim_{\varepsilon\to 0+} \int\limits_{|x|\geq\varepsilon} \frac{\varphi(x)}{x}\mathrm{d}x \tag{C.4}$$

for $\varphi \in C_\mathrm{c}^\infty(\mathbb{R})$. *[66.1.6]* It is a singular distribution on $\mathbb{R}$, but regular on $\mathbb{R}\setminus\{0\}$ where it coincides with the function $1/x$.

*[66.2.1]* Equation (C.2) illustrates how distributions circumvent the limitations of differentiation for ordinary functions. *[66.2.2]* The basic idea is the formula for partial integration

$$\int\limits_{\mathbb{G}} \partial_i f(x)\varphi(x)\mathrm{d}x = -\int\limits_{\mathbb{G}} f(x)\partial_i\varphi(x)\mathrm{d}x \tag{C.5}$$

valid for $f \in C_\mathrm{c}^1(\mathbb{G})$, $\varphi \in C^1(\mathbb{G})$, $i = 1, ..., d$ and $\mathbb{G} \subset \mathbb{R}^d$ an open set. *[66.2.3]* The formula is proved by extending $f\varphi$ as 0 to all of $\mathbb{R}^d$ and using Leibniz' product rule. *[66.2.4]* Rewriting the formula as

$$\langle \partial_i f, \varphi\rangle = -\langle f, \partial_i\varphi\rangle \tag{C.6}$$

suggests to view $\partial_i f$ again as a linear continuous mapping (integral) on a space $X$ of test functions $\varphi \in X$. *[66.2.5]* Then the formula is a rule for differentiating $f$ given that $\varphi$ is differentiable.

*[66.3.1]* Distributions on the test function space $\mathcal{S}(\mathbb{R}^d)$ are called *tempered distributions*. *[66.3.2]* The space of tempered distributions is the dual space $\mathcal{S}(\mathbb{R}^d)'$. *[66.3.3]* Tempered distributions generalize locally integrable functions growing at most polynomially for $|x| \to \infty$. *[66.3.4]* All distributions with compact support are tempered. Square integrable functions are tempered distributions. *[66.3.5]* The derivative of a tempered distribution is again a tempered distribution. *[66.3.6]* $\mathcal{S}(\mathbb{R}^d)$ is dense in $L^p(\mathbb{R}^d)$ for all $1 \leq p < \infty$ but not in $L^\infty(\mathbb{R}^d)$. *[66.3.7]* The Fourier transform and its inverse are continous maps of the Schwartz space onto itself. *[66.3.8]* A distribution $f$ belongs to $\mathcal{S}(\mathbb{R}^d)'$ if and only if it is the derivative of a continuous function with slow growth, i.e. it is of the form $f = \mathrm{D}^\gamma[(1+|x|^2)^{k/2}g(x)]$

*[page 67, §0]* where $k \in \mathbb{N}$, $\gamma \in \mathbb{N}^d$ and $g$ is a bounded continuous function on $\mathbb{R}^d$. *[67.0.1]* Note that the exponential function is not a tempered distribution.

*[67.1.1]* A distribution $f \in \mathcal{S}(\mathbb{R}^d)'$ is said to have *compact support* if there exists a compact subset $K \subset \mathbb{R}^d$ such that $\langle f, \varphi \rangle = 0$ for all test functions with $\operatorname{supp} \varphi \cap K = \emptyset$. *[67.1.2]* The Dirac $\delta$-function is an example. *[67.1.3]* Other examples are Radon measures on a compact set $K$. *[67.1.4]* They can be described as linear functionals on $C^0(K)$. *[67.1.5]* If the set $K$ is sufficiently regular (e.g. if it is the closure of a region with piecewise smooth boundary) then every distribution with compact support in $K$ can be written in the form

$$f = \sum_{|\gamma| \le N} \mathrm{D}^\gamma f_\gamma \tag{C.7}$$

where $\gamma = (\gamma_1, ..., \gamma_d)$, $\gamma_j \ge 0$ is a multiindex, $|\gamma| = \sum \gamma_i$ and $f_\gamma$ are continuous functions of compact support. *[67.1.6]* Here $N \ge 0$ and the partial derivatives in $\mathrm{D}^\gamma$ are distributional derivatives defined above. *[67.1.7]* A special case are distributions with support in a single point taken as $\{0\}$. *[67.1.8]* Any such distributions can be written in the form

$$f = \sum_{|\gamma| \le N} c_\gamma \, \mathrm{D}^\gamma \, \delta \tag{C.8}$$

where $\delta$ is the Dirac $\delta$-function and $c_\gamma$ are constants.

*[67.2.1]* The multiplication of a distribution $f$ with a smooth function $g$ is defined by the formula $\langle gf, \varphi \rangle = \langle f, g\varphi \rangle$ where $g \in C^\infty(\mathbb{G})$. *[67.2.2]* A combination of multiplication by a smooth function and differentiation allows to define differential operators

$$A = \sum_{|\gamma| \le m} a_\gamma(x) \, \mathrm{D}^\gamma \tag{C.9}$$

with smooth $a_\gamma(x) \in C^\infty(\mathbb{G})$. *[67.2.3]* They are well defined for all distributions in $C^\infty_{\mathrm{c}}(\mathbb{G})'$.

*[67.3.1]* A distribution is called homogeneous of degree $\alpha \in \mathbb{C}$ if

$$f(\lambda x) = \lambda^\alpha f(x) \tag{C.10}$$

for all $\lambda > 0$. *[67.3.2]* Here $\lambda^\alpha = \exp(\alpha \log \lambda)$ is the standard definition. *[67.3.3]* The Dirac $\delta$-distribution is homogeneous of degree $-d$. *[67.3.4]* For regular distributions the definition coincides with homogeneity of functions $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$. *[67.3.5]* The convolution kernels $K^\alpha_\pm$ from eq. (2.39) are homogeneous of degree $\alpha - 1$. *[67.3.6]* Homogeneous distributions remain homogeneous under differentiation. *[67.3.7]* A homogeneous locally integrable function $g$ on $\mathbb{R}^d \backslash \{0\}$ of degree $\alpha$ can be extended to homogeneous distributions $f$ on all of $\mathbb{R}^d$. *[67.3.8]* The degree of homogeneity of $f$ must again be $\alpha$. *[67.3.9]* As long as $\alpha \neq -d, -d-1, -d-2, ...$ the integral

$$\langle g_\beta, \varphi \rangle = \int g\left(\frac{x}{|x|}\right) |x|^\beta \mathrm{d}^d x \tag{C.11}$$

*[page 68, §0]* which converges absolutely for $\operatorname{Re}\beta > -d$ can be used to define $f = g_\alpha$ by analytic continuation from the region $\operatorname{Re}\beta > -d$ to the point $\alpha$. *[68.0.1]* For $\alpha = -d, -d-1, ...$, however, this is not always possible. *[68.0.2]* An example is the function $1/|x|$ on $\mathbb{R} \setminus \{0\}$. *[68.0.3]* It cannot be extended to a homogeneous distribution of degree $-1$ on all of $\mathbb{R}$.

*[68.1.1]* For $f \in L^1_{\mathrm{loc}}(\mathbb{G}_1)$ and $g \in L^1_{\mathrm{loc}}(\mathbb{G}_2)$ their tensor product is the function $(f \otimes g)(x, y) = f(x)g(y)$ defined on $\mathbb{G}_1 \times \mathbb{G}_2$. *[68.1.2]* The function $f \otimes g$ gives a functional

$$\langle f \otimes g, \varphi(x, y)\rangle = \langle f(x), \langle g(y), \varphi(x, y)\rangle\rangle \tag{C.12}$$

for $\varphi \in C^\infty_{\mathrm{c}}(\mathbb{G}_1 \times \mathbb{G}_2)$. *[68.1.3]* For two distributions this formula defines the their tensor product. *[68.1.4]* An example is a measure $\mu(x) \otimes \delta(y)$ concentrated on the surface $y = 0$ in $\mathbb{G}_1 \otimes \mathbb{G}_2$ where $\mu(x)$ is a measure on $\mathbb{G}_1$. *[68.1.5]* The convolution of distribution defined in the main text (see eq. (2.52) can then be defined by the formula

$$\langle f * g, \varphi\rangle = \langle (f \otimes g)(x, y), \varphi(x + y)\rangle \tag{C.13}$$

whenever one of the distributions $f$ or $g$ has compact support.

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