Applications of Fractional Calculus in Physics

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ποταμοῖς τοῖς αὐτοῖς ἐμβαίνομέν τε καὶ οὐκ ἐμβαίνομεν,
εἶμέν τε καὶ οὐκ εἶμεν.
Ἡράκλειτος

Preface

Although fractional calculus is a natural generalization of calculus, and although its mathematical history is equally long, it has, until recently, played a negligible role in physics. One reason could be that, until recently, the basic facts were not readily accessible even in the mathematical literature. This book intends to increase the accessibility of fractional calculus by combining an introduction to the mathematics with a review of selected recent applications in physics.

Many applications of fractional calculus amount to replacing the time derivative in an evolution equation with a derivative of fractional order. This is not merely a phenomenological procedure providing an additional fit parameter. Rather the chapters of this book illustrate that fractional derivatives seem to arise generally and universally, and for deep mathematical reasons. One central theme of this book is the fact that fractional derivatives arise as the infinitesimal generators of a class of translation invariant convolution semigroups. These semigroups appear universally as attractors for coarse graining procedures or scale changes. They are parametrized by a number in the unit interval corresponding to the order of the fractional derivative.

Despite their common theme all chapters are self contained and can be read independently of the rest. Editing has been kept to a minimum in order to preserve the diverse style and levels of formalization in the different areas of application. Its diversity shows that the field is still evolving and workers have not even agreed on a common notation for fractional integrals and derivatives.

Given the long mathematical history of fractional calculus it is appropriate that the book begins with a mathematical introduction to fractional calculus. Chapter I [BW00] provides such an introduction, and reviews also mathematical applications to special functions, Euler, Bernoulli, and Stirling numbers. Chapter II [Hil00b] discusses fractional evolution equations and their emergence from coarse graining. It stresses the general importance of fractional semigroups for applications in physics, and gives explicit solutions for some fractional differential equations. Chapter III [Wes00] continues the mathematical discussion of fractional semigroups and their infinitesimal generators from a functional analytic point of view. Chapters IV [WG00] and V [Zas00] review phenomenological and physical arguments for the general importance of fractional derivatives. The arguments are based mainly on the ubiquity of long time memory in nonequilibrium processes and on the behaviour of trajectories in chaotic Hamiltonian systems. Polymer science applications of fractional calculus are discussed in Chapters VI [Dou00] and VII [SFB00] Chapter VI focusses on surface interacting polymers and the decimation transformation of random walk models. Chapter VII discusses the Rouse model and rheological constitutive modelling. Applications to relaxation and diffusion models for biophysical phenomena are presented in Chapter VIII [NM00]. Finally the last chapter (IX) reviews a somewhat unorthodox application in which fractional calculus is used to generalize the Ehrenfest classification of phase transitions in equilibrium thermodynamics [Hil00a].

Let me conclude this preface by wishing all readers the joy and excitement that I felt many times when wandering and wondering in the fields of fractional calculus and its applications. Last but not least it is a pleasant task to thank Marc Lätzel, Martin Ottmann and Marlies Parsons for their help with typesetting the manuscript.

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References

[BW00] P.L. Butzer and U. Westphal. Introduction to fractional calculus. In R. Hilfer, editor, *Applications of Fractional Calculus in Physics*, pages 1–86, Singapore, 2000. World Scientific.

[Dou00] J. Douglas. Polymer science applications of path-integration, integral equations and fractional calculus. In R. Hilfer, editor, *Applications of Fractional Calculus in Physics*, pages 241–330, Singapore, 2000. World Scientific.

[Hil00a] R. Hilfer. Fractional calculus and regular variation in thermodynamics. In R. Hilfer, editor, *Applications of Fractional Calculus in Physics*, pages 429–463, Singapore, 2000. World Scientific.

[Hil00b] R. Hilfer. Fractional time evolution. In R. Hilfer, editor, *Applications of Fractional Calculus in Physics*, pages 87–130, Singapore, 2000. World Scientific.

[NM00] T. F. Nonnenmacher and R. Metzler. Applications of fractional calculus techniques to problems in biophysics. In R. Hilfer, editor, *Applications of Fractional Calculus in Physics*, pages 377–428, Singapore, 2000. World Scientific.

[SFB00] H. Schiessel, C. Friedrich, and A. Blumen. Applications to problems in polymer physics and rheology. In R. Hilfer, editor, *Applications of Fractional Calculus in Physics*, pages 331–378, Singapore, 2000. World Scientific.

[Wes00] U. Westphal. Fractional powers of infinitesimal generators of semigroups. In R. Hilfer, editor, *Applications of Fractional Calculus in Physics*, pages 131–170, Singapore, 2000. World Scientific.

[WG00] B.J. West and P. Grigolini. Fractional differences, derivatives and fractal time series. In R. Hilfer, editor, *Applications of Fractional Calculus in Physics*, pages 171–202, Singapore, 2000. World Scientific.

[Zas00] G. Zaslavsky. Fractional kinetics of hamiltonian chaotic systems. In R. Hilfer, editor, *Applications of Fractional Calculus in Physics*, pages 203–240, Singapore, 2000. World Scientific.