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Fractional Dynamics, Irreversibility and Ergodicity Breaking

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Abstract—Time flow in dynamical systems is analysed within the framework of ergodic theory from the perspective of a recent classification theory of phase transitions. Induced automorphisms are studied on subsets of measure zero. The induced transformations are found to be stable convolution semigroups rather than translation groups. This implies non-uniform flow of time, time irreversibility and ergodicity breaking. The induced semigroups are generated by fractional time derivatives. Stationary states with respect to fractional dynamics are dissipative in the sense that the measure of regions in phase space may decay algebraically with time although the measure is time transformation invariant.

1. INTRODUCTION

A recent development of rapidly increasing interest in the area of fractals and nonlinear dynamics are fractional dynamical systems [1–10]. A fractional dynamical system is defined as a dynamical system involving fractional time derivatives.

Many authors have recently and in the past proposed to use fractional time derivatives on heuristic or aesthetic grounds [1, 3–6, 8, 10–16]. None of these proposals is convincing from the viewpoint of basic physics because fractional derivatives, contrary to integer ones, are non-local operators. This is at variance with the principle of locality in physics [17] and, consequently, the proposals must be viewed as postulates justifiable only a posteriori in a particular modelling context. The purpose of the present paper is to give a general model-independent justification for introducing fractional time derivatives into dynamical systems. It is shown that non-local fractional dynamics arise generically from local integer order dynamics in a certain limit. As a consequence it is found that the appearance of fractional dynamics is related to the problems of time irreversibility [18] and ergodicity breaking [19]. The results of the present paper are direct consequences of a recent classification of phase transitions in statistical mechanics [7, 9].

Dynamical systems are discussed here in the sense of abstract ergodic theory as flows or semiflows on measure spaces [20, 21]. The present paper introduces the concept of ergodicity breaking into abstract ergodic theory as the phenomenon that the time evolution induced by an ergodic time evolution on subsets with zero measure may not be ergodic. Almost all dynamical systems occuring in physics are subsystems or larger systems. The dynamics of the supersystem induces the dynamics of the subsystem of interest. The basic idea is that there exists a microscopic theory describing the dynamical evolution of a large supersystem from which the dynamical laws of each of its subsystems can be derived by a process of systematic restriction or approximation.

Given any dynamical system describing a physical reality it is necessary to choose a basic

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time scale associated with the subsystem of interest. In a rarefied gas, for example, the typical time between successive collisions is 10^{-10} s, and hydrodynamical processes, such as gas flow in pipes or containers may reasonably be idealized as a long time limit. On the other hand, the microscopic time scale of 10^{-10} s may itself be considered as the long time limit of processes occurring on much shorter time scales. It is therefore necessary to study the passage from one long time limit to another mathematically. The appropriate mathematical idealization for moving up or down in the hierarchy of temporal (or spatial) scales was recently introduced in the form of the so-called ensemble limit [7, 22, 23].

Let me conclude the introduction with a statement of the main result. Define T^t to be the reversible microscopic time evolution operator acting on density matrices or probability distributions $\rho(s)$ in the state or phase space Γ of the dynamical system as a group of translations through

$$T^t\rho(s) = \rho(s - t), \tag{1}$$

where s, $t \in \mathbb{R}$ represent the microscopic time. The induced transformation S^t on a subset $G \subset \Gamma$ is obtained from T^t by recording the recurrence of the same fixed state in $\Gamma \backslash G$. The main result of this paper concerns the case when G has zero measure. In that case S^t will generally have the form of a stable convolution semigroup whose action is given as

$$S^t_{\varpi}\rho(s) = \frac{1}{t}\int_0^{\infty} T^{\tau}\rho(s)h_{\varpi}(\tau/t)\,\mathrm{d}\tau, \tag{2}$$

where $t \geqslant 0$ and $s \in \mathbb{R}$ now represent a renormalized macroscopic time, and $h_{\varpi}(x)$ denotes a stable one sided probability density with stability index $0 < \varpi \leqslant 1$. While T^t can in general be defined for all $t \in \mathbb{R}$ the induced transformations S^t_{ϖ} exist only for $t \geqslant 0$, and thus they form only a semigroup. For $\varpi = 1$ the infinitesimal generator of S^t_1 is found to be identical to that of T^t and it is given by the ordinary time derivative $-\mathrm{d}/\mathrm{d}t$. For $0 < \varpi < 1$ on the other hand one obtains fractional time derivatives of order ϖ. In the latter case the time evolution is not given by a uniform translation but involves memory effects as evident from (2).

2. DEFINITIONS AND STATEMENT OF THE PROBLEM

Consider a dynamical system with phase or state space Γ. Let $\mathscr{G}$ denote a σ-algebra of measurable subsets of Γ, and μ a countably additive non-negative set function on $\mathscr{G}$ such that $\mu(\Gamma) = 1$. The time evolution of the system is given as a flow, i.e. as the action of the additive group $\mathbb{R}$ of real numbers on Γ. A flow is defined as a one-parameter family of automorphisms T^t: $\Gamma \to \Gamma$ such that $T^0 = I$ is the identity, $T^{s+t} = T^sT^t$ for all t, $s \in \mathbb{R}$ and such that for every measurable function f the function $f(T^tx)$ is measurable on the direct product $\Gamma \times \mathbb{R}$. An automorphism T is defined as an invertible map T: $\Gamma \to \Gamma$ such that for every $G \in \mathscr{G}$ also TG, $T^{-1}G \in \mathscr{G}$. The measure μ is called invariant under the flow T^t if at all times $t \in \mathbb{R}$ one has $\mu(G) = \mu(T^tG) = \mu((T^t)^{-1}G)$ for all $G \in \mathscr{G}$. An invariant measure is called ergodic if it cannot be written as a non-trivial convex combination of invariant measures, i.e. if $\mu = \lambda\mu_1 + (1 - \lambda)\mu_2$ with μ_1, μ_2 invariant and $0 \leqslant \lambda \leqslant 1$ implies $\lambda = 1$, $\mu_1 = \mu$ or $\lambda = 0$, $\mu_2 = \mu$. The existence of the inverse $(T^t)^{-1} = T^{-t}$ for a flow is an expression for microscopic reversibility. Viewed actively (Heisenberg picture) the flow acts on measurable functions (observables) f: $\Gamma \to \mathbb{R}$ through $T^sf(x(t)) = f(x(t + s))$. Viewed passively (Schrödinger picture) the observables are time independent, and now the flow acts on time dependent measures $\rho(G, t)$ through right translation $T^t\rho(G, s) = \rho(G, s - t)$ as given in (1). The infinitesimal generator of T^t is defined as the strong limit

$$A = \lim_{t \to 0^+} \frac{T^t - I}{t}, \tag{3}$$

where $I = T^0$ denotes the identity. One has $A = -\mathrm{d}/\mathrm{d}t$ for right translations. The invariance of the measure μ can be expressed as $A\mu = -\mathrm{d}\mu/\mathrm{d}t = 0$ and it implies

$$\mu(G, t) = \mu(G) \tag{4}$$

for all $G \in \mathcal{G}$, $t \in \mathbb{R}$.

A concrete example of the general definitions is provided by a differentiable dynamical system. In continuous time it may be defined as a set of differential equations on the phase space $\Gamma = \mathbb{R}^n$

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{F}(\mathbf{x}(t)) \tag{5}$$

in discrete time as a set of difference equations

$$\mathbf{x}(t + \Delta t) = \tilde{\mathbf{F}}(\mathbf{x}(t)), \tag{6}$$

where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{F}, \tilde{\mathbf{F}}: \mathbb{R}^n \to \mathbb{R}^n$ are in general non-linear mappings. The flow T^t is then defined such that $T^t\mathbf{x}$ is the solution of (5) for which $\mathbf{x}(0) = \mathbf{x}$. Liouville's theorem furnishes an invariant measure as a solution to the stationary Liouville equation. Often the continuous time flow T^t is replaced with a discrete time evolution T^N with $N \in \mathbb{N}$ as in (6) which is generated by the automorphism $T = T^{\Delta t}$. The continuous time evolution T^t is recovered from the discrete one T^N in the limit $N \to \infty$, $\Delta t \to 0$ with $\Delta t \propto N^{-1}$.

Consider now the discretized time evolution $T = T^{\Delta t}$ on an arbitrary subset $G \subset \Gamma$. A point $x \in G$ is called recurrent with respect to G if there exists a $k \geq 1$ for which $T^k x \in G$. The Poincarè recurrence theorem asserts that if μ is invariant under T and $G \in \mathcal{G}$ then almost every point of G is recurrent with respect to G. A set $G \in \mathcal{G}$ is called a μ-recurrent set if μ-almost every $x \in G$ is recurrent with respect to G. By virtue of Poincarè's recurrence theorem the transformation T defines an induced transformation or induced automorphism S on subsets G of positive measure, $\mu(G) > 0$, through

$$Sx(s) = T^{\tau_G(x)}x(s) = x(s + \tau_G(x)) \tag{7}$$

for almost every $x \in G$. The recurrence time $\tau_G(x)$ of the point x, defined as

$$\tau_G(x) = \Delta t \min\{k \geq 1: T^k x \in G\}, \tag{8}$$

is positive and finite for almost every point $x \in G$. Because G has positive measure it is again a probability measure space with the induced measure $\nu = \mu/\mu(G)$. If μ was invariant under T then ν is invariant under S, and ergodicity of μ implies ergodicity also for ν [20].

The definition of S above is x-dependent. This raises the problem how to define S on measures. A natural idea is to average over the x-dependence. If μ is ergodic, and G a recurrent set with positive measure then the average recurrence time $\langle \tau \rangle$ is defined as

$$\langle \tau \rangle = \frac{1}{\mu(G)} \int_G \tau_G \, \mathrm{d}\mu = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \tau(S^{i-1}x), \tag{9}$$

where the last equality holds by virtue of the ergodic theorem. Kac's lemma [20] gives

$$\langle \tau \rangle = \frac{\Delta t}{\mu(G)} \tag{10}$$

and it suggests to interpret S formally as an average

$$S = \langle T^{\tau} \rangle = \frac{1}{\mu(G)} \int_G T^{\tau_G} \, \mathrm{d}\mu \approx T^{\langle \tau \rangle}, \tag{11}$$

where the last relation resembles a mean field type approximation. Accepting this approximation S^t becomes formally

$$S^t = \lim_{\substack{\Delta t \to 0, N \to \infty \\ N\Delta t \to t}} S^N \approx \lim_{\substack{\Delta t \to 0, N \to \infty \\ N\Delta t \to t}} T^{N\langle \tau \rangle} = T^{t/\mu(G)} \tag{12}$$

in the continous time limit. Thus S^t and T^t are identical except for a renormalization of time scale.

Often one is interested in the dynamics induced on subsystems $G \subset \Gamma$ such as subspaces of lower dimensionality in which many or all points are recurrent with respect to G but which have measure zero, $\mu(G) = 0$. A typical example occurs when passing to a description involving a reduced number of degrees of freedom. The problem of interest in this paper arises from the observation that for a set G with ergodic measure $\mu(G) = 0$ in which all points are recurrent the mean recurrence time diverges while the recurrence time of every point $x \in G$ exists and is finite. This suggests that the induced transformation S remains well defined on G but its action cannot simply be given as $T^{\langle \tau \rangle}$. The divergence of $\langle \tau \rangle$ indicates that the flow of time must be erratic and strongly fluctuating, and that any mathematical description must involve a renormalization of time scale.

3. AVERAGING THE INDUCED DYNAMICS

The problem then is to attach a meaning to (11) in the case where G has measure zero. The necessity to define S as an average $\langle T^{\tau} \rangle$ results from the x-dependence of S and represents a fundamental difference between the original flow and an induced transformation which is not emphasized in traditional ergodic theory. The idea of the present approach originated in [7, 9] and consists in defining $\langle \ldots \rangle$ as a time average rather than a phase space average as in (11). The advantage is that this can be done also for subsets of measure zero, as long as these are again measure spaces.

Let $(\Gamma, \mathscr{G}, \mu)$ denote the original measure space on which the time flow T^t is defined and discretized into $T = T^{\Delta t}$. Let $(G, \mathfrak{G}, \nu)$ denote a subspace $G \subset \Gamma$ of measure $\mu(G) = 0$ with σ-algebra $\mathfrak{G}$. For concreteness assume that Γ is a subset of $\mathbb{R}^m$ and μ is an m-dimensional measure, while G is a subset of $\mathbb{R}^n$ with an n-dimensional Lebesgue measure ν where $n < m$. Let $\mu(\Gamma) = 1$ and $\nu(G) = 1$. One has $\mathfrak{G} \subset \mathscr{G}$, in the sense that $B \in \mathfrak{G}$ for all $B \in \mathscr{G}$. Moreover $\mu(B) = 0$ for all $B \in \mathfrak{G}$ while $\nu(B) = \infty$ for all sets $B \in \mathscr{G}$ with $\mu(B) > 0$. Assume that G is ν-recurrent under T in the sense that ν-almost every point (rather than μ) is recurrent with respect to G. Define

$$G_k = \{x \in G : \tau(x) \leqslant k\Delta t\} \tag{13}$$

as the set of points whose recurrence time is $\leqslant k\Delta t$. Then the step function defined on the time intervals $k\Delta t \leqslant t < (k+1)\Delta t$ for all $k \in \mathbb{N}$ as

$$P(t) = \frac{\nu(G_k)}{\nu(G)}, \tag{14}$$

and as $P(t) = 0$ for all $t < \Delta t$, is a probability distribution function for the recurrence times on G.

The probability distribution of recurrence times $P(t)$ allows the problem stated in the previous section to be overcome, and to define S on measures as the mathematical expectation

$$S\rho(B, t) = \langle T^{\tau}\rho(B, t)\rangle = \int_0^{\infty} T^{\tau}\rho(B, t)\,\mathrm{d}P(\tau) = \int_0^{\infty} \rho(B, t-\tau)\,\mathrm{d}P(\tau), \tag{15}$$

where $B \subset G$ and ρ is a measure on G. Thus S is a convolution operator in time

$$S\rho(B) = \rho(B) * P, \tag{16}$$

and repeated application yields

$$S^{N}\rho(B) = (S^{N-1}\rho(B)) * P = \rho(B) * \underbrace{P * \ldots * P}_{N \text{ factors}} = \rho(B) * P_N, \tag{17}$$

where the last equation defines the N-fold convolution $P_N(t)$. This may be interpreted as the probability density function for a random variable

$$\mathcal{T}_N = \tau_1 + \ldots + \tau_N \tag{18}$$

representing the sum of N independent and identically distributed random recurrence times τ_j with common distribution $P(t)$.

4. CLASSIFICATION OF INDUCED DYNAMICS

The last section has shown that S acts on measures as a convolution rather than a simple shift. This section studies the continuous time limit for S defined in (15). This amounts to studying the limit $N \to \infty$, $\Delta t \to 0$ for the distributions $P_N(t)$. The question is whether it is possible to choose $\Delta t \to 0$ in such a way that $P_N(t)$ converges. This question is answered positively in probability theory [24], and it is well known [7] that the possible limit distributions

$$P_{\infty}(t) = \lim_{\substack{N\to\infty \\ \Delta t \to 0}} P_N(t) \tag{19}$$

must be stable distribution functions if the limit exists. Note that a trivial centering of the variable $\mathcal{T}_N$ to zero is implicit in this formulation (see [7, 9]). The positivity of the recurrence times $\tau_i \geq 0$ for all $i \in \mathbb{N}$ puts the constraint

$$P_{\infty}(t) = 0 \qquad \text{for} \qquad t \leq 0 \tag{20}$$

on the possible limiting distributions. The remaining stable limit distributions are then characterized by two numbers $0 < \varpi \leq 1$ and $D \geq 0$. The number ϖ is called the index of stability while D is the width of the distribution. For $0 < \varpi < 1$ the limiting densities can be written as

$$p_{\infty}(t) = \frac{\mathrm{d}P_{\infty}(t)}{\mathrm{d}t} = \frac{1}{D^{1/\varpi}} h_{\varpi}\left(\frac{t}{D^{1/\varpi}}\right), \tag{21}$$

where

$$h_{\varpi}(x) = \frac{1}{\varpi x} H^{10}_{11}\left(\frac{1}{x}\,\middle|\,\begin{matrix}(0, 1)\\(0, 1/\varpi)\end{matrix}\right) \tag{22}$$

is defined in terms of H-functions [25] whose definition is given in the appendix. For $\varpi = 1$ the limiting density

$$h_1(x) = \lim_{\varpi \to 1^-} h_{\varpi}(x) = \delta(x-1) \tag{23}$$

is the Dirac distribution concentrated at $x = 1$. If the limit exists, and $D \neq 0$, the discretization Δt must have the form

$$\Delta t \sim (N\Lambda(N))^{-1/\varpi}, \tag{24}$$

where $\Lambda(N)$ is a slowly varying function [24], i.e.

$$\lim_{x\to\infty} \frac{\Lambda(bx)}{\Lambda(x)} = 1 \tag{25}$$

for all $b > 0$.

Two very different cases arise in the continuous time limit $N \to \infty$, $\Delta t \to 0$ for induced automorphisms S on subsets $G \subset \Gamma$ of measure $\mu(G) = 0$: firstly, for $0 < \varpi < 1$ and ρ a measure on G the continuous time transformation reads

$$\begin{aligned} S^{t^*}_{\varpi}\rho(B, t) &= \int_0^\infty \rho(B, t - s) h_\varpi\left(\frac{s}{t^*}\right)\frac{\mathrm{d}s}{t^*} \\ &= \frac{1}{t^*}\int_0^\infty T^s\rho(B, t) h_\varpi(s/t^*)\,\mathrm{d}s, \end{aligned} \tag{26}$$

where the macroscopic time parameter t^* was introduced as

$$t^* = D^{1/\varpi} \geq 0. \tag{27}$$

Secondly, for $\varpi = 1$ it follows from (23) that

$$S^{t^*}_1\rho(B, t) = \int_{-\infty}^{\infty} \rho(B, t - s)\delta\left(\frac{s}{t^*} - 1\right)\frac{\mathrm{d}s}{t^*} = \rho(B, t - t^*) = T^{t^*}\rho(B, t), \tag{28}$$

which resembles (12). Note, however, that the result (28) differs from (12) in two important aspects. First, $\mu(G) = 0$ for (28) and thus the tranformation S^{t^*} corresponds to a time shift by an infinite amount in the sense of (12). Secondly, while (12) remains well defined for negative t the transformation S^{t^*} does not exist for $t^* < 0$ by virtue of (27). Therefore, $S^{t/\mu(G)}$ in (12) is time reversible while the renormalized induced automorphism S^{t^*} is time irreversible. Of course this irreversibility holds true also in the case $0 < \varpi < 1$.

5. INDUCED GENERATORS AND FRACTIONAL DYNAMICS

This section investigates the condition of invariance or stationarity for the induced renormalized semigroups $S^{t^*}_\varpi$, whose renormalized macroscopic time parameter $t^* \geq 0$ is given by (27). Invariance of the measure ν on G under $S^{t^*}_\varpi$ requires

$$S^{t^*}_\varpi \nu(B, t) = \nu(B, t) \tag{29}$$

for $t > 0$ and $B \subset G$. For $0 < \varpi < 1$ (29) may be called the condition of fractional invariance or fractional stationarity. Using (3) the invariance condition becomes

$$A_\varpi \nu(B, t) = 0 \tag{30}$$

for $t > 0$ where A_ϖ is the infinitesimal generator of the semigroup $S^{t^*}_\varpi$. For $\varpi = 1$ the relation (28) implies $A_1\nu(B, t) = -\mathrm{d}\nu(B, t)/\mathrm{d}t = 0$, and thus in this case invariant measures conserve volumes in phase space as usual. A very different situation arises for $\varpi < 1$.

For $0 < \varpi < 1$ it is well known [24] that the infinitesimal generators of $S^{t^*}_\varpi$ may be interpreted as the distribution $s_+^{-\varpi-1}$ [26] evaluated on the time translation group T^s

$$A_\varpi\rho(t) = c^+\int_0^\infty s^{-\varpi-1}(T^s - T^0)\,\mathrm{d}s\rho(t) = c^+\int_0^\infty s_+^{-\varpi-1} T^s\,\mathrm{d}s\rho(t), \tag{31}$$

where $c^+ > 0$ is a constant. Comparing (31) with the Balakrishnan algorithm [27–29] for fractional powers of the generator of a semigroup T^t

$$
\begin{aligned}
(-A)^\alpha \rho(t) &= \lim_{t\to 0^+}\left(\frac{I - T^t}{t}\right)^\alpha \rho \\
&= \frac{1}{\Gamma(-\alpha)}\int_0^\infty s^{-\alpha-1}(I - T^s)\rho(t)\,\mathrm{d}s
\end{aligned}
\tag{32}
$$

shows that if $A = -\mathrm{d}/\mathrm{d}t$ denotes the infinitesimal generator of the original time evolution T^t then $A_\varpi = (-A)^\varpi$ is the infinitesimal generator of the induced time evolution $S^{t^*}_\varpi$. For $0 < \varpi < 1$ the generators A_ϖ for $S^{t^*}_\varpi$ are fractional time derivatives [15, 28, 26]. The differential form (30) of the fractional invariance condition for v becomes

$$\frac{\mathrm{d}^\varpi}{\mathrm{d}t^\varpi} v(B, t) = 0 \tag{33}$$

for $t > 0$ which was first derived in [7, 9]. Its solution is

$$v(B, t) = C_0 t^{\varpi-1} \tag{34}$$

for $t > 0$ with C_0 a constant. This shows that $v(B)$ for a fractional stationary dynamical state is not constant. Fractional stationarity or fractional invariance of a measure v implies that phase space volumes $v(B)$ shrink with time. Thus fractional dynamics is dissipative. More generally (33) reads $A_\varpi v(B, t) = \delta(t)$ with solution $v(B, t) = C_0 t_+^{\varpi-1}$ for $t \geqslant 0$ in the sense of distributions. The stationary solution with $\varpi = 1$ has a jump discontinuity at $t = 0$, and is not simply constant.

The transition from an original invariant measure μ on Γ to a fractional invariant measure v on a subset G of measure $\mu(G) = 0$ is called invariance breaking or ergodicity breaking because invariance is a prerequisite for ergodicity. Note, however, that the resulting fractional dynamical system $(G, \mathfrak{G}, v, S^{t^*}_\varpi)$ may again exhibit fractional ergodicity. Fractional ergodicity is defined analogously to ergodicity by replacing invariance with fractional invariance.

The solution (34) develops a singularity when extrapolated backwards in time $t \to 0$ because it holds only after renormalization in the limit $\mu(G) \to 0$, $N \to \infty$, $\Delta t \to 0$. This singular behaviour is not unphysical but expresses ergodicity breaking as a spontaneous dynamically generated reduction of the accessible regions in phase space.

The foregoing results provide a general and model-independent justification for studying fractional dynamical systems such as the following formal modification of (5)

$$\frac{\mathrm{d}^\varpi \mathbf{x}(t)}{\mathrm{d}t^\varpi} = \mathbf{F}_\omega(\mathbf{x}(t)), \tag{35}$$

which can be interpreted by applying the inverse operator $\mathrm{d}^{-\varpi}/\mathrm{d}t^{-\varpi}$ to obtain an integral equation

$$\mathbf{x}(t) = \mathbf{x}_0 + \frac{1}{\Gamma(\varpi)}\int_0^t (t - s)^{\varpi-1}\mathbf{F}_\varpi(\mathbf{x}(s))\,\mathrm{d}s \tag{36}$$

taking the initial condition $\mathbf{x}(0) = \mathbf{x}_0$ into account. A particular example is the fractional Liouville equation for a density matrix $\rho(t)$

$$\frac{\mathrm{d}^\varpi \rho(t)}{\mathrm{d}t^\varpi} = \mathscr{L}^\varpi \rho(t) \tag{37}$$

discussed first in [7]. Here $\mathscr{L}^\varpi$ denotes a fractional Liouville operator acting linearly on ρ.

The solution of (37) is obtained as

$$\rho(t) = \frac{1}{\varpi} H^{11}_{12}\left(-\mathscr{L}t \,\middle|\, \begin{matrix}(0,\ 1/\varpi)\\(0,\ 1/\varpi)(0,\ 1)\end{matrix}\right)\rho_0, \tag{38}$$

where ρ_0 is the initial density matrix $\rho(0) = \rho_0$.

6. CONCLUSIONS

The present paper extends the definition of induced automorphisms within ergodic theory from subsets of positive measure to subsets of measure zero. The extension requires a renormalization of time scale because the recurrence times are state dependent and their averages diverge. It is found that the renormalized induced time evolution forms a semigroup even if the original time evolution was a group. This provides a new general and model-independent mechanism for the origin of macroscopic time irreversibility. The typical recurrence times remain finite, and thus the origin of irreversibility is related to the difference between the typical behaviour of a recurrent trajectory and its average behaviour [18].

An equally important result of this work is that not only time translation invariant states are time evolution invariant. Instead time evolution invariant (or stationary) states are more generally those which are stable time convolution invariant. This has implications for the observation of time flow itself which requires a stationary (time evolution invariant) measurement apparatus. The measurement of discrete time is based on counting periods or recurrences of a periodic (deterministic) or recurrent (stochastic) process (the clock). A continuous time is obtained by adding the duration of single periods or recurrences. This method of measuring time assumes that the laws governing the clock itself are invariant under the evolution of time, i.e. that the periodic or recurrent clock process is stationary. Traditionally this assumption is expressed by postulating that the period or average recurrence time $\langle\tau\rangle$ of the clock defined in (9) is finite and constant. The present paper points out that the clock is in general a subsystem $G \subset \Gamma$ of a larger dynamical system with more degrees of freedom which implies $\mu(G) = 0$ and therefore $\langle\tau\rangle = \infty$. After renormalization the possible stationarity conditions are classified by the stability index ϖ. The renormalization may be iterated allowing different clock processes in a hierarchy of time scales. As a consequence the measured time may appear uniform or accelerated depending upon whether the renormalized time evolution of a 'clock' process has the same stability index as the time evolution used as a reference in the definition of stationarity. Uniformity of time flow is therefore a relative concept.

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APPENDIX: DEFINITION OF *H*-FUNCTIONS

The general H-function is defined as the inverse Mellin transform [25]

$$H^{mn}_{PQ}\left(z\,\middle|\,\begin{matrix}(a_1, A_1)\ldots(a_P, A_P)\\(b_1, B_1)\ldots(b_Q, B_Q)\end{matrix}\right) = \frac{1}{2\pi i}\int_{\mathscr{C}} \frac{\prod_{j=1}^{m}\Gamma(b_j - B_j s)\prod_{j=1}^{n}\Gamma(1 - a_j + A_j s)}{\prod_{j=m+1}^{Q}\Gamma(1 - b_j + B_j s)\prod_{j=n+1}^{P}\Gamma(a_j - A_j s)} z^s\,ds, \tag{A1}$$

where the contour $\mathscr{C}$ runs from $c - i\infty$ to $c + i\infty$ separating the poles of $\Gamma(b_j - B_j s)$, $(j = 1, \ldots, m)$ from those of $\Gamma(1 - a_j + A_j s)$, $(j = 1, \ldots, n)$. Empty products are interpreted as unity. The integers m, n, P, Q satisfy $0 \leq m \leq Q$ and $0 \leq n \leq P$. The coefficients A_j and B_j are positive real numbers and the complex parameters a_j, b_j are such that no poles in the integrand coincide. If

$$\Omega = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{P} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{Q} B_j > 0 \tag{A2}$$

then the integral converges absolutely and defines the H-function in the sector $|\arg z| < \Omega\pi/2$. The H-function is also well defined when either

$$\delta = \sum_{j=1}^{q} B_j - \sum_{j=1}^{P} A_j > 0 \quad \text{with} \quad 0 < |z| < \infty \tag{A3}$$

or

$$\delta = 0 \qquad \text{and} \qquad 0 < |z| < R = \prod_{j=1}^{P} A_j^{-A_j} \prod_{j=1}^{Q} B_j^{B_j}. \tag{A4}$$

The H-function is a generalization of Meijers G-function and many of the known special functions are special cases of it.

Recently the results obtained here were shown to hold also in the ultra long time limit [30].