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Fractional and operational calculus with generalized fractional derivative operators and Mittag–Leffler type functions

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Dedicated to Professor Rudolf Gorenflo on the Occasion of his Eightieth Birth Anniversary

In this paper, we study a certain family of generalized Riemann–Liouville fractional derivative operators $D_{a\pm}^{\alpha,\beta}$ of order α and type β, which were introduced and investigated in several earlier works [R. Hilfer (ed.), *Applications of Fractional Calculus in Physics*, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 2000; R. Hilfer, *Fractional time evolution*, in *Applications of Fractional Calculus in Physics*, R. Hilfer, ed., World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 2000, pp. 87–130; R. Hilfer, *Experimental evidence for fractional time evolution in glass forming materials*, J. Chem. Phys. 284 (2002), pp. 399–408; R. Hilfer, *Threefold introduction to fractional derivatives*, in *Anomalous Transport*: *Foundations and Applications*, R. Klages, G. Radons, and I.M. Sokolov, eds., Wiley-VCH Verlag, Weinheim, 2008, pp. 17–73; R. Hilfer and L. Anton, *Fractional master equations and fractal time random walks*, Phys. Rev. E 51 (1995), pp. R848–R851; R. Hilfer, Y. Luchko, and Ž. Tomovski, *Operational method for solution of the fractional differential equations with the generalized Riemann-Liouville fractional derivatives*, Fract. Calc. Appl. Anal. 12 (2009), pp. 299–318; F. Mainardi and R. Gorenflo, *Time-fractional derivatives in relaxation processes: A tutorial survey*, Fract. Calc. Appl. Anal. 10 (2007), pp. 269–308; T. Sandev and Ž. Tomovski, *General time fractional wave equation for a vibrating string*, J. Phys. A Math. Theor. 43 (2010), 055204; H.M. Srivastava and Ž. Tomovski, *Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel*, Appl. Math. Comput. 211 (2009), pp. 198–210]. In particular, we derive various compositional properties, which are associated with Mittag–Leffler functions and Hardy-type inequalities for the generalized fractional derivative operator $D_{a\pm}^{\alpha,\beta}$. Furthermore, by using the Laplace transformation methods, we provide solutions of many different classes of fractional differential equations with constant and variable coefficients and some general Volterra-type differintegral equations in the space of Lebesgue integrable functions. Particular cases of these general solutions and a brief discussion about some recently investigated fractional kinetic equations are also given.

Keywords: Riemann–Liouville fractional derivative operator; generalized Mittag–Leffler function; Hardy-type inequalities; Laplace transform method; Volterra differintegral equations; fractional differential equations; fractional kinetic equations; Lebesgue integrable functions; Fox–Wright hypergeometric functions

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1. Introduction, Definitions and Preliminaries

Applications of fractional calculus require fractional derivatives of different kinds [5–9,11,16,20,24]. Differentiation and integration of fractional order are traditionally defined by the right-sided Riemann–Liouville fractional integral operator I_{a+}^{p} and the left-sided Riemann–Liouville fractional integral operator I_{a-}^{p}, and the corresponding Riemann–Liouville fractional derivative operators D_{a+}^{p} and D_{a-}^{p}, as follows [3, Chapter 13,14, pp. 69–70, 19]:

$$\left(I_{a+}^{\mu} f\right)(x) = \frac{1}{\Gamma(\mu)} \int_{a}^{x} \frac{f(t)}{(x-t)^{1-\mu}} \mathrm{d}t \quad \left(x > a;\ \Re(\mu) > 0\right), \tag{1.1}$$

$$\left(I_{a-}^{\mu} f\right)(x) = \frac{1}{\Gamma(\mu)} \int_{x}^{a} \frac{f(t)}{(t-x)^{1-\mu}} \mathrm{d}t \quad \left(x < a;\ \Re(\mu) > 0\right) \tag{1.2}$$

and

$$\left(D_{a\pm}^{\mu} f\right)(x) = \left(\pm \frac{\mathrm{d}}{\mathrm{d}x}\right)^{n} \left(I_{a\pm}^{n-\mu} f\right)(x) \quad \left(\Re(\mu) \geqq 0;\ n = [\Re(\mu)] + 1\right), \tag{1.3}$$

where the function f is locally integrable, $\Re(\mu)$ denotes the real part of the complex number $\mu \in \mathbb{C}$ and $[\Re(\mu)]$ means the greatest integer in $\Re(\mu)$.

Recently, a remarkably large family of generalized Riemann–Liouville fractional derivatives of order α $(0 < \alpha < 1)$ and type β $(0 \leqq \beta \leqq 1)$ was introduced as follows [5–7,9,11,20].

Definition 1 *The right-sided fractional derivative $D_{a+}^{\alpha,\beta}$ and the left-sided fractional derivative $D_{a-}^{\alpha,\beta}$ of order α $(0 < \alpha < 1)$ and type β $(0 \leqq \beta \leqq 1)$ with respect to x are defined by*

$$\left(D_{a\pm}^{\alpha,\beta} f\right)(x) = \left(\pm I_{a\pm}^{\beta(1-\alpha)} \frac{\mathrm{d}}{\mathrm{d}x} \left(I_{a\pm}^{(1-\beta)(1-\alpha)} f\right)\right)(x), \tag{1.4}$$

whenever the second member of (1.4) *exists. This generalization* (1.4) *yields the classical Riemann–Liouville fractional derivative operator when $\beta = 0$. Moreover, for $\beta = 1$, it gives the fractional derivative operator introduced by Liouville* [15, p. 10]*, which is often attributed to Caputo now-a-days and which should more appropriately be referred to as the Liouville–Caputo fractional derivative. Several authors* [16, 24] *called the general operators in* (1.4) *the Hilfer fractional derivative operators. Applications of $D_{a\pm}^{\alpha,\beta}$ are given in* [7].

The purpose of this paper is to investigate the generalized fractional derivative operators $D_{a\pm}^{\alpha,\beta}$ of order α and type β. In particular, several fractional differential equations arising in applications are solved.

Using the formulas (1.1) and (1.2) in conjunction with (1.3) when $n = 1$, the fractional derivative operator $D_{a\pm}^{\alpha,\beta}$ can be rewritten in the form:

$$\left(D_{a\pm}^{\alpha,\beta} f\right)(x) = \left(\pm I_{a\pm}^{\beta(1-\alpha)} \left(D_{a\pm}^{\alpha+\beta-\alpha\beta} f\right)\right)(x). \tag{1.5}$$

The difference between fractional derivatives of different types becomes apparent from their Laplace transformations. For example, it is found for $0 < \alpha < 1$ that [5, 6, 24]

$$\mathcal{L}\left[\left(D_{0+}^{\alpha,\beta} f\right)(x)\right](s) = s^{\alpha} \mathcal{L}[f(x)](s) - s^{\beta(\alpha-1)} \left(I_{0+}^{(1-\beta)(1-\alpha)} f\right)(0+) \tag{1.6}$$

$$(0 < \alpha < 1),$$

where

$$\left(I_{0+}^{(1-\beta)(1-\alpha)} f\right)(0+)$$

is the Riemann–Liouville fractional integral of order $(1-\beta)(1-\alpha)$ evaluated in the limit as $t \to 0+$, it being understood (as usual) that [2, Chapters 4 and 5]

$$\mathcal{L}[f(x)](s) := \int_0^\infty \mathrm{e}^{-sx} f(x)\,\mathrm{d}x =: F(s), \tag{1.7}$$

provided that the defining integral in (1.7) exists.

The familiar Mittag–Leffler functions $E_\mu(z)$ and $E_{\mu,\nu}(z)$ are defined by the following series:

$$E_\mu(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\mu n+1)} =: E_{\mu,1}(z) \qquad \big(z \in \mathbb{C};\ \Re(\mu) > 0\big) \tag{1.8}$$

and

$$E_{\mu,\nu}(z) := \sum_{n=0}^\infty \frac{z^n}{\Gamma(\mu n+\nu)} \qquad \big(z, \nu \in \mathbb{C};\ \Re(\mu) > 0\big), \tag{1.9}$$

respectively. These functions are natural extensions of the exponential, hyperbolic and trigonometric functions, since

$$E_1(z) = \mathrm{e}^z, \quad E_2\left(z^2\right) = \cosh z, \quad E_2\left(-z^2\right) = \cos z,$$
$$E_{1,2}(z) = \frac{\mathrm{e}^z - 1}{z} \quad \text{and} \quad E_{2,2}(z^2) = \frac{\sinh z}{z}.$$

For a detailed account of the various properties, generalizations and applications of the Mittag–Leffler functions, the reader may refer to the recent works by, for example, Gorenflo *et al*. [4] and Kilbas *et al*. [12–14, Chapter 1]. The Mittag–Leffler function (1.1) and some of its various generalizations have only recently been calculated numerically in the whole complex plane [10,22]. By means of the series representation, a generalization of the Mittag–Leffler function $E_{\mu,\nu}(z)$ of (1.2) was introduced by Prabhakar [18] as follows:

$$E_{\mu,\nu}^{\lambda}(z) = \sum_{n=0}^\infty \frac{(\lambda)_n}{\Gamma(\mu n+\nu)} \frac{z^n}{n!} \qquad \big(z, \nu, \lambda \in \mathbb{C};\ \Re(\mu) > 0\big), \tag{1.10}$$

where (and throughout this investigation) $(\lambda)_\nu$ denotes the familiar Pochhammer symbol or the *shifted* factorial, since

$$(1)_n = n! \qquad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \mathbb{N} := \{1, 2, 3, \ldots\}),$$

defined (for $\lambda, \nu \in \mathbb{C}$ and in terms of the familiar Gamma function) by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}). \end{cases}$$

Clearly, we have the following special cases:

$$E_{\mu,\nu}^1(z) = E_{\mu,\nu}(z) \qquad \text{and} \qquad E_{\mu,1}^1(z) = E_\mu(z).$$

Indeed, as already observed earlier by Srivastava and Saxena [23, p. 201, Equation (1.6)], the generalized Mittag–Leffler function $E_{\mu,\nu}^{\lambda}(z)$ itself is actually a very specialized case of a rather extensively investigated function ${}_p\Psi_q$ as indicated below [14, p. 45, Equation (1.9.1)]:

$$E_{\mu,\nu}^{\lambda}(z) = \frac{1}{\Gamma(\lambda)}\, {}_1\Psi_1 \left[\begin{array}{c} (\lambda, 1); \\ \\ (\nu, \mu); \end{array} z \right].$$

Here, *and in what follows*, ${}_p\Psi_q$ denotes the Wright (or, more appropriately, the Fox–Wright) generalization of the hypergeometric ${}_pF_q$ function, which is defined as follows [14, p. 56 et seq.]:

$${}_p\Psi_q \left[\begin{array}{c} (a_1, A_1), \ldots, (a_p, A_p); \\ \\ (b_1, B_1), \ldots, (b_q, B_q); \end{array} z \right] := \sum_{k=0}^{\infty} \frac{\Gamma(a_1 + A_1 k) \cdots \Gamma(a_p + A_p k)}{\Gamma(b_1 + B_1 k) \cdots \Gamma(b_q + B_q k)} \frac{z^k}{k!}$$

$$\left(\Re(A_j) > 0 \;\; (j = 1, \ldots, p); \Re(B_j) > 0 \;\; (j = 1, \ldots, q); 1 + \Re\left(\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \right) \geqq 0 \right),$$

in which we have assumed, in general, that

$$a_j, A_j \in \mathbb{C} \quad (j = 1, \ldots, p) \qquad \text{and} \qquad b_j, B_j \in \mathbb{C} \quad (j = 1, \ldots, q)$$

and that the equality holds true only for suitably bounded values of $|z|$.

Special functions of the Mittag–Leffler and the Fox–Wright types are known to play an important role in the theory of fractional and operational calculus and their applications in the basic processes of evolution, relaxation, diffusion, oscillation and wave propagation. Just as we have remarked above, the Mittag–Leffler type functions have only recently been calculated numerically in the whole complex plane [10,22].

The following Laplace transform formula for the generalized Mittag–Leffler function $E_{\mu,\nu}^{\lambda}(z)$ was given by Prabhakar [18]:

$$\mathcal{L}\left[x^{\nu-1} E_{\mu,\nu}^{\lambda}(\omega x^{\mu})\right](s) = \frac{s^{\lambda\mu-\nu}}{(s^{\mu} - \omega)^{\lambda}} \tag{1.11}$$

$$\left(\lambda, \mu, \omega \in \mathbb{C}; \Re(\nu) > 0; \Re(s) > 0; \left|\frac{\omega}{s^{\mu}}\right| < 1\right).$$

Prabhaker [18] also introduced the following fractional integral operator:

$$\left(\mathbf{E}_{\mu,\nu,\omega;a+}^{\lambda}\varphi\right)(x) = \int_a^x (x-t)^{\nu-1} E_{\mu,\nu}^{\lambda}\left(\omega(x-t)^{\mu}\right)\varphi(t)\,\mathrm{d}t \qquad (x > a) \tag{1.12}$$

in the space $L(\mathfrak{a}, \mathfrak{b})$ of Lebesgue integrable functions on a finite closed interval $[\mathfrak{a}, \mathfrak{b}]$ $(\mathfrak{b} > \mathfrak{a})$ of the real line $\mathbb{R}$ given by

$$L(\mathfrak{a}, \mathfrak{b}) = \left\{ f : \|f\|_1 = \int_{\mathfrak{a}}^{\mathfrak{b}} |f(x)|\,\mathrm{d}x < \infty \right\}, \tag{1.13}$$

it being *tacitly* assumed (throughout the present investigation) that, in situations such as those occurring in (1.12) and in conjunction with the usages of the definitions in (1.3), (1.4) and (1.5), $\mathfrak{a}$ in all such function spaces as, for example, the function space $L(\mathfrak{a}, \mathfrak{b})$ coincides precisely with the *lower* terminal a in the integrals involved in the definitions (1.3), (1.4) and (1.5).

The fractional integral operator (1.12) was investigated earlier by Kilbas *et al*. [13] and its generalization involving a family of more general Mittag–Leffler type functions than $E_{\mu,\nu}^{\lambda}(z)$ was studied recently by Srivastava and Tomovski [24].

2. Properties of the generalized fractional derivative operator and relationships with the Mittag–Leffler functions

In this section, we derive several continuity properties of the generalized fractional derivative operator $D_{a+}^{\alpha,\beta}$. Each of the following results (Lemma 1 as well as Theorems 1 and 2) are easily derivable by suitably specializing the corresponding general results proven recently by Srivastava and Tomovski [24].

LEMMA 1 [24] *The following fractional derivative formula holds true*:

$$\left(D_{a+}^{\alpha,\beta}\left[(t-a)^{\nu-1}\right]\right)(x)=\frac{\Gamma(\nu)}{\Gamma(\nu-\alpha)}(x-a)^{\nu-\alpha-1} \tag{2.1}$$
$$\left(x>a;\, 0<\alpha<1;\, 0\leqq\beta\leqq 1;\, \Re(\nu)>0\right).$$

THEOREM 1 [24] *The following relationship holds true*:

$$\left(D_{a+}^{\alpha,\beta}\left[(t-a)^{\nu-1}E_{\mu,\nu}^{\lambda}\left[\omega(t-a)^{\mu}\right]\right]\right)(x)=(x-a)^{\nu-\alpha-1}E_{\mu,\nu-\alpha}^{\lambda}\left[\omega(x-a)^{\mu}\right] \tag{2.2}$$
$$\left(x>a;\, 0<\alpha<1;\, 0\leqq\beta\leqq 1;\, \lambda,\omega\in\mathbb{C};\, \Re(\mu)>0;\, \Re(\nu)>0\right).$$

THEOREM 2 [24] *The following relationship holds true for any Lebesgue integrable function* $\varphi\in L(\mathfrak{a},\mathfrak{b})$:

$$D_{a+}^{\alpha,\beta}\left(\mathbf{E}_{\mu,\nu,\omega;a+}^{\lambda}\varphi\right)=\mathbf{E}_{\mu,\nu-\alpha,\omega;a+}^{\lambda}\varphi \tag{2.3}$$
$$\left(x>a\ (a=\mathfrak{a});\, 0<\alpha<1;\, 0\leqq\beta\leqq 1;\, \lambda,\omega\in\mathbb{C};\, \Re(\mu)>0;\, \Re(\nu)>0\right).$$

In addition to the space $L(\mathfrak{a},\mathfrak{b})$ given by (1.13), we shall need the weighted L^p-space

$$X_{\mathfrak{c}}^{p}(\mathfrak{a},\mathfrak{b}) \qquad \left(\mathfrak{c}\in\mathbb{R};\ 1\leqq p\leqq\infty\right),$$

which consists of those complex-valued Lebesgue integrable functions f on $(\mathfrak{a},\mathfrak{b})$ for which

$$\|f\|_{X_{\mathfrak{c}}^{p}}<\infty,$$

with

$$\|f\|_{X_{\mathfrak{c}}^{p}}=\left(\int_{\mathfrak{a}}^{\mathfrak{b}}\left|t^{\mathfrak{c}}f(t)\right|^{p}\frac{\mathrm{d}t}{t}\right)^{1/p} \qquad \left(1\leqq p<\infty\right).$$

In particular, when $\mathfrak{c}=\dfrac{1}{p}$, the space $X_{\mathfrak{c}}^{p}(\mathfrak{a},\mathfrak{b})$ coincides with the $L^p(\mathfrak{a},\mathfrak{b})$-space, that is,

$$X_{1/p}^{p}(\mathfrak{a},\mathfrak{b})=L^{p}(\mathfrak{a},\mathfrak{b}).$$

We also introduce here a suitable *fractional* Sobolev space $W_{a+}^{\alpha,p}(\mathfrak{a},\mathfrak{b})$ defined, for a closed interval $[\mathfrak{a},\mathfrak{b}]$ $(\mathfrak{b}>\mathfrak{a})$ in $\mathbb{R}$, by

$$W_{a+}^{\alpha,p}(\mathfrak{a},\mathfrak{b})=\left\{f:f\in L^{p}(\mathfrak{a},\mathfrak{b}) \quad \text{and} \quad D_{a+}^{\alpha}f\in L^{p}(\mathfrak{a},\mathfrak{b}) \quad (0<\alpha\leqq 1)\right\},$$

where $D_{a+}^{\alpha}f$ denotes the *fractional* derivative of f of order α $(0<\alpha\leqq 1)$. Alternatively, in Theorems 3 and 4, we can make use of a suitable p-variant of the space $\mathbf{L}_{a+}^{\alpha}(\mathfrak{a},\mathfrak{b})$ which was defined, for $\Re(\alpha)>0$, by Kilbas *et al*. [14, p. 144, Equation (3.2.1)] as follows

$$\mathbf{L}_{a+}^{\alpha}(\mathfrak{a},\mathfrak{b})=\left\{f:f\in L(\mathfrak{a},\mathfrak{b}) \quad \text{and} \quad D_{a+}^{\alpha}f\in L(\mathfrak{a},\mathfrak{b}) \quad \left(\Re(\alpha)>0\right)\right\}.$$

See also the notational convention mentioned in connection with (1.13).

THEOREM 3 *For* $0 < \alpha < 1$ *and* $0 < \beta < 1$, *the operator* $D_{a+}^{\alpha,\beta}$ *is bounded in the space* $W_{a+}^{\alpha+\beta-\alpha\beta,1}(\mathfrak{a},\mathfrak{b})$ *and*

$$\left\| D_{a+}^{\alpha,\beta} \right\|_1 \leqq A \left\| D_{a+}^{\alpha+\beta-\alpha\beta} \right\|_1 \qquad \left(A := \frac{(b-a)^{\beta(1-\alpha)}}{\beta(1-\alpha)\,\Gamma[\beta(1-\alpha)]} \right). \tag{2.4}$$

Proof Using a known result [19, Equation (2.72)], we get

$$\begin{aligned} \left\| D_{a+}^{\alpha,\beta}\varphi \right\|_1 &= \left\| I_{a+}^{\beta(1-\alpha)} \left(D_{a+}^{\alpha+\beta-\alpha\beta}\varphi \right) \right\|_1 \\ &\leqq \frac{(b-a)^{\beta(1-\alpha)}}{\beta(1-\alpha)\,\Gamma[\beta(1-\alpha)]} \left\| D_{a+}^{\alpha+\beta-\alpha\beta}\varphi \right\|_1 . \end{aligned}$$

■

The weighted Hardy-type inequality for the integral operator I_{a+}^{α} is stated as the following lemma.

LEMMA 2 [14, p. 82] *If* $1 < p < \infty$ *and* $\alpha > 0$, *then the integral operator* I_{0+}^{α} *is bounded from* $L^p(0,\infty)$ *into* $X_{1/p-\alpha}^{p}(0,\infty)$ *as follows*:

$$\begin{aligned} &\left(\int_0^\infty x^{-\alpha p} \left| \left(I_{0+}^{\alpha} f \right)(x) \right|^p \mathrm{d}x \right)^{1/p} \\ &\qquad \leqq \frac{\Gamma\left(1/p'\right)}{\Gamma(\alpha+1/p)} \left(\int_0^\infty |f(x)|^p \,\mathrm{d}x \right)^{1/p} \qquad \left(\frac{1}{p} + \frac{1}{p'} = 1 \right). \end{aligned} \tag{2.5}$$

Applying the last two inequalities to the fractional derivative operator $D_{a+}^{\alpha,\beta}$, we get

$$\begin{aligned} &\left(\int_0^\infty x^{-\alpha p} \left| \left(D_{0+}^{\alpha,\beta} f \right)(x) \right|^p \mathrm{d}x \right)^{1/p} \\ &\leqq \frac{\Gamma\left(1/p'\right)}{\Gamma\big(\beta(1-\alpha)+1/p\big)} \left(\int_0^\infty \left| \left(D_{0+}^{\alpha+\beta-\alpha\beta} f \right)(x) \right|^p \mathrm{d}x \right)^{1/p} \qquad \left(\frac{1}{p} + \frac{1}{p'} = 1 \right). \end{aligned} \tag{2.6}$$

Hence, we arrive at the following result.

THEOREM 4 *If* $1 < p < \infty$, $0 < \alpha < 1$ *and* $0 \leqq \beta \leqq 1$, *then the fractional derivative operator* $D_{0+}^{\alpha,\beta}$ *is bounded from* $W_{0+}^{\alpha+\beta-\alpha\beta,p}(0,\infty)$ *into* $X_{1/p-\alpha}^{p}(0,\infty)$.

3. Fractional differential equations with constant and variable coefficients

The eigenfunctions of the Riemann–Liouville fractional derivatives are defined as the solutions of the following fractional differential equation:

$$\left(D_{0+}^{\alpha} f \right)(x) = \lambda f(x), \tag{3.1}$$

where λ is the eigenvalue. The solution of (3.1) is given by

$$f(x) = x^{1-\alpha} E_{\alpha,\alpha}(\lambda x^{\alpha}). \tag{3.2}$$

More generally, the eigenvalue equation for the fractional derivative $D_{0+}^{\alpha,\beta}$ of order α and type β reads as follows:

$$\left(D_{0+}^{\alpha,\beta} f\right)(x) = \lambda f(x) \tag{3.3}$$

and its solution is given by [6, Equation (124)]

$$f(x) = x^{(1-\beta)(1-\alpha)} E_{\alpha,\alpha+\beta(1-\alpha)}\left(\lambda x^{\alpha}\right), \tag{3.4}$$

which, in the special case when $\beta = 0$, corresponds to (3.2). A second important special case of (3.3) occurs when $\beta = 1$:

$$\left(D_{0+}^{\alpha,1} f\right)(x) = \lambda f(x). \tag{3.5}$$

In this case, the eigenfunction is given by

$$f(x) = E_{\alpha}\left(\lambda x^{\alpha}\right). \tag{3.6}$$

We now divide this section into the following five subsections.

3.1. *A General Family of Fractional Differential Equations*

In this section, we assume that

$$0 < \alpha_1 \leqq \alpha_2 < 1,\ 0 \leqq \beta_1 \leqq 1,\ 0 \leqq \beta_2 \leqq 1 \quad \text{and} \quad a, b, c \in \mathbb{R},$$

and consider the following fractional differential equation:

$$a\left(D_{0+}^{\alpha_1,\beta_1} y\right)(x) + b\left(D_{0+}^{\alpha_2,\beta_2} y\right)(x) + cy(x) = f(x) \tag{3.7}$$

in the space of Lebesgue integrable functions $y \in L(0,\infty)$ with the initial conditions:

$$\left(I_{0+}^{(1-\beta_i)(1-\alpha_i)} y\right)(0+) = c_i \qquad (i = 1, 2), \tag{3.8}$$

where, without loss of generality, we assume that

$$(1-\beta_1)(1-\alpha_1) \leqq (1-\beta_2)(1-\alpha_2).$$

If $c_1 < \infty$, then

$$c_2 = 0 \quad \text{unless} \quad (1-\beta_1)(1-\alpha_1) = (1-\beta_2)(1-\alpha_2).$$

An equation of the form (3.7) was introduced in [7] for dielectric relaxation in glasses. Although the Laplace transformed relaxation function and the corresponding dielectric susceptibility were found, its general solution was not given in [7]. Here we proceed to find its general solution.

THEOREM 5 *The fractional differential equation* (3.7) *with the initial conditions* (3.8) *has its solution in the space* $L(0,\infty)$ *given by*

$$\begin{aligned} y(x) = \frac{1}{b}\sum_{m=0}^{\infty}\left(-\frac{a}{b}\right)^m \Bigg[& ac_1 x^{(\alpha_2-\alpha_1)m+\alpha_2+\beta_1(1-\alpha_1)-1} E_{\alpha_2,(\alpha_2-\alpha_1)m+\alpha_2+\beta_1(1-\alpha_1)}^{m+1}\left(-\frac{c}{b}x^{\alpha_2}\right) \\ & + bc_2 x^{(\alpha_2-\alpha_1)m+\alpha_2+\beta_2(1-\alpha_2)-1} E_{\alpha_2,(\alpha_2-\alpha_1)m+\alpha_2+\beta_2(1-\alpha_2)}^{m+1}\left(-\frac{c}{b}x^{\alpha_2}\right) \\ & + \left(\mathbf{E}_{\alpha_2,(\alpha_2-\alpha_1)m+\alpha_2,-\frac{c}{b};0+}^{m+1} f\right)\left(-\frac{c}{b}x^{\alpha_2}\right)\Bigg]. \end{aligned} \tag{3.9}$$

Proof Our demonstration of Theorem 5 is based upon the Laplace transform method. Indeed, if we make use of the notational convention depicted in (1.7) and the Laplace transform formula (1.6), by applying the operator $\mathcal{L}$ to both the sides of (3.7), it is easily seen that

$$Y(s) = ac_1 \frac{s^{\beta_1(\alpha_1-1)}}{as^{\alpha_1}+bs^{\alpha_2}+c} + bc_2 \frac{s^{\beta_2(\alpha_2-1)}}{as^{\alpha_1}+bs^{\alpha_2}+c} + \frac{F(s)}{as^{\alpha_1}+bs^{\alpha_2}+c}.$$

Furthermore, since

$$\begin{aligned}\frac{s^{\beta_i(\alpha_i-1)}}{as^{\alpha_1}+bs^{\alpha_2}+c} &= \frac{1}{b}\left(\frac{s^{\beta_i(\alpha_i-1)}}{s^{\alpha_2}+\frac{c}{b}}\right)\left(\frac{1}{1+\frac{a}{b}\left(\frac{s^{\alpha_1}}{s^{\alpha_2}+\frac{c}{b}}\right)}\right) = \frac{1}{b}\sum_{m=0}^{\infty}\left(-\frac{a}{b}\right)^m \frac{s^{\alpha_1 m+\beta_i\alpha_i-\beta_i}}{\left(s^{\alpha_2}+\frac{c}{b}\right)^{m+1}} \\ &= \mathcal{L}\Bigg[\frac{1}{b}\sum_{m=0}^{\infty}(-1)^m\left(\frac{a}{b}\right)^m x^{(\alpha_2-\alpha_1)m+\alpha_2+\beta_i(1-\alpha_i)-1} \\ &\qquad \cdot E^{m+1}_{\alpha_2,(\alpha_2-\alpha_1)m+\alpha_2+\beta_i(1-\alpha_i)}\left(-\frac{c}{b}x^{\alpha_2}\right)\Bigg](s) \qquad (i=1,2)\end{aligned}$$

and

$$\begin{aligned}\frac{F(s)}{as^{\alpha_1}+bs^{\alpha_2}+c} &= \frac{1}{b}\sum_{m=0}^{\infty}\left(-\frac{a}{b}\right)^m\left(\frac{s^{\alpha_1 m}}{\left(s^{\alpha_2}+\frac{c}{b}\right)^{m+1}}\,F(p)\right) = \mathcal{L}\Bigg[\frac{1}{b}\sum_{m=0}^{\infty}\left(-\frac{a}{b}\right)^m \\ &\qquad \cdot\left(x^{(\alpha_2-\alpha_1)m+\alpha_2-1}E^{m+1}_{\alpha_2,(\alpha_2-\alpha_1)m+\alpha_2}\left(-\frac{c}{b}x^{\alpha_2}\right) * f(x)\right)\Bigg](s) \\ &= \mathcal{L}\left[\frac{1}{b}\sum_{m=0}^{\infty}(-1)^m\left(\frac{a}{b}\right)^m\left(\mathbf{E}^{m+1}_{\alpha_2,(\alpha_2-\alpha_1)m+\alpha_2,-\frac{c}{b};0+}f\right)\left(-\frac{c}{b}x^{\alpha_2}\right)\right](s)\end{aligned}$$

in terms of the Laplace convolution, by applying the inverse Laplace transform, we get the solution (3.9) asserted by Theorem 5. ■

3.2. *An Application of Theorem 5*

The next problem is to solve the fractional differential equation (3.7) in the space of Lebesgue integrable functions $y \in L(0,\infty)$ when

$$\alpha_1 = \alpha_2 = \alpha \quad \text{and} \quad \beta_1 \neq \beta_2,$$

that is, the following fractional differential equation:

$$a\left(D^{\alpha,\beta_1}_{0+}y\right)(x) + b\left(D^{\alpha,\beta_2}_{0+}y\right)(x) + cy(x) = f(x) \tag{3.10}$$

under the initial conditions given by

$$\left(I^{(1-\beta_i)(1-\alpha)}_{0+}y\right)(0+) = c_i \qquad (i=1,2). \tag{3.11}$$

We now assume, without loss of generality, that $\beta_2 \leqq \beta_1$. If $c_1 < \infty$, then

$$c_2 = 0 \quad \text{unless} \quad \beta_1 = \beta_2.$$

COROLLARY 1 *The fractional differential equation* (3.10) *under the initial conditions* (3.11) *has its solution in the space* $L(0, \infty)$ *given by*

$$y(x) = \left(\frac{ac_1}{a+b}\right) x^{\beta_1+\alpha(1-\beta_1)-1} E_{\alpha,\beta_1+\alpha(1-\beta_1)}\left(-\frac{c}{a+b}x^{\alpha}\right) + \left(\frac{bc_2}{a+b}\right) x^{\beta_2+\alpha(1-\beta_2)-1} E_{\alpha,\beta_2+\alpha(1-\beta_2)}\left(-\frac{c}{a+b}x^{\alpha}\right) + \left(\frac{1}{a+b}\right)\left(\mathbf{E}^{1}_{\alpha,1,-\frac{c}{a+b};0+}f\right)(x). \tag{3.12}$$

Proof Our proof of Corollary 1 is much akin to that of Theorem 5. We choose to omit the details involved. ■

3.3. *A Fractional Differential Equation Related to the Process of Dielectric Relaxation*

Let

$$0 < \alpha_1 \leqq \alpha_2 \leqq \alpha_3 < 1 \quad \text{and} \quad 0 \leqq \beta_i \leqq 1 \qquad (i = 1, 2, 3;\ a, b, c, e \in \mathbb{R}).$$

Consider the following fractional differential equation:

$$a\left(D_{0+}^{\alpha_1,\beta_1}y\right)(x) + b\left(D_{0+}^{\alpha_2,\beta_2}y\right)(x) + c\left(D_{0+}^{\alpha_3,\beta_3}y\right)(x) + ey(x) = f(x) \tag{3.13}$$

in the space of Lebesgue integrable functions $y \in L(0, \infty)$ with the initial conditions given by

$$\left(I_{0+}^{(1-\beta_i)(1-\alpha_i)}y\right)(0+) = c_i \qquad (i = 1, 2, 3). \tag{3.14}$$

Without loss of generality, we assume that

$$(1-\beta_1)(1-\alpha_1) \leqq (1-\beta_2)(1-\alpha_2) \leqq (1-\beta_3)(1-\alpha_3).$$

If $c_1 < \infty$, then

$$c_2 = 0 \quad \text{unless} \quad (1-\beta_1)(1-\alpha_1) = (1-\beta_2)(1-\alpha_2)$$

and

$$c_3 = 0 \quad \text{unless} \quad (1-\beta_1)(1-\alpha_1) = (1-\beta_2)(1-\alpha_2) = (1-\beta_3)(1-\alpha_3).$$

Hilfer [7] observed that a particular case of the fractional differential equation (3.13) when

$$\alpha_1 = 1,\ \beta_i = 1 \quad (i = 1, 2, 3),\ e = 1 \quad \text{and} \quad f(x) = 0$$

describes the process of dielectric relaxation in glycerol over 12 decades in frequency.

THEOREM 6 *The fractional differential equation* (3.13) *with the initial conditions* (3.14) *has its solution in the space* $L(0,\infty)$ *given by*

$$\begin{aligned} y(x) = &\sum_{m=0}^{\infty} \frac{(-1)^m}{c^{m+1}} \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k} x^{(\alpha_3-\alpha_2)m+(\alpha_2-\alpha_1)k+\alpha_3-1} \\ &\cdot \Big[a c_1 x^{\beta_1(1-\alpha_1)} E_{\alpha_3,(\alpha_3-\alpha_2)m+(\alpha_2-\alpha_1)k+\alpha_3+\beta_1(1-\alpha_1)} \left(-\frac{e}{c} x^{\alpha_3}\right) \\ &\quad + b c_2 x^{\beta_2(1-\alpha_2)} E_{\alpha_3,(\alpha_3-\alpha_2)m+(\alpha_2-\alpha_1)k+\alpha_3+\beta_2(1-\alpha_2)} \left(-\frac{e}{c} x^{\alpha_3}\right) \\ &\quad + c c_3 x^{\beta_3(1-\alpha_3)} E_{\alpha_3,(\alpha_3-\alpha_2)m+(\alpha_2-\alpha_1)k+\alpha_3+\beta_3(1-\alpha_3)} \left(-\frac{e}{c} x^{\alpha_3}\right) \Big] \\ &+ \sum_{m=0}^{\infty} \frac{(-1)^m}{c^{m+1}} \sum_{k=0}^{m} \binom{m}{k} a^k b^{m-k} \left(\mathbf{E}^{m+1}_{\alpha_3,(\alpha_3-\alpha_2)m+(\alpha_2-\alpha_1)k+\alpha_3} f\right) \left(-\frac{e}{c} x^{\alpha_3}\right). \end{aligned} \tag{3.15}$$

Proof Making use of the above-demonstrated technique based upon the Laplace and the inverse Laplace transformations once again, it is not difficult to deduce the solution (3.15) just as we did in our proof of Theorem 5. ■

3.4. *An Interesting Consequence of Theorem 6*

Let

$$0 < \alpha < 1 \quad \text{and} \quad 0 \leqq \beta_i \leqq 1 \quad (i = 1, 2, 3).$$

In the space of Lebesgue integrable functions $y \in L(0,\infty)$, we consider special case of the fractional differential equation (3.13) when

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha.$$

$$a\left(D^{\alpha,\beta_1}_{0+} y\right)(x) + b\left(D^{\alpha,\beta_2}_{0+} y\right)(x) + c\left(D^{\alpha,\beta_3}_{0+} y\right)(x) + e y(x) = f(x) \tag{3.16}$$

under the initial conditions given by

$$\left(I^{(1-\beta_i)(1-\alpha)}_{0+} y\right)(0+) = c_i \qquad (i = 1, 2, 3). \tag{3.17}$$

We assume, without loss of generality, that $\beta_3 \leqq \beta_2 \leqq \beta_1$. If $c_1 < \infty$, then

$$c_2 = 0 \quad \text{unless} \quad \beta_1 = \beta_2$$

and

$$c_3 = 0 \quad \text{unless} \quad \beta_1 = \beta_2 = \beta_3.$$

We thus arrive easily at the following consequence of Theorem 6.

COROLLARY 2 *The fractional differential equation* (3.16) *with the initial conditions* (3.17) *has its solution in the space* $L(0,\infty)$ *given by*

$$\begin{aligned} y(x) = &\left(\frac{a c_1}{a+b+c}\right) x^{\beta_1+\alpha(1-\beta_1)-1} E_{\alpha,\beta_1+\alpha(1-\beta_1)} \left(-\frac{e}{a+b+c} x^{\alpha}\right) \\ &+ \left(\frac{b c_2}{a+b+c}\right) x^{\beta_2+\alpha(1-\beta_2)-1} E_{\alpha,\beta_2+\alpha(1-\beta_2)} \left(-\frac{e}{a+b+c} x^{\alpha}\right) \\ &+ \left(\mathbf{E}^1_{\alpha,1,-\frac{e}{a+b+c};0+} f\right)(x). \end{aligned} \tag{3.18}$$

Remark 1 Podlubny [17] used the Laplace transform method in order to give an explicit solution for an arbitrary fractional linear ordinary differential equation with constant coefficients involving Riemann–Liouville fractional derivatives in series of *multinomial* Mittag–Leffler functions.

3.5. *A Fractional Differential Equation with Variable Coefficient*

Kilbas *et al*. [14] used the Laplace transform method to derive an explicit solution for the following fractional differential equation with variable coefficients:

$$x\left(D_{0+}^{\alpha}y\right)(x)=\lambda y(x)\qquad\left(x>0,\lambda\in\mathbb{R};\alpha>0;l-1<\alpha\leqq l;l\in\mathbb{N}\setminus\{1\}\right). \tag{3.19}$$

They proved that the differential equation (3.19) with $0<\alpha<1$ is solvable and that its solution is given by [14]

$$y(x)=cx^{\alpha-1}\,\phi\left(\alpha-1,\alpha;-\frac{\lambda}{1-\alpha}x^{\alpha-1}\right),$$

where (see, for details, Section 1)

$$\phi={}_0\Psi_1$$

is the Wright function defined by the following series [14, p. 54]:

$$\phi(\alpha,\beta;z)=\sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!}={}_0\Psi_1[-;(\beta,\alpha);z]\quad(\alpha,\beta,z\in\mathbb{C})$$

and c is an arbitrary real constant.

In the space of Lebesgue integrable functions $y\in L(0,\infty)$, we consider the following more general fractional differential equation (3.19):

$$x\left(D_{0+}^{\alpha,\beta}y\right)(x)=\lambda y(x)\qquad\left(x>0;\lambda\in\mathbb{R};0<\alpha<1;0\leqq\beta\leqq1)\right) \tag{3.20}$$

under the initial condition given by

$$\left(I_{0+}^{(1-\beta)(1-\alpha)}y\right)(0+)=c_1. \tag{3.21}$$

THEOREM 7 *The fractional differential equation* (3.20) *with the initial condition* (3.21) *has its solution in the space* $L(0,\infty)$ *given by*

$$\begin{aligned}y(x)=c_1\beta x^{(\alpha-1)(1-\beta)}\sum_{n=0}^{\infty}\frac{\left(\frac{\lambda}{1-\alpha}\,x^{\alpha-1}\right)^n}{n!\,(\beta-n)}\,\phi\left(\alpha-1,(\beta-n)(1-\alpha)+\alpha,\frac{\lambda}{\alpha-1}\,x^{\alpha-1}\right)\\+c_2x^{\alpha-1}\phi\left(\alpha-1,\alpha,\frac{\lambda}{\alpha-1}\,x^{\alpha-1}\right),\end{aligned} \tag{3.22}$$

where c_1 and c_2 are arbitrary constants.

Proof We first apply the Laplace transform operator $\mathcal{L}$ to each member of the fractional differential equation (3.22) and use the special case $n=1$ of the following formula [2, p. 129, Entry 4.1 (6)]:

$$\frac{\partial^n}{\partial s^n}\left(\mathcal{L}\left[f(x)\right](s)\right)=(-1)^n\ \mathcal{L}\left[x^n f(x)\right](s) \qquad (n\in\mathbb{N})\,. \tag{3.23}$$

We thus find from (3.20) and (3.23) that

$$\frac{\partial}{\partial s}\left(s^{\alpha}Y(s)-c_1 s^{\beta(\alpha-1)}\right)=-\lambda Y(s)\,,$$

which leads us to the following ordinary linear differential equation of the first order:

$$Y'(s)+\left(\frac{\alpha}{s}+\frac{\lambda}{s^{\alpha}}\right)Y(s)-c_1\beta(\alpha-1)\,s^{\beta(\alpha-1)-\alpha-1}=0.$$

Its solution is given by

$$Y(s)=\frac{1}{s^{\alpha}}\,\mathrm{e}^{\left(\frac{\lambda}{\alpha-1}\right)s^{1-\alpha}}\left(c_2+c_1\beta(\alpha-1)\int_0^s x^{\beta(\alpha-1)-1}\mathrm{e}^{-\frac{\lambda}{\alpha-1}x^{1-\alpha}}\,\mathrm{d}x\right), \tag{3.24}$$

where c_1 and c_2 are arbitrary constants.

Upon expanding the exponential function in the integrand of (3.24) in a series, if we use term-by-term integration in conjunction with the above Laplace transform method, we eventually arrive at the solution (3.22) asserted by Theorem 7. ■

4. Fractional differintegral equations of the Volterra type

4.1. *A General Volterra-Type Fractional Differintegral Equation*

Al-Saqabi and Tuan [1] made use of an operational method to solve a general Volterra-type differintegral equation of the form:

$$\left(D_{0+}^{\alpha}f\right)(x)+\frac{a}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}f(t)\,\mathrm{d}t=g(x) \qquad \left(\Re(\alpha)>0;\ \Re(\nu)>0\right), \tag{4.1}$$

where $a\in\mathbb{C}$ and $g\in L(0,\mathfrak{b})$ $(\mathfrak{b}>0)$. Here, in this section, we consider the following general class of differintegral equations of the Volterra type involving the generalized fractional derivative operators:

$$\left(D_{0+}^{\alpha,\mu}f\right)(x)+\frac{a}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}f(t)\,\mathrm{d}t=g(x) \tag{4.2}$$

$$\left(0<\alpha<1;\ 0\leqq\mu\leqq1;\ \Re(\nu)>0\right)$$

in the space of Lebesgue integrable functions $f\in L(0,\infty)$ with the initial condition given by

$$\left(I_{0+}^{(1-\mu)(1-\alpha)}f\right)(0+)=c. \tag{4.3}$$

THEOREM 8 *The fractional differintegral equation* (4.2) *with the initial condition* (4.3) *has its solution in the space* $L(0,\infty)$ *given by*

$$f(x)=cx^{\alpha-\mu(\alpha-1)-1}E_{\alpha+\nu,\alpha-\mu(\alpha-1)}\left(-ax^{\alpha+\nu}\right)+\left(\mathbf{E}^{1}_{\alpha+\nu,\alpha,-a;0+}g\right)(x)\,, \tag{4.4}$$

where c is an arbitrary constant.

Proof By applying the Laplace transform operator $\mathcal{L}$ to both sides of (4.2) and using the formula (1.6), we readily get

$$F(s) = c\,\frac{s^{\mu(\alpha-1)+\nu}}{s^{\alpha+\nu}+a} + \frac{s^{\nu}}{s^{\alpha+\nu}+a}\,G(s),$$

which, in view of the Laplace transform formula (1.11) and Laplace convolution theorem, yields

$$\begin{aligned} F(s) &= c\mathcal{L}\left[x^{\alpha-\mu(\alpha-1)-1}E_{\alpha+\nu,\alpha-\mu(\alpha-1)}\left(-ax^{\alpha+\nu}\right)\right](s) \\ &\quad + \mathcal{L}\left[\left(x^{\alpha-1}E_{\alpha+\nu,\alpha}\left(-ax^{\alpha+\nu}\right)\right) * g(x)\right](s). \end{aligned}$$

The solution (4.4) asserted by Theorem 8 would now follow by appealing to the inverse Laplace transform to each member of this last equation. ■

Next, we consider some illustrative examples of the solution (4.4).

Example 1 If we put

$$g(x) = x^{\mu-1}$$

in Theorem 8 and apply the special case of the following integral formula when $\gamma = 1$ [8], then

$$\int_0^x (x-t)^{\alpha-1}\,E_{\rho,\alpha}^{\gamma}\big(\omega(x-t)^{\rho}\big)t^{\mu-1}\,\mathrm{d}t = \Gamma(\mu)\,x^{\alpha+\mu-1}E_{\rho,\alpha+\mu}^{\gamma}(\omega x^{\rho}), \tag{4.5}$$

we can deduce a particular case of the solution (4.4) given by Corollary 3.

COROLLARY 3 *The following fractional differintegral equation*:

$$\left(D_{0+}^{\alpha,\mu}f\right)(x) + \frac{a}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}f(t)\,\mathrm{d}t = x^{\mu-1} \tag{4.6}$$

$$\big(0<\alpha<1;\ 0\leqq\mu\leqq1;\ \Re(\nu)>0\big)$$

with the initial condition (4.3) *has its solution in the space* $L(0,\infty)$ *given by*

$$f(x) = x^{\alpha-\mu(\alpha-1)-1}\left[cE_{\alpha+\nu,\alpha-\mu(\alpha-1)}\left(-ax^{\alpha+\nu}\right) + \Gamma(\mu)\,x^{\alpha\mu}E_{\alpha+\nu,\alpha+\mu}\left(-ax^{\alpha+\nu}\right)\right]. \tag{4.7}$$

Example 2 If, in Theorem 8, we put

$$g(x) = x^{\mu-1}E_{\alpha+\nu,\mu}\left(-ax^{\alpha+\nu}\right)$$

and apply the special case of the following integral formula when $\gamma = \sigma = 1$ [24], then

$$\int_0^x (x-t)^{\mu-1}\,E_{\rho,\mu}^{\gamma}\big(\omega(x-t)^{\rho}\big)t^{\nu-1}E_{\rho,\nu}^{\sigma}(\omega t^{\rho})\,\mathrm{d}t = x^{\mu+\nu-1}E_{\rho,\mu+\nu}^{\gamma+\sigma}(\omega x^{\rho}), \tag{4.8}$$

and we get another particular case of the solution (4.4) given by Corollary 4.

COROLLARY 4 *The following fractional differintegral equation*:

$$\left(D_{0+}^{\alpha,\mu}f\right)(x) + \frac{a}{\Gamma(\nu)}\int_0^x (x-t)^{\nu-1}f(t)\,\mathrm{d}t = x^{\mu-1}E_{\alpha+\nu,\mu}\left(-ax^{\alpha+\nu}\right) \tag{4.9}$$

$$\big(0<\alpha<1;\ 0\leqq\mu\leqq1;\ \Re(\nu)>0\big)$$

with the initial condition (4.3) *has its solution in the space* $L(0,\infty)$ *given by*

$$f(x) = x^{\alpha-\mu(\alpha-1)-1}\left[cE_{\alpha+\nu,\alpha-\mu(\alpha-1)}\left(-ax^{\alpha+\nu}\right) + x^{\alpha\mu}E_{\alpha+\nu,\alpha+\mu}^{2}\left(-ax^{\alpha+\nu}\right)\right], \tag{4.10}$$

where c is an arbitrary constant.

Example 3 If we put

$$g(x) = x^{\beta+\nu-1} E_{\alpha+\nu,\beta+\nu}\left(-bx^{\alpha+\nu}\right)$$

and apply the following integral formula [23], then

$$\int_0^x (x-t)^{\alpha-1} E_{\alpha+\nu,\alpha}\left(-a(x-t)^{\alpha+\nu}\right) t^{\beta+\nu-1} E_{\alpha+\nu,\beta+\nu}\left(-bt^{\alpha+\nu}\right) \mathrm{d}t$$

$$= \frac{E_{\alpha+\nu,\beta}\left(-bx^{\alpha+\nu}\right) - E_{\alpha+\nu,\beta}\left(-ax^{\alpha+\nu}\right)}{a-b} x^{\beta-1} \quad (a \neq b), \tag{4.11}$$

and we get yet another particular case of the solution (4.4) given by Corollary 5.

COROLLARY 5 *The following fractional differintegral equation*:

$$\left(D_{0+}^{\alpha,\mu} f\right)(x) + \frac{a}{\Gamma(\nu)} \int_0^x (x-t)^{\nu-1} f(t)\,\mathrm{d}t = x^{\beta+\nu-1} E_{\alpha+\nu,\beta+\nu}\left(-bx^{\alpha+\nu}\right) \tag{4.12}$$

$$\left(0 < \alpha < 1;\ 0 \leqq \mu \leqq 1;\ \Re(\nu) > 0\right)$$

with the initial condition (4.3) *has its solution in the space* $L(0,\infty)$ *given by*

$$f(x) = cx^{\alpha-\mu(\alpha-1)-1} E_{\alpha+\nu,\alpha-\mu(\alpha-1)}\left(-ax^{\alpha+\nu}\right)$$

$$+ \frac{E_{\alpha+\nu,\beta}\left(-bx^{\alpha+\nu}\right) - E_{\alpha+\nu,\beta}\left(-ax^{\alpha+\nu}\right)}{a-b} x^{\beta-1} \quad (a \neq b), \tag{4.13}$$

where c is an arbitrary constant.

4.2. *A Cauchy-Type Problem for Fractional Differential Equations*

Kilbas *et al*. [10] established the explicit solution of the Cauchy-type problem for the following fractional differential equation:

$$\left(D_{a+}^{\alpha} f\right)(x) = \lambda\left(\mathbf{E}_{\rho,\alpha,\nu;a+}^{\gamma} f\right)(x) + g(x) \qquad (a \in \mathbb{C};\ g \in L[0,b]) \tag{4.14}$$

in the space of Lebesgue integrable functions $f \in L(0,\infty)$ with the initial conditions given by

$$\left(D_{a+}^{\alpha} f\right)(a+) = b_k \qquad \left(b_k \in \mathbb{C} \quad (k = 1, \cdots, n)\right). \tag{4.15}$$

Let us consider the following more general Volterra-type fractional differintegral equation:

$$\left(D_{0+}^{\alpha,\mu} f\right)(x) = \lambda\left(\mathbf{E}_{\rho,\alpha,\nu;0+}^{\gamma} f\right)(x) + g(x) \tag{4.16}$$

in the space of Lebesgue integrable functions $f \in L(0,\infty)$ with initial condition given by

$$\left(I_{0+}^{(1-\mu)(1-\alpha)} f\right)(0+) = c. \tag{4.17}$$

THEOREM 9 *The fractional differintegral equation* (4.16) *with the initial condition* (4.17) *has its solution in the space* $L(0,\infty)$ *given by*

$$f(x) = c\sum_{k=0}^{\infty} \lambda^k x^{2\alpha k+\alpha+\mu-\mu\alpha-1} E_{\rho,2\alpha k+\alpha+\mu-\mu\alpha}^{\gamma k}\left(\nu x^{\rho}\right)$$

$$+ \sum_{k=0}^{\infty} \lambda^k \left(\mathbf{E}_{\rho,2\alpha k+\alpha,\nu;0+}^{\gamma k} g\right)(x), \tag{4.18}$$

where c is an arbitrary constant.

Proof Applying Laplace transforms on both sides of (4.16), we get

$$F(s) = c\,\frac{s^{\mu(\alpha-1)}}{s^{\alpha} - \lambda\left[\frac{s^{\rho\gamma-\alpha}}{(s^{\rho}-\nu)^{\gamma}}\right]} + \frac{G(s)}{s^{\alpha} - \lambda\left[\frac{s^{\rho\gamma-\alpha}}{(s^{\rho}-\nu)^{\gamma}}\right]}. \tag{4.19}$$

On the other hand, by virtue of (1.11), it is not difficult to see that

$$\frac{s^{\mu(\alpha-1)}}{s^{\alpha} - \lambda\left[\frac{s^{\rho\gamma-\alpha}}{(s^{\rho}-\nu)^{\gamma}}\right]} = \mathcal{L}\left[\sum_{k=0}^{\infty}\lambda^{k}x^{2\alpha k+\alpha+\mu-\mu\alpha-1}E^{\gamma k}_{\rho,2\alpha k+\alpha+\mu-\mu\alpha}(\nu x^{\rho})\right](s)$$

and

$$\frac{G(s)}{s^{\alpha} - \lambda\left[\frac{s^{\rho\gamma-\alpha}}{(s^{\rho}-\nu)^{\gamma}}\right]} = \mathcal{L}\left[\left(\sum_{k=0}^{\infty}\lambda^{k}x^{2\alpha k+\alpha-1}E^{\gamma k}_{\rho,2\alpha k+\alpha}(\nu x^{\rho})\right) * g(x)\right](s).$$

Upon substituting these last two relations into (4.19), if we apply the inverse Laplace transforms, we arrive at the solution (4.18) asserted by Theorem 9. The details involved are being omitted here. ■

Each of the following particular cases of Theorem 9 are worthy of note here.

Example 4 If we put

$$g(x) = x^{\mu-1}$$

and use the integral formula (4.5), we get the following particular case of the solution (4.18).

COROLLARY 6 *The following fractional differintegral equation*:

$$\left(D^{\alpha,\mu}_{0+}f\right)(x) = \lambda\left(\mathbf{E}^{\gamma}_{\rho,\alpha,\nu;0+}f\right)(x) + x^{\mu-1} \tag{4.20}$$
$$\left(0 < \alpha < 1;\ 0 \leqq \mu \leqq 1;\ \Re(\nu) > 0\right)$$

with the initial condition (4.17) *has its solution in the space* $L(0,\infty)$ *given by*

$$f(x) = x^{\alpha+\mu-\mu\alpha-1}\left[c\sum_{k=0}^{\infty}\left(\lambda x^{2\alpha}\right)^{k}E^{\gamma k}_{\rho,2\alpha k+\alpha+\mu-\mu\alpha}(\nu x^{\rho}) + \Gamma(\mu)\,x^{\mu\alpha}\sum_{k=0}^{\infty}\left(\lambda x^{2\alpha}\right)^{k}E^{\gamma k}_{\rho,2\alpha k+\alpha+\mu}(\nu x^{\rho})\right]. \tag{4.21}$$

where c is an arbitrary constant.

Example 5 If we put

$$g(x) = cx^{\mu-\mu\alpha-1}E_{\rho,\mu-\mu\alpha}(\nu x^{\rho})$$

and use the integral formula (4.8), we get the following particular case of the solution (4.18).

COROLLARY 7 *The following fractional differintegral equation*:

$$\left(D_{0+}^{\alpha,\mu} f\right)(x) = \lambda \left(\mathbf{E}_{\rho,\alpha,\nu;0+}^{\gamma} f\right)(x) + c x^{\mu-\mu\alpha-1} E_{\rho,\mu-\mu\alpha}\left(\nu x^{\rho}\right) \tag{4.22}$$

$$\left(0 < \alpha < 1;\ 0 \leqq \mu \leqq 1;\ \Re(\nu) > 0\right)$$

with the initial condition (4.17) *has its solution in the space* $L(0,\infty)$ *given by*

$$f(x) = c \sum_{k=0}^{\infty} \lambda^k x^{2\alpha k+\alpha+\mu-\mu\alpha-1} \left[E_{\rho,2\alpha k+\alpha+\mu-\mu\alpha}^{\gamma k}\left(\nu x^{\rho}\right) + E_{\rho,2\alpha k+\alpha+\mu-\mu\alpha}^{\gamma k+1}\left(\nu x^{\rho}\right)\right], \tag{4.23}$$

where c is an arbitrary constant.

5. Fractional kinetic equations

Fractional kinetic equations have gained popularity during the past decade mainly due to the discovery of their relation with the CTRW-theory in [9]. These equations are investigated in order to determine and interpret certain physical phenomena which govern such processes as diffusion in porous media, reaction and relaxation in complex systems, anomalous diffusion, and so on [5,6].

In a recent investigation by Saxena and Kalla [21, p. 506, Equation (2.1)] (see also references to many closely related works cited in [21]), the following fractional kinetic equation was considered:

$$N(t) - N_0\, f(t) = -c^{\nu}\left(I_{0+}^{\nu} N\right)(t) \qquad \left(\Re(\nu) > 0\right), \tag{5.1}$$

where $N(t)$ denotes the number density of a given species at time t, $N_0 = N(0)$ is the number density of that species at time $t = 0$, c is a constant and (for convenience) $f \in L(0,\infty)$, it being tacitly assumed that $f(0) = 1$ in order to satisfy the initial condition $N(0) = N_0$. By applying the Laplace transform operator $\mathcal{L}$ to each member of (5.1), we readily obtain

$$\mathcal{L}[N(t)](s) = N_0\, \frac{F(s)}{1 + c^{\nu} s^{-\nu}} = N_0 \left(\sum_{k=0}^{\infty} (-c^{\nu})^k\ s^{-k\nu}\right) F(s) \qquad \left(\left|\frac{c}{s}\right| < 1\right). \tag{5.2}$$

Remark 2 Because

$$\mathcal{L}\left[t^{\mu-1}\right](s) = \frac{\Gamma(\mu)}{s^{\mu}} \qquad \left(\Re(s) > 0;\ \Re(\mu) > 0\right), \tag{5.3}$$

it is not possible to compute the inverse Laplace transform of $s^{-k\nu}$ $(k \in \mathbb{N}_0)$ by setting $\mu = k\nu$ in (5.3), simply because the condition $\Re(\mu) > 0$ would obviously be violated when $k = 0$. Consequently, the claimed solution of the fractional kinetic equation (5.1) by Saxena and Kalla [21, p. 506, Equation (2.2)] should be corrected to read as follows:

$$N(t) = N_0 \left(f(t) + \sum_{k=1}^{\infty} \frac{(-c^{\nu})^k}{\Gamma(k\nu)} \left(t^{k\nu-1} * f(t)\right)\right) \tag{5.4}$$

or, equivalently,

$$N(t) = N_0 \left(f(t) + \sum_{k=1}^{\infty} (-c^{\nu})^k \left(I_{0+}^{k\nu} f\right)(t)\right), \tag{5.5}$$

where we have made use of the following relationship between the Laplace convolution and the Riemann–Liouville fractional integral operator $\left(I_{0+}^{\mu} f\right)(x)$ defined by (1.1) with $a = 0$:

$$t^{k\nu-1} * f(t) := \int_0^t (t-\tau)^{k\nu-1} f(\tau)\mathrm{d}\tau =: \Gamma(k\nu)\left(I_{0+}^{k\nu} f\right)(t) \qquad \left(k \in \mathbb{N};\ \Re(\nu) > 0\right). \tag{5.6}$$

Remark 3 The solution (5.5) would provide the *corrected* version of the obviously erroneous solution of the fractional kinetic equation (5.1) given by Saxena and Kalla [21, p. 508, Equation (3.2)] by applying a technique which was employed earlier by Al-Saqabi and Tuan [1] for solving fractional differeintegral equations.

We conclude this section by considering the following general fractional kinetic differintegral equation:

$$a\left(D_{0+}^{\alpha,\beta} N\right)(t) - N_0\, f(t) = b\left(I_{0+}^{\nu} N\right)(t) \tag{5.7}$$

under the initial condition

$$\left(I_{0+}^{(1-\beta)(1-\alpha)} f\right)(0+) = c, \tag{5.8}$$

where a, b and c are constants and $f \in L(0,\infty)$.

By suitably making use of the Laplace transform method as in our demonstrations of the results proven in the preceding sections, we can obtain the following explicit solution of (5.7) under the initial condition (5.8).

Theorem 10 *The fractional kinetic differintegral equation* (5.7) *with the initial condition* (5.8) *has its explicit solution given by*

$$N(t) = \frac{N_0}{a}\sum_{k=0}^{\infty}\left(\frac{b}{a}\right)^k \frac{\left(t^{\alpha+k(\nu+\alpha)-1} * f(t)\right)}{\Gamma\left(\alpha+k(\nu+\alpha)\right)} + c\sum_{k=0}^{\infty}\left(\frac{b}{a}\right)^k \frac{t^{\alpha-\beta(1-\alpha)+k(\nu+\alpha)-1}}{\Gamma\left(\alpha-\beta(1-\alpha)+k(\nu+\alpha)\right)} \qquad (a \neq 0) \tag{5.9}$$

or, equivalently, by

$$N(t) = \frac{N_0}{a}\sum_{k=0}^{\infty}\left(\frac{b}{a}\right)^k \left(I_{0+}^{\alpha+k(\nu+\alpha)} f\right)(t) + c\sum_{k=0}^{\infty}\left(\frac{b}{a}\right)^k \frac{t^{\alpha-\beta(1-\alpha)+k(\nu+\alpha)-1}}{\Gamma\left(\alpha-\beta(1-\alpha)+k(\nu+\alpha)\right)} \qquad (a \neq 0), \tag{5.10}$$

where a, b and c are constants and $f \in L(0,\infty)$.

The case of the explicit solution of the fractional kinetic differintegral equation (5.7) with the initial condition (5.8) when $b \neq 0$ can be considered similarly. Several illustrative examples of Theorem 10 involving appropriately chosen special values of the function $f(t)$ can also be derived fairly easily.

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