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Article

Time Automorphisms on C*-Algebras

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Abstract: Applications of fractional time derivatives in physics and engineering require the existence of nontranslational time automorphisms on the appropriate algebra of observables. The existence of time automorphisms on commutative and noncommutative C*-algebras for interacting many-body systems is investigated in this article. A mathematical framework is given to discuss local stationarity in time and the global existence of fractional and nonfractional time automorphisms. The results challenge the concept of time flow as a translation along the orbits and support a more general concept of time flow as a convolution along orbits. Implications for the distinction of reversible and irreversible dynamics are discussed. The generalized concept of time as a convolution reduces to the traditional concept of time translation in a special limit.

Keywords: fractional time derivatives; fractional time evolution; C*-algebra; local stationarity; irreversibility problem

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1. Introduction

Applications of fractional time derivatives in physics and engineering assume the existence of a physical time automorphism (time evolution) of observables, which for closed quantum many-body systems is usually given as a Hamiltonian-generated one-parameter group of unitary operators on a Hilbert space. Dissipative processes, irreversible phenomena, the decay of unstable particles, the approach to thermodynamic equilibrium or quantum measurement processes are difficult to accommodate within this traditional mathematical framework [1–3].

Many theoretical approaches to these problems consider an "open" system (or subsystem) S coupled to a "reservoir" R, often viewed as a heat bath or as an apparatus for measurement [3,4]. A different physical interpretation with the same mathematical structure is to identify S with a selection of macroscopic degrees of freedom of a large or infinite many-body system $S \cup R$, while R corresponds to the large or infinite number of microscopic degrees of freedom. It has remained difficult to find physical conditions that rigorously imply irreversibility for the time evolution of the subsystem [4,5]. One expects intuitively that the separation of time scales will be important. Relaxation processes in the reservoir R are usually much faster than the characteristic time scale for the evolution of the system S of interest. Equally important for macroscopic dynamics and thermodynamic behavior is scale separation in the size of R and S. Memory effects are expected to arise from interaction between the system and the reservoir.

Dynamical equations of motion for closed systems are frequently formulated as abstract Cauchy problems on some Banach space $\mathfrak{B}$ of states or observables $A \in \mathfrak{B}$

$$\tau\epsilon\frac{\mathrm{d}}{\mathrm{d}t}A(t/\tau) = \mathscr{L}A(t/\tau) \tag{1a}$$

$$A(t_0/\tau) = A_{0,\tau} \tag{1b}$$

where $A_{0,\tau}$ is the initial value, t, t_0 are time instants measured in units of τ seconds (such that $t/\tau \in \mathbb{R}$) and ϵ provides energy units (Joule) for the infinitesimal generator $\mathscr{L}$ (Liouvillian), which is a linear, often unbounded, operator with domain $D(\mathscr{L}) \subset \mathfrak{B}$. The existence of a physical time evolution is equivalent to the existence of global solutions of Equation (1) under various circumstances and assumptions, such as physical constraints and boundary conditions. It is well known that global solutions do not always exist, particularly when the system is infinite.

Given a kinematical structure describing the states and observables of a physical system, the infinitesimal generator $\mathscr{L}$ in Equation (1) describes infinitesimal changes of these states and observables with time starting from an initial condition $A_{0,\tau} \in \mathfrak{B}$. Let me briefly recall the kinematical structures for classical mechanics, quantum mechanics and field theory [2,6,7]. Observables and states in classical mechanics of point particles correspond to functions over and points in a differentiable manifold. Rays in a Hilbert space and operators acting on them are the kinematical structure in quantum mechanics. In field theory, the observables form a C*-algebra of field operators, and the states correspond to positive linear functionals on this algebra. Automorphisms of the algebra of field operators in field theory, unitary operators on the Hilbert space in quantum mechanics and diffeomorphisms of the differentiable manifold in classical mechanics represent the time evolution of the system as a flow on the kinematical structure. Many theories of interacting particles are based on some Hamiltonian formalisms as in Equation (1) with a Hamiltonian $\mathscr{L}$ corresponding to a vector field in classical mechanics, a self-adjoint operator in quantum mechanics and some form of derivation on the algebra in field theories.

Let $\mathfrak{B} = \mathfrak{A}$ be the C*-algebra of observables of a physical system. Unless otherwise stated, all C*-algebras will be assumed to have an identity. Formally, integrating Equation (1) gives

$$\mathscr{T}^{t/\tau}\mathcal{K}_{A_{0,\tau}}\left(\frac{t_0}{\tau}\right) = T^{t/\tau}A_{0,\tau} \tag{2}$$

where the maps $T^s : \mathfrak{A} \to \mathfrak{A}$ and $\mathscr{T}^s : \mathfrak{A} \to \mathfrak{A}$ are

$$T^{t/\tau} A = \exp\left(\frac{\mathscr{L} t}{\epsilon \tau}\right) A \tag{3a}$$

$$\mathscr{T}^{t/\tau} \mathcal{K}_A(s) = \mathcal{K}_A\left(s + \frac{t}{\tau}\right) \tag{3b}$$

and the orbit maps $\mathcal{K}_A : \mathbb{R} \to \mathfrak{A}$ are defined as

$$\mathcal{K}_A(s) : s \mapsto T^s A \tag{4}$$

for each fixed $A \in \mathfrak{A}$, if T^s with $s \in \mathbb{R}$ is a one-parameter family of *-automorphisms of $\mathfrak{A}$. Of course, the problem is to give meaning to the formal exponential in Equation (3a), such that the orbit maps $\mathcal{K}_A : \mathbb{R} \to \mathfrak{A}$ are continuous for every $A \in \mathfrak{A}$.

The one-parameter family $(T^s)_{s \in \mathbb{R}}$ of *-automorphisms is expected to obey the time evolution law

$$T^{t/\tau} T^{s/\tau} = T^{(t+s)/\tau} \tag{5}$$

with $T^0 = \mathbf{1}$ being the identity. The continuity of the orbit maps may be rephrased as continuity of the maps $t \mapsto T^t$ from $\mathbb{R}$ into the space $\mathfrak{B}(\mathfrak{A})$ of all bounded operators on $\mathfrak{A}$ endowed with the strong operator topology [8,9]. The operator family $(T^s)_{s \in \mathbb{R}}$ is then a strongly continuous one-parameter group (C_0-group) on $\mathfrak{A}$.

The time evolution of states is obtained from the time evolution of observables by passing to adjoints [10,11]. States are elements of the topological dual $\mathfrak{A}^* = \{\mathsf{z} : \mathfrak{A} \to \mathbb{C} : \mathsf{z} \text{ is linear and continuous}\}$. The notation $\langle \mathsf{z}, A \rangle$ is used for the value $\mathsf{z}(A) \in \mathbb{R}$ of a self-adjoint $A \in \mathfrak{A}$ in the state z. States are positive, $\langle \mathsf{z}, A^* A \rangle \geq 0$ for all $A \in \mathfrak{A}$, and normalized, $\|\mathsf{z}\| = \sup\{|\langle \mathsf{z}, A \rangle|, \|A\| = 1\} = 1$, linear functionals on the algebra $\mathfrak{A}$ of observables [6]. The adjoint time evolution $T^{*t} : \mathfrak{A}^* \to \mathfrak{A}^*$ with $t \in \mathbb{R}$ consists of all adjoint operators $(T^t)^*$ on the dual space $\mathfrak{A}^*$ [10,12]. Let $\mathsf{Z} \subset \mathfrak{A}^*$ denote the set of all states. The orbit maps for states $\mathcal{K}_{\mathsf{z}} : \mathbb{R} \to \mathsf{Z}$ are defined as

$$\mathcal{K}_{\mathsf{z}}(s) : s \mapsto (T^s)^* \mathsf{z} = T^{*s} \mathsf{z} \tag{6}$$

for states $\mathsf{z} \in \mathsf{Z} \subset \mathfrak{A}^*$. If T^t is strongly continuous, then

$$|\left\langle (T^{*t} - \mathbf{1})\mathsf{z}, A \right\rangle| = |\left\langle \mathsf{z}, T^t A - A \right\rangle| \leq \|\mathsf{z}\| \, \|T^t A - A\| \tag{7}$$

shows that the adjoint time evolution T^{*t} is weak*-continuous in the sense that the maps $\langle A \rangle_{\mathsf{z}} : \mathbb{R} \to \mathbb{R}$

$$t \mapsto \langle A \rangle_{\mathsf{z}}(t) = \left\langle \mathsf{z}, T^t A \right\rangle = \left\langle T^{*t} \mathsf{z}, A \right\rangle \tag{8}$$

are continuous for all $A \in \mathfrak{A}, \mathsf{z} \in \mathsf{Z}$. These maps are the time evolutions of all expectation values. In other words, for a C_0-group $(T^s)_{s \in \mathbb{R}}$, the orbit maps $\mathcal{K}_{\mathsf{z}}(s)$ are continuous from $\mathbb{R}$ into the space $\mathfrak{B}(\mathfrak{A}^*)$ of all bounded operators on $\mathfrak{A}^*$ endowed with the weak* topology [8,13], and the adjoint family $(T^{*s})_{s \in \mathbb{R}}$ is a C_0^*-group. Note, that the adjoint time evolution T^{*t} need not be strongly continuous unless $\mathfrak{A}$ is reflexive. The relation between the time evolution of states and observables is

$$\begin{aligned} \left\langle \mathcal{K}_{\mathsf{z}}(t_0), T^t \mathcal{K}_A(t_0) \right\rangle &= \left\langle \mathcal{K}_{\mathsf{z}}(t_0), \mathscr{T}^t \mathcal{K}_A(t_0) \right\rangle = \langle \mathcal{K}_{\mathsf{z}}(t_0), \mathcal{K}_A(t_0 + t) \rangle \\ &= \langle \mathcal{K}_{\mathsf{z}}(t_1 - t), \mathcal{K}_A(t_1) \rangle = \left\langle \mathscr{T}^{-t} \mathcal{K}_{\mathsf{z}}(t_1), \mathcal{K}_A(t_1) \right\rangle \\ &= \left\langle T^{*t} \mathcal{K}_{\mathsf{z}}(t_1), \mathcal{K}_A(t_1) \right\rangle \end{aligned} \tag{9}$$

where $t_1 = t_0 + t \in \mathbb{R}$. The adjoint time evolution of states is related to right translations along the orbits in state space in the same way as the time evolution of observables is related to left translations along orbits in the algebra.

Equation (1a) combined with Equation (9) for the adjoint time evolution states formally the proportionality

$$\pm\tau\frac{\mathrm{d}}{\mathrm{d}t} = \pm\frac{\mathscr{L}}{\epsilon} \tag{10}$$

of the infinitesimal generator $\mathrm{d}/\mathrm{d}t$ of time translations and the infinitesimal generator $\mathscr{L}$ of changes of the physical system. Independently of the manner in which one attaches a meaning to the formal exponential, Equation (2) says that the time evolution of a physical system is a translation along orbits corresponding to the changes of the system, specifically

(left shift along $\mathfrak{A}$-orbit) = (change of observable) (11a)

(right shift along Z-orbit) = (change of state) (11b)

where the first equation reflects the Heisenberg picture, while the second corresponds to the Schrödinger picture.

2. Problems and Objective

There are several unsolved problems with the mathematical framework described in the Introduction, particularly when the system is infinite. The following examples of open problems stem from three different areas of theoretical physics.

(A) Open quantum systems: For open quantum systems with infinite reservoirs, $\mathfrak{A} = \mathfrak{B}(\mathfrak{H})$ is the C*-algebra of bounded operators on the Hilbert space $\mathfrak{H}$ of the system $S \cup R$ in a suitably-chosen GNS-representation. The states $\langle \rho, A \rangle = \mathrm{Tr}\,\rho A$ can be identified with trace class operators on $\mathfrak{H}$ if Tr denotes the trace operation. The unitary time-automorphisms $T^t : \mathfrak{B}(\mathfrak{H}) \to \mathfrak{B}(\mathfrak{H})$ of $S \cup R$ are generated by a Hamiltonian H of $S \cup R$

$$H = H_S \otimes \mathbf{1} + \mathbf{1} \otimes H_R + H_{SR} = -\mathrm{i}\mathscr{L} \tag{12}$$

written formally in terms of Hamiltonians H_S of S, H_R of R and their interaction H_{SR}. It is well known [2] that the automorphisms T^t on $\mathfrak{B}(\mathfrak{H})$ do not induce automorphisms on the algebra $\mathfrak{B}(\mathfrak{H}_S)$ of the open subsystem, because the time evolution will mix the Hilbert spaces $\mathfrak{H}_S$ and $\mathfrak{H}_R$ of the system and the reservoir. Even if the initial state is prepared as a product state $\rho_S \otimes \rho_R$, the subsystem evolutions defined by

$$T_S^t A = \mathop{\mathrm{Tr}}_R (\mathbf{1} \otimes \rho_R) T^{-t} (A \otimes \mathbf{1}) T^t \tag{13a}$$

$$T_S^{*\,t} \rho_S = \mathop{\mathrm{Tr}}_R \; T^t (\rho_S \otimes \rho_R) T^{-t} \tag{13b}$$

do not form groups:

$$T_S^{h_1} T_S^{h_2} \neq T_S^{h_1+h_2} \tag{14a}$$

$$T_S^{*\,h_1} T_S^{*\,h_2} \neq T_S^{*\,h_1+h_2} \tag{14b}$$

because of memory effects accumulating from the mixing of the system and the reservoir whenever there is a nonvanishing interaction [2]. Here, ρ_S, ρ_R denote the density matrices of the system and the reservoir. The trace Tr_R integrates out the reservoir degrees of freedom.

(B) Classical dynamical systems: In classical systems, it is well known [14] that the orbits in the abelian algebra $\mathfrak{A}$ of functions on phase space cannot always be defined for all $t \in \mathbb{R}$ and for all initial conditions A_0 in the thermodynamics limit. The integration of Equation (1) does not generally give a dynamical flow of time for all initial conditions, and the problem is to find sufficiently large subsets of $\mathfrak{A}$, such that catastrophic behavior is absent and a unique orbit exists for all $t \in \mathbb{R}$.

(C) Quantum field theory: For quantum field theories or infinite systems, the Stone–von Neumann uniqueness breaks down. Haag's theorem shows that the determination of a suitable representation of the canonical commutation relations becomes a dynamical problem, if the vacuum states for different couplings are different. Non-normal states arise that yield representations assigning different values to global observables, like densities. Due to the problem of inequivalent representations, it is not possible to represent the time evolution as a group of unitary transformations within a single representation, because the representation algebra may change into an inequivalent representation as time evolves.

One objective of this paper is to suggest that these three open problems are, in fact, related to each other, even though they seem to be unrelated at first sight.

The present article suggests that the common denominator of Problems (A)–(C) associated with the mathematical framework described in the Introduction is the concept of time flow as a translation, implicitly assumed on the left-hand side of Equation (1). The common origin of Problems (A)–(C) emerges from studying the following two general questions associated with Equation (1).

Problem 1. *Are there global solutions of equation* (1)*, i.e., solutions for all $t \in \mathbb{R}$?*

If global solutions and, hence, a group of *-automorphisms on $\mathfrak{A}$ exist, then this implies a continuous time evolution for all states $\mathsf{z} \in \mathsf{Z}$. This means a time evolution independent of the state, which is not to be expected for general infinite systems without rescaling time. Rescaling of time is also expected to be necessary for establishing hydrodynamic limits governing invariant states.

Problem 2. *If global solutions of Equation* (1) *exist, how can invariant solutions still change with time?*

Local stationarity (invariance) in time arises from the underlying dynamics. Local stationarity in time is necessary, if thermodynamic observables, such as temperature, pressure or densities, are to provide an approximate representation of the physical system that changes slowly on long time scales. Hence, one has to study the set of stationary states that are invariant under the time evolution. If the thermodynamic observables change, then there must exist many invariant states and many possible time averages, *i.e.*, the time averages are not unique.

The objective of this paper is to introduce a framework in which questions concerning the abundance of time-invariant states and their embedding in the set of all states can be posed mathematically in a proper way.

3. Almost Invariant States

Strictly stationary or invariant states [15] are an idealization. In experiments, stationarity is never ideal, but only approximate. Expectation values are uncertain within the accuracy of the experiment. Experimental accuracy depends on the response and integration times of the experimental apparatus.

These experimental restrictions suggest to focus on a class of states that are stationary (invariant) only up to a given experimental accuracy ε. To do so, recall the definition of invariant (stationary) states. A state $\mathsf{z} \in \mathfrak{A}^*$ is called invariant if

$$\big\langle \mathsf{z}, T^t A)\big\rangle = \langle \mathsf{z}, A\rangle \tag{15}$$

holds for all $A \in \mathfrak{A}$ and $t \in \mathbb{R}$, *i.e.*, if the expectation values $\langle A\rangle_{\mathsf{z}}(t) = \langle \mathsf{z}, A\rangle$ of all observables $A \in \mathfrak{A}$ are constant. The set of invariant states $\mathsf{B}_0 \subset \mathfrak{A}^*$ over $\mathfrak{A}$ is convex and compact in the weak*-topology [6]. The same holds for the set of all states $\mathsf{Z} \supset \mathsf{B}_0$. Invariant states are fixed points of the adjoint time evolution T^{*t}, as seen from Equation (8). Because invariant states are fixed points of T^{*t}, they are of limited benefit for a proper mathematical formulation of the problems discussed above. Once an orbit in state space reaches an invariant state, it remains forever in that state and cannot leave it.

Almost invariant states are based on states whose expectation values are of bounded mean oscillation (BMO). A state $\mathsf{z} \in \mathsf{Z}$ is called a BMO-state if all maps $\langle A\rangle_{\mathsf{z}} : \mathbb{R} \to \mathbb{R}$ have bounded mean oscillation for all $A \in \mathfrak{A}$. The Banach space $\mathrm{BMO}(\mathbb{R})$ of functions with bounded mean oscillation on $\mathbb{R}$ is defined as the linear space

$$\mathrm{BMO}(\mathbb{R}) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}), \|f\|_{\mathrm{BMO}} < \infty\} \tag{16}$$

where $L^1_{\mathrm{loc}}(\mathbb{R})$ is the space of locally-integrable functions $f : \mathbb{R} \to \mathbb{R}$. The BMO-norm is defined as

$$\|f\|_{\mathrm{BMO}} = \inf_C \left\{ \int_I |f(x) - f_I| \mathrm{d}x \leq C|I|, \text{for all } I \right\} \tag{17}$$

where $I \subset \mathbb{R}$ denotes intervals of length $|I|$ and

$$f_I = \frac{1}{|I|} \int_I f(x)\mathrm{d}x \tag{18}$$

denotes the average of f over the interval I. The set of all BMO-states

$$\mathsf{B} = \big\{\mathsf{z} \in \mathsf{Z} : \|\langle A\rangle_{\mathsf{z}}\|_{\mathrm{BMO}} < \infty \;\text{ for all } A \in \mathfrak{A}\big\} \tag{19}$$

is convex by linearity. As a subset $\mathsf{B} \subset \mathsf{Z}$ of a weak* compact set, it is itself weak* compact. Hence, a decomposition theory into extremal BMO-states exists by virtue of the Krein–Milman theorem. The set of invariant states is identified through

$$\mathsf{B}_0 = \big\{\mathsf{z} \in \mathsf{B} : \|\langle A\rangle_{\mathsf{z}}\|_{\mathrm{BMO}} = 0 \;\text{ for all } A \in \mathfrak{A}\big\} \tag{20}$$

as a subset $\mathsf{B}_0 \subset \mathsf{B}$.

A BMO-state will be called ε-almost invariant or almost invariant with accuracy ε if the expectation of all observables are stationary to within experimental accuracy ε. More precisely, the set B_ε of all ε-almost invariant states is defined as

$$\mathsf{B}_\varepsilon = \left\{ \mathsf{z} \in \mathsf{B} : \|\langle A \rangle_{\mathsf{z}}\|_{\mathrm{BMO}} < \varepsilon \ \text{ for all } A \in \mathfrak{A} \right\} \tag{21}$$

as a family of subsets of B. For small $\varepsilon \to 0$, these states are almost invariant. The accuracy ε measures temporal fluctuations away from the time average.

The following inclusions of classes of states used in the following are summarized for orientation and convenience

$$\mathsf{K}_\beta \subset \mathsf{B}_0 \subset \mathsf{B}_\varepsilon \subset \mathsf{B}_\infty = \mathsf{B} \subset \mathsf{Z} \subset \mathfrak{A}^* \tag{22}$$

where $0 < \varepsilon < \infty$ and the set of KMS-states K_β at inverse temperature $\beta > 0$ are defined as states $\mathsf{z} \in \mathsf{Z}$, such that the KMS-condition [16]

$$\left\langle \mathsf{z}, T^{t/\tau}(A)B \right\rangle = \left\langle \mathsf{z}, BT^{t/\tau + \mathrm{i}\epsilon\beta}(A) \right\rangle \tag{23}$$

holds for all $t/\tau \in \mathbb{R}$ and $A, B \in \mathfrak{A}$. The KMS-states are invariant states for all $\beta \geq 0$, but KMS-states for infinite volume systems at different β are often disjoint [16]. For $\beta = 0$, the KMS-states are trace states, *i.e.*, $\langle \mathsf{z}, AB \rangle = \langle \mathsf{z}, BA \rangle$ holds for all $A, B \in \mathfrak{A}$. Because KMS-states are Gibbs states, they are usually interpreted as equilibrium states with extremal states corresponding to pure thermodynamic phases [16].

4. Indistinguishability of States

Experimental uncertainties limit also the ability to distinguish different states. Two states are experimentally indistinguishable (or metrologically equivalent) if they cannot be distinguished by measurements. Let $m < \infty$ denote the maximal number of experiments that can be performed to distinguish the states of the system. Let $\{A_i\}_1^m \subset \mathfrak{A}$ with $i = 1, ..., m$ denote the observables in these experiments, and let η_i $(i = 1, ..., m)$ be the experimental resolutions or accuracy that can be attained for A_i. Two states $\mathsf{z}, \mathsf{z}' \in \mathsf{Z}$ with

$$|\,\langle \mathsf{z}, A_i \rangle - \langle \mathsf{z}', A_i \rangle\,| = |\,\langle \mathsf{z} - \mathsf{z}', A_i \rangle\,| < \eta_i \leq \eta = \max_{i=1,...,m} \eta_i \tag{24}$$

for all $i = 1, ..., m$ are called metrologically equivalent or experimentally indistinguishable with respect to the observables $A_1, ..., A_m$. The sets of indistinguishable states

$$\mathsf{N}(\mathsf{z}; \{A_i\}_1^m; \eta) = \{\mathsf{z}' \in \mathfrak{A}^* : |\,\langle \mathsf{z} - \mathsf{z}', A_i \rangle\,| < \eta_i, i = 1, ..., m\} \tag{25}$$

are η-neighborhoods of z in the weak* topology [17]. The algebra $\mathfrak{M}$ generated by the elements $A_1, ..., A_m \in \mathfrak{A}$ will be called the macroscopic algebra.

In the following, $0 < \eta_i < \infty$ and $0 < \eta = \max_i \eta_i < \infty$ will be assumed. The η-neighborhoods of ε-almost invariant states, *i.e.*, the sets $\mathsf{N}(\mathsf{z}; \{A\}_1^m, \eta) \cap \mathsf{B}_\varepsilon$ with $\mathsf{z} \in \mathsf{B}_0$ for small $\varepsilon, \eta \to 0$, will be the candidates for local (in time) stationary states.

5. Invariant Measures on BMO-states

The set of BMO-states B is weak*-compact. Its open subsets are the elements of the weak*-topology restricted to B. They generate the σ-algebra $\mathcal{B}$ of Borel sets on B. Let $\mathsf{z} \in \mathsf{B}_0 \subset \mathsf{B}$ denote an invariant state, so that Equation (15) holds for all $t \in \mathbb{R}, A \in \mathfrak{A}$. An invariant probability measure on B corresponding to the invariant z can be constructed with the help of a resolution of the identity on $\mathcal{B}$.

Let $(\mathfrak{H}_\mathsf{z}, \pi_\mathsf{z}, \Omega_\mathsf{z}, U_\mathsf{z}^t)$ denote the cyclic representation canonically associated with an invariant state $\mathsf{z} \in \mathsf{B}$ and the time evolution T^t on $\mathfrak{A}$. It is uniquely determined by the two requirements

$$U_\mathsf{z}^t \, \pi_\mathsf{z}(A) \, U_\mathsf{z}^{-t} = \pi_\mathsf{z}\left(T^t A\right) \tag{26}$$

for $A \in \mathfrak{A}$, $t \in \mathbb{R}$ and

$$U_\mathsf{z}^t \, \Omega_\mathsf{z} = \Omega_\mathsf{z} \tag{27}$$

for $t \in \mathbb{R}$. Let $(\ ,\ )$ denote the scalar product in $\mathfrak{H}_\mathsf{z}$.

A resolution of the identity ([13] p. 301) on the Borel σ-algebra $\mathcal{B}$ is a mapping

$$P : \mathcal{B} \to \mathfrak{B}(\mathfrak{H}_\mathsf{z}) \tag{28}$$

with the properties

1. $P(\emptyset) = 0$, $P(\mathsf{B}) = \mathbf{1}$
2. Each $P(\mathsf{G})$ is a self-adjoint projector.
3. $P(\mathsf{G} \cap \mathsf{G}') = P(\mathsf{G})P(\mathsf{G}')$
4. If $\mathsf{G} \cap \mathsf{G}' = \emptyset$, then $P(\mathsf{G} \cup \mathsf{G}') = P(\mathsf{G}) + P(\mathsf{G}')$
5. For every $\psi \in \mathfrak{H}_\mathsf{z}$ and $\phi \in \mathfrak{H}_\mathsf{z}$, the set function $P_{\psi,\phi} : \mathcal{B} \to \mathbb{C}$ defined by

$$P_{\psi,\phi}(\mathsf{G}) = (P(\mathsf{G})\psi, \phi) \tag{29}$$

is a complex regular Borel measure on $\mathcal{B}$.

Because the projectors are self-adjoint, the set function $P_{\psi,\psi}$ is a positive measure for every $\psi \in \mathfrak{H}_\mathsf{z}$. For $\psi = \phi = \Omega_\mathsf{z}$, the resulting measure

$$P_{\Omega_\mathsf{z},\Omega_\mathsf{z}} = (P(\mathsf{G})\Omega_\mathsf{z}, \Omega_\mathsf{z}) =: P_\mathsf{z} \tag{30}$$

is an invariant probability measure on the measurable space $(\mathsf{B}, \mathcal{B})$ associated with the invariant BMO-state $\mathsf{z} \in \mathsf{B}$. The triple $(\mathsf{B}, \mathcal{B}, P_\mathsf{z})$ is a probability space. The probability measure P_z is invariant under the adjoint time evolution T^{*t} on B.

6. Almost Invariance and Recurrence

To discuss the question of how invariant states can evolve in time (Problem 2), consider two invariant states $\mathsf{u}, \mathsf{v} \in \mathsf{B}_0$ and the straight line segment

$$\mathsf{S} = \{\mathsf{z} = \lambda\mathsf{u} + (1-\lambda)\mathsf{v}, 0 \leq \lambda \leq 1, \mathsf{u} \in \mathsf{B}_0, \mathsf{v} \in \mathsf{B}_0\} \tag{31}$$

connecting u and v. Of course, $\mathsf{S} \subset \mathsf{B}_0$. In practical applications, S might be a more or less general subset of B_0, e.g., a KMS-state in $\bigcup_\beta \mathsf{K}_\beta$. Straight line segments of invariant states are expected to be physically important for phase transformations at thermodynamic coexistence. Define a weak*-neighborhood

$$\mathsf{G} = \mathsf{B}_\varepsilon \cap \left(\bigcup_{\mathsf{z} \in \mathsf{S}} \mathsf{N}(\mathsf{z}, \{A\}_1^m; \eta) \right) \tag{32}$$

of ε-almost invariant η-indistinguishable states near S. Depending on the invariant states S and the macroscopic algebra $\mathfrak{M}$ of interest, a similar weak*-neighborhood $\mathsf{G} = \mathsf{G}(\mathsf{S}, \mathfrak{M}, \varepsilon, \eta)$ can be defined for other subsets of B_0.

The time translations $\mathscr{T}^{-t/\tau}$ with time scale τ translate any initial state $\mathsf{z} \in \mathsf{G}$ along its orbit according to

$$\mathscr{T}^{-t/\tau}\mathcal{K}_\mathsf{z}\left(\frac{t_0}{\tau}\right) = \mathcal{K}_\mathsf{z}\left(\frac{t_0 - t}{\tau}\right) \tag{33}$$

where t_0 denotes the initial instant, $\mathcal{K}_\mathsf{z}(t_0/\tau) = \mathsf{z}$ and $\tau > 0$ the time scale. Discretizing time as

$$t = k\tau \tag{34}$$

with $k \in \mathbb{Z}$, such that $t_0 = 0$ produces discretized orbits $\mathcal{K}_\mathsf{z}(-k)$, $k \in \mathbb{N}$ for all $\mathsf{z} \in \mathsf{G}$ as iterates of $\mathscr{T}^{-1}$. For every initial state $\mathsf{z} \in \mathsf{G}$, define

$$w_\mathsf{G}(\mathsf{z}) = \min\left\{k \geq 1 : \mathscr{T}^{-k}\mathcal{K}_\mathsf{z}(0) \in \mathsf{G}\right\} \tag{35}$$

as the first return time of z into the set G. For all invariant $\mathsf{z} \in \mathsf{B}_0$, one has $w_\mathsf{G}(\mathsf{z}) = 1$. For states z that never return to G, one sets $w_\mathsf{G}(\mathsf{z}) = \infty$. For all $k \geq 1$, let

$$\mathsf{G}_k = \{\mathsf{z} \in \mathsf{G} : w_\mathsf{G}(\mathsf{z}) = k\} \tag{36}$$

denote the subset of states with recurrence time $1 \leq k \leq \infty$ with $k = \infty$ interpreted as

$$\mathsf{G}_\infty = \mathsf{G} \setminus \bigcup_{k \in \mathbb{N}} \mathsf{G}_k. \tag{37}$$

The states $\mathsf{z} \in \mathsf{S}$ generate a one-parameter family of resolutions of the identity resulting in a one-parameter family of measures on $(\mathsf{B}, \mathcal{B})$ denoted as $P_{\lambda\mathsf{u}+(1-\lambda)\mathsf{v}}$ with $\lambda \in [0,1]$. Their mixture

$$Q = \int_0^1 P_{\lambda\mathsf{u}+(1-\lambda)\mathsf{v}} \mathrm{d}\lambda \tag{38}$$

is again an invariant measure on $(\mathsf{B}, \mathcal{B})$. The numbers

$$p(k) = \frac{Q(\mathsf{G}_k)}{Q(\mathsf{G})} \tag{39}$$

define a discrete probability density on $\mathbb{N}\cup\{\infty\}$. It may be interpreted as a properly-weighted probability of recurrence into the neighborhood G of the straight line segment $\mathsf{S} \subset \mathsf{B}_0$.

7. Results

The time evolution of almost invariant states can be defined by the addition of random recurrence times. Let $p_N(k)$ be the probability density of the sum

$$W_N = w_1 + \cdots + w_N \tag{40}$$

of $N \geq 1$ independent and identically-distributed random recurrence times $w_i \geq 1$. Let $p(k)$ from Equation (39) be the common probability density of all w_i. Then, with $N \geq 2$ and $p_1(k) = p(k)$,

$$p_N(k) = (p_{N-1} * p)(k) = \sum_{m=0}^{k} p_{N-1}(m)p(k-m) \tag{41}$$

is an N-fold convolution of the discrete recurrence time density in Equation (39). The family of distributions $p_N(k)$ obeys

$$p_N(\infty) + \sum_{k=1}^{\infty} p_N(k) = 1 \tag{42}$$

for all $N \geq 1$, and the discrete analogue of Equation (5)

$$p_{N+M}(k) = (p_N * p_M)(k) \tag{43}$$

holds for all $N, M \geq 1$. Because the individual states in G are indistinguishable within the given accuracy η, but may evolve very differently in time, it is natural to define the duration of time needed for the first recurrence (a single time step) as an average

$$\mathscr{S}^{-1} = \sum_{k=1}^{\infty} p(k)\ \mathscr{T}^{-k} \tag{44}$$

over recurrence times. If a macroscopic time evolution with a rescaled time exists, then one has to rescale the sums W_N and the iterations

$$\mathscr{S}^{-N} = \mathscr{S}^{-(N-1)}\mathscr{S}^{-1} = \sum_{k=1}^{\infty} p_N(k)\ \mathscr{T}^{-k} \tag{45}$$

in the limit $N \to \infty$ with suitable norming constants $D_N \geq 0$.

Theorem 3. *Let $p_N(k)$ be the probability density of W_N specified above in (41). If the distributions of W_N/D_N converge to a limit as $N \to \infty$ for suitable norming constants $D_N \geq 0$, then there exist constants $D \geq 0$ and $0 < \alpha \leq 1$, such that*

$$\lim_{N\to\infty} \sup_k \left| D_N p_N(k) - \frac{\tau}{D^{1/\alpha}} \mathrm{h}_\alpha \left(\frac{k\tau}{D_N D^{1/\alpha}} \right) \right| = 0 \tag{46}$$

where

$$\alpha = \sup\{0 < \beta < 1 : \sum_{k=1}^{\infty} k^{\beta} p(k) < \infty\} \tag{47}$$

if $\sum_{k=1}^{\infty} kp(k)$ *diverges, while*

$$\alpha = 1 \tag{48}$$

if $\sum_{k=1}^{\infty} kp(k)$ *converges. For* $\alpha = 1$*, the function* $\mathrm{h}_\alpha(x)$ *is* $\mathrm{h}_1(x) = \delta(x-1)$*. For* $0 < \alpha < 1$*, the function* $\mathrm{h}_\alpha(x) = 0$ *for* $x \leq 0$ *and*

$$\mathrm{h}_\alpha(x) = \frac{1}{x} \sum_{j=0}^{\infty} \frac{(-1)^j x^{-\alpha j}}{j!\,\Gamma(-\alpha j)} \tag{49}$$

for $x > 0$*.*

Proof. The existence of a limiting distribution for $W_N/D_N > 0$ is known to be equivalent to the stability of the limit [18]. If the limit distribution is nondegenerate, this implies that the rescaling constants D_N have the form

$$D_N = (N\Lambda(N))^{1/\alpha} \tag{50}$$

where $\Lambda(N)$ is a slowly varying function [19], defined by the requirement that

$$\lim_{x\to\infty} \frac{\Lambda(bx)}{\Lambda(x)} = 1 \tag{51}$$

holds for all $b > 0$. That the number α obeys Equation (47) is proven in [18] (p. 179). It is bounded as $0 < \alpha \leq 1$, because the rescaled random variables $W_N/D_N > 0$ are positive.

To prove Equation (46), note that the characteristic function of W_N is the N-th power

$$\langle \mathrm{e}^{\mathrm{i}\xi W_N} \rangle = [p(\xi)]^N = \sum_k \mathrm{e}^{\mathrm{i}\xi y} p_N(k) \tag{52}$$

because the characteristic functions $p(\xi) = \langle \mathrm{e}^{\mathrm{i}\xi w_j} \rangle$ of w_j are identical for all $j = 1, ..., N$. Inverse Fourier transformation gives

$$p_N(k) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{e}^{-\mathrm{i}\xi k} [p(\xi)]^N \mathrm{d}\xi = \frac{\tau}{2\pi D_N D^{1/\alpha}} \int_{-\pi D_N D^{1/\alpha}}^{\pi D_N D^{1/\alpha}} \mathrm{e}^{-\mathrm{i}\xi x} \left[p\left(\frac{\xi\tau}{D_N D^{1/\alpha}}\right)\right]^N \mathrm{d}\xi \tag{53}$$

where

$$x = x_{kN} = \frac{k\tau}{D_N D^{1/\alpha}} \tag{54}$$

and ξ was substituted with $(\xi\tau)/(D_N D^{1/\alpha})$. Let $\mathrm{h}_\alpha(\xi)$ denote the characteristic function of $\mathrm{h}_\alpha(x)$, so that

$$\mathrm{h}_\alpha(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}x\xi} \mathrm{h}_\alpha(\xi) \mathrm{d}\xi \tag{55}$$

holds.

Following [20], the difference $\Delta_N(k)$ in (46) can be decomposed and bounded from above as

$$\begin{aligned}\Delta_N(k) &= \left|D_N p_N(k) - \frac{\tau}{D^{1/\alpha}} \mathrm{h}_\alpha\left(\frac{k\tau}{D_N D^{1/\alpha}}\right)\right| = \left|D_N p_N(k) - \frac{\tau}{D^{1/\alpha}} \mathrm{h}_\alpha(x)\right| \\ &= \frac{\tau}{2\pi D^{1/\alpha}} \left| \int_{-\pi D_N D^{1/\alpha}}^{\pi D_N D^{1/\alpha}} \mathrm{e}^{-\mathrm{i}\xi x} \left[p\left(\frac{\xi\tau}{D_N D^{1/\alpha}}\right)\right]^N \mathrm{d}\xi - \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\xi x}\, \mathrm{h}_\alpha(\xi)\mathrm{d}\xi \right| \\ &= \frac{\tau}{2\pi D^{1/\alpha}} \left| \int_{|\xi|<B} \mathrm{e}^{-\mathrm{i}\xi x} \left[p\left(\frac{\xi\tau}{D_N D^{1/\alpha}}\right)^N - \mathrm{h}_\alpha(\xi)\right] \mathrm{d}\xi \right. \\ &\quad + \int_{B\le|\xi|<\eta D_N D^{1/\alpha}} \mathrm{e}^{-\mathrm{i}\xi x} \left[p\left(\frac{\xi\tau}{D_N D^{1/\alpha}}\right)^N - \mathrm{h}_\alpha(\xi)\right] \mathrm{d}\xi \\ &\quad \left. + \int_{\eta\le\frac{|\xi|}{D_N D^{1/\alpha}}<\pi} \mathrm{e}^{-\mathrm{i}\xi x} \left[p\left(\frac{\xi\tau}{D_N D^{1/\alpha}}\right)^N - \mathrm{h}_\alpha(\xi)\right] \mathrm{d}\xi - \int_{|\xi|\ge\pi D_N D^{1/\alpha}} \mathrm{e}^{-\mathrm{i}\xi x}\, \mathrm{h}_\alpha(\xi)\mathrm{d}\xi \right| \\ &\le \frac{\tau}{2\pi D^{1/\alpha}} \left(\int_{|\xi|<B} \left| p\left(\frac{\xi\tau}{D_N D^{1/\alpha}}\right)^N - \mathrm{h}_\alpha(\xi)\right| \mathrm{d}\xi + \int_{B\le|\xi|<\eta D_N D^{1/\alpha}} \left| p\left(\frac{\xi\tau}{D_N D^{1/\alpha}}\right)\right|^N \mathrm{d}\xi \right. \\ &\quad \left. + \int_{\eta\le\frac{|\xi|}{D_N D^{1/\alpha}}<\pi} \left| p\left(\frac{\xi\tau}{D_N D^{1/\alpha}}\right)\right|^N \mathrm{d}\xi + \int_{|\xi|\ge B} \mathrm{h}_\alpha(\xi)\mathrm{d}\xi \right)\end{aligned} \tag{56}$$

with constants B, η to be specified below. The terms involving $\mathrm{h}_\alpha(\xi)$ from the second and third integral have been absorbed in the fourth integral. The four integrals are now discussed further individually.

The first integral converges uniformly to zero for $N \to \infty$, because $p(k)$ belongs to the domain of attraction of a stable law with index α, as already noted above.

To estimate the second integral, note that the characteristic function $p(\xi)$ belongs to the domain of attraction for index α if and only if it behaves for $|\xi| \to 0$ as [20]

$$|p(\xi)| = \exp\left\{-c|\xi|^\alpha \Lambda\left(\frac{1}{|\xi|}\right)\right\} \tag{57}$$

where $c > 0$ and $\Lambda(x)$ is a slowly varying function at infinity obeying

$$\lim_{N\to\infty} \frac{N\Lambda(D_N)}{D_N^\alpha} = 1 \tag{58}$$

By the representation theorem for slowly varying functions ([21] p. 12), there exist functions $d(y)$ and $\varepsilon(y)$, such that the function $\Lambda(y)$ can be represented as

$$\Lambda(y) = d(y) \exp\left\{-\int_b^y \frac{\varepsilon(u)}{u} \mathrm{d}u\right\} \tag{59}$$

for some $b > 0$ where $d(y)$ is measurable and $d(y) \to d \in (0, \infty)$, as well as $\varepsilon(u) \to 0$ hold for $y \to \infty$. As a consequence

$$\frac{\Lambda(\lambda y)}{\Lambda(y)} = \frac{d(\lambda y)}{d(y)} \exp\left\{ -\int_{y}^{\lambda y} \frac{\varepsilon(u)}{u} \mathrm{d}u \right\} \tag{60}$$

so that with $\lambda = |\xi|^{-1}$ and $y = D_N$

$$\frac{\Lambda(D_N/|\xi|)}{\Lambda(D_N)} = |\xi|^{o(1)}(1 + o(1)) \tag{61}$$

is obtained for $N \to \infty$. Therefore, there exists for any $\gamma < \alpha$ a positive number $c(\gamma)$ independent of N, such that

$$|p_N(\xi)| = \left| p\left(\frac{\xi}{D_N}\right)\right|^N = \left| \exp\left\{ -\frac{cN}{D_N^\alpha} \frac{\Lambda(D_N)}{\Lambda(D_N)} \Lambda\left(\frac{D_N}{\xi}\right) |\xi|^\alpha \right\} \right| \leq \exp\left\{ -c(\gamma)|\xi|^\gamma \right\} \tag{62}$$

for sufficiently large N. If N is sufficiently large, it is then possible to choose an $\eta > 0$ (and find $\widetilde{c}(\gamma)$), such that

$$\begin{aligned} \int_{B \leq |\xi| < \eta D_N D^{1/\alpha}} \left| p\left(\frac{\xi\tau}{D_N D^{1/\alpha}}\right)\right|^N \mathrm{d}\xi &\leq \int_{B \leq |\xi| < \eta D_N D^{1/\alpha}} \exp\left\{ -\widetilde{c}\left(\frac{\alpha}{2}\right) |\xi|^{\frac{\alpha}{2}} \right\} \mathrm{d}\xi \\ &\leq \int_{|\xi| \geq B} \exp\left\{ -\widetilde{c}\left(\frac{\alpha}{2}\right) |\xi|^{\frac{\alpha}{2}} \right\} \mathrm{d}\xi \end{aligned} \tag{63}$$

and this converges to zero for $B \to \infty$.

The third integral is estimated by noting that $|p(\xi)| < 1$ for $0 < |\xi| < 2\pi/\tau$. Hence, there is a positive constant $c > 0$, such that

$$|p(\xi)| \leq \mathrm{e}^{-c} \tag{64}$$

for $\eta \leq |\xi| \leq \pi$. Consequently, with Equation (50),

$$\int_{\eta \leq \frac{|\xi|}{D_N D^{1/\alpha}} < \pi} \left| p\left(\frac{\xi\tau}{D_N D^{1/\alpha}}\right)\right|^N \mathrm{d}\xi \leq 2\pi \mathrm{e}^{-cN} \left[N\Lambda(N)D\right]^{1/\alpha} \tag{65}$$

converges to zero as $N \to \infty$

Finally, the fourth integral converges to zero, because the characteristic function $\mathrm{h}_\alpha(\xi)$ is integrable on $\mathbb{R}$. In summary, all four terms in Equation (56) vanish for $N \to \infty$, and Equation (46) holds. □

Equation (46) implies

$$p_N(k) \approx \frac{\tau}{D_N D^{1/\alpha}} \mathrm{h}_\alpha\left(\frac{k\tau}{D_N D^{1/\alpha}}\right) \tag{66}$$

for sufficiently large N and all τ. Inserting this into Equation (45) gives

$$\begin{aligned} \mathscr{S}^{-N} &= \sum_{k=1}^{\infty} p_N(k)\ \mathscr{T}^{-k} \approx \sum_{k=1}^{\infty} \frac{\tau}{D_N D^{1/\alpha}} \mathrm{h}_\alpha\left(\frac{k\tau}{D_N D^{1/\alpha}}\right)\ \mathscr{T}^{-k} \\ &= \sum_{k=1}^{\infty} \mathrm{h}_\alpha\left(\frac{k\tau}{D_N D^{1/\alpha}}\right)\ \mathscr{T}^{-k} \frac{[k-(k-1)]\tau}{D_N D^{1/\alpha}}. \end{aligned} \tag{67}$$

For $\alpha = 1$, the average return time $\tau \sum_k kp(k) < \infty$ is proportional to the discretization τ. In the case $0 < \alpha < 1$, the average time $\tau \sum_k kp(k) = \infty$ for return into the set G in a single step diverges. This suggests an infinite rescaling of time as $\tau \to \infty$ for $0 < \alpha < 1$. This rescaling of time combined with $N \to \infty$ was called the ultra-long-time limit in [22]. In the ultra-long-time limit $N \to \infty, \tau \to \infty$ with

$$\lim_{\substack{\tau\to\infty \\ N\to\infty}} \frac{D_N D^{1/\alpha}}{\tau} = \lim_{\substack{\tau\to\infty \\ N\to\infty}} \frac{[N\Lambda(N)D]^{1/\alpha}}{\tau} = h \tag{68}$$

one finds from Equation (67) the result

$$\lim_{\substack{\tau\to\infty, N\to\infty \\ [N\Lambda(N)D]^{1/\alpha}/\tau = h}} \mathscr{S}^{-N} \approx \sum_{k=1}^{\infty} \mathrm{h}_\alpha\left(\frac{k}{h}\right) \mathscr{T}^{-kh} \frac{[k-(k-1)]}{h} \approx \int_0^\infty \mathrm{h}_\alpha(x) \ \mathscr{T}^{-xh} \, \mathrm{d}x \tag{69}$$

for sufficiently large N and τ. The limit gives rise to a family of one-parameter semigroups $\mathscr{T}_\alpha^h$ (with family index α and parameter h) of ultra-long-time evolution operators

$$\lim_{\substack{\tau\to\infty, N\to\infty \\ [N\Lambda(N)D]^{1/\alpha}/\tau = h}} \mathscr{S}^{-N} = \mathscr{T}_\alpha^{-h} = \int_0^\infty \mathrm{h}_\alpha(x) \ \mathscr{T}^{-xh} \, \mathrm{d}x \tag{70}$$

which are convolutions instead of translations. Note that $h \geq 0$ because $D_N \geq 0$ and $D \geq 0$. The rescaled age evolutions $\mathscr{T}_\alpha^{-h}$ are called fractional time evolutions, because their infinitesimal generators are fractional time derivatives [22,23].

The result shows that a proper mathematical formulation of local stationarity requires a generalization of the left-hand side in Equation (1), because Equation (1) assumes implicitly a translation along the orbit. In general, the integration of infinitesimal system changes leads to convolutions instead of just translations along the orbit [22,23]. Of course, translations are a special case of convolutions, to which they reduce in the case when the parameter α approaches unity. For $\alpha \to 1^-$, one finds

$$\mathrm{h}_1(x) = \lim_{\alpha\to 1^-} \mathrm{h}_\alpha(x) = \delta(x-1) \tag{71}$$

and therefore

$$\mathscr{T}_1^{-h} = \int_0^\infty \delta(x-1) \ \mathscr{T}^{-xh} \, \mathrm{d}x = \mathscr{T}^{-h} \tag{72}$$

is a right translation. Here, $h \geq 0$ is an age or duration. This shows that also the special case of induced right translations does not give a group, but only a semigroup.

8. Discussion

The introduction of the sets B_ε of ε-almost invariant BMO-states with $0 \leq \varepsilon \leq \infty$ has provided a mathematical framework in which questions concerning the abundance of time-invariant states and their embedding in the set of all states can be posed mathematically in a proper way. The class of BMO-states reflects in its definition the experimental reality that observations are always performed by integration of experimental data over time intervals. BMO-states allow for singular expectation values, thereby establishing a general framework to discuss Problems 1 and 2 above.

There exists a direct relation between Theorem 3 and the BMO-states. It is given by Equation (32), which directly determines the values of α and D, as well as the function Λ in Theorem 3 and Equation (70).

The result in Equation (70) shows that the left-hand side in a coarse-grained or rescaled version of Equation (1) may not always be a time translation along the orbits of the original unscaled dynamics. Instead, the left-hand side is in general the infinitesimal generator of a convolution along time rescaled orbits of ε-almost invariant states. The orbits of ε-almost invariant states can approach the manifold of invariant states of the physical system or subsystem of interest at every point for any length of time without being trapped.

As discussed above, the result in Equation (70) implies a general concept of time flow and, hence, provides a new perspective on the issue of irreversibility [22,24,25]. It suggest a reformulation [25,26] of the much discussed irreversibility problem. The normal problem can be stated as:

Problem 4 (The normal irreversibility problem)**.** *Assume that time is reversible. Explain how and why time irreversible equations arise in physics.*

The assumption that time is reversible, *i.e.*, $t \in \mathbb{R}$, is made in all fundamental theories of modern physics. The explanation of macroscopically irreversible behavior for macroscopic nonequilibrium states of subsystems is due to Boltzmann. It is based on the applicability of statistical mechanics and thermodynamics, the large separation of scales, the importance of low entropy initial conditions and probabilistic reasoning [27].

The problem with with assuming $t \in \mathbb{R}$ is that an experiment (*i.e.*, the preparation of an initial state within an infinity of η-indistinguishable initial states for a dynamical system) cannot be repeated yesterday, but only tomorrow [25]. While it is possible to translate the spatial position of a physical system forward and backward in space, it is not possible to translate the temporal position of a physical system backwards in time. Translating an experiment backward in time is not the same as reversing the momenta of all particles in a physical system, as emphasized in [25,26]. These observations combined with Equations (71) and (72) suggest to reformulate the normal irreversibility problem above as:

Problem 5 (The reversed irreversibility problem)**.** *Assume that time evolution is always irreversible. Explain why time reversible equations are more frequent in physics.*

The reversed irreversibility problem was introduced in [25]. Its solution is given by Theorem 3 combined with two additional facts. Firstly, ultra-long-time evolutions with $0 < \alpha < 1$ are always irreversible, while those with $\alpha = 1$ may be irreversible or reversible, depending on the operator on the right-hand side of Equation (1). Secondly, the set of recurrence time distributions $p(k)$ in the domain of attraction for the case $\alpha = 1$ comprises all distributions whose first moment $\sum_k kp(k)$ exists, independent of their tail behavior. Contrary to this, the domain of attraction for the case $0 < \alpha < 1$ is restricted to those $p(k)$ with the correct tail behavior. Thus, the domain of attraction is much larger for $\alpha = 1$ than for $0 < \alpha < 1$. This explains why equations of motion with time reversal symmetry arise more frequently.

Because anomalous time evolutions from Equation (70) with $0 < \alpha < 1$ must be expected on theoretical grounds, they are attracting increasing experimental interest [15,28]. For the example of broadband dielectric spectroscopy in glasses, generalized relaxation functions and susceptibilities based on Equation (70) have already been successfully compared to experiments [23,29–32]. Theoretical, mathematical and experimental studies are encouraged to further explore the consequences of the generalized concept.

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