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ON FINITELY RAMIFIED FRACTALS AND THEIR EXTENSIONS

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We construct deterministic fractal lattices using generators with tetrahedral symmetry. From the corresponding master equation we determine the spectral dimension $\tilde{d}$ and prove that $\tilde{d}<2$. Furthermore we extend our set of fractals (with $\tilde{d}$ dense in [1,2]) by direct multiplication, thus obtaining fractals whose $\tilde{d}$ are dense in $[1,\infty[$.

It was suggested that many disordered media are fractal structures.[1] This motivates us to study deterministic fractal lattices by investigating random walks on them. We concentrate on lattices with finite order of ramification.[1,2] These are characterized by the property that the elimination of a preassigned finite number of lattice bonds is sufficient to isolate an arbitrarily large compact subset of the infinite structure.

We start from the class of finitely ramified fractals having tetrahedral symmetry whose best known representative is the Sierpinski-gasket. We briefly describe their construction and give their fractal (Hausdorff-) dimensions. Analysing the master equation we then turn to their spectral (fracton) dimension. Finally we consider also a class of infinitely ramified fractals obtained by direct multiplication.

Deterministic fractal lattices are completely described through a geometrical generator and an iteration prescription. Figure 1 shows a variety of two-dimensional (d=2) generators whose side-length is called b. The Sierpinski-gasket corresponds to the special case b=2. In Figure 2 we exemplify one step of the iterative construction for the generator with b=5 of Figure 1. Infinite repetition yields the full fractal lattice. A finite number of iterations will be called a stage-n-structure, where n=1 corresponds to the generator.

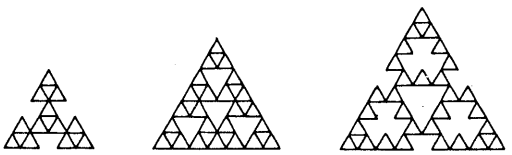

FIGURE 1
Three generators for finitely ramified fractals

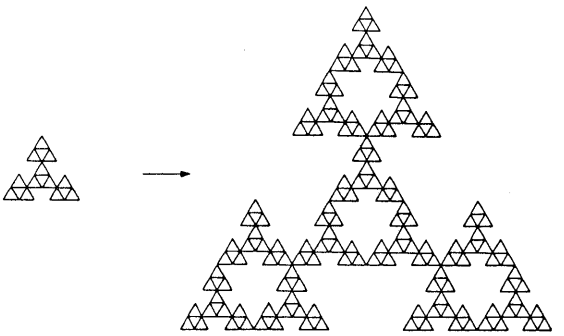

FIGURE 2
One step in the iterative construction of the fractal

We now turn to the Hausdorff-dimension $\bar{d}$ of the fractal[1] which may be expressed as $b^{\bar{d}} = \lim_{n\to\infty} N(n+1)/N(n)$ where N(n) is the number of lattice points in a stage-n-structure. From the relation $N(n+1) = N(1) + N[N(n)-d-1]$ we have

$$\bar{d} = \ln N/\ln b \qquad (1)$$

where N denotes the number of upward pointing triangles (resp. hypertetrahedrons) inside a generator. Note that the number E(n) of edges in a stage-n-structure obeys the recursion relation

$$E(n+1) = N\,E(n) \qquad (2)$$

which allows to determine $\bar{d}$ through E(n).

To obtain the spectral dimension $\tilde{d}$, we use the master equation describing a nearest neighbour random walk on the lattice. Suppose the walker starts at site $\vec{r}_0$ at time t=0. The probability $P(\vec{r}_i,t)$ to find the walker at site $\vec{r}_i$ at time t obeys the master equation

$$\frac{d}{dt} P(\vec{r}_i,t) = \sum_{j(i)} [w_{ij}P(\vec{r}_j,t) - w_{ji}P(\vec{r}_i,t)] \qquad (3a)$$

where the sum runs over all $\vec{r}_j$ that are nearest neighbours to $\vec{r}_i$. We specify the transition rates w_{ij} from $\vec{r}_j$ to $\vec{r}_i$ through

$$z(\vec{r}_j)\, w_{ij} = w = \text{const} \qquad (3b)$$

where w is a constant rate and $z(\vec{r}_j)$ is the number of nearest neighbours of $\vec{r}_j$. Laplace-transforming Eq.(3) with initial condition $P(\vec{r}_i,0) = \delta_{\vec{r}_i,\vec{r}_0}$ gives

$$(1-\alpha)P(\vec{r}_i,u) - \sum_{j(i)} P(\vec{r}_j,u)/z(\vec{r}_j) = \delta_{\vec{r}_i,\vec{r}_0}/w \qquad (4)$$

where we have set $\alpha=-u/w$. Due to the self-similarity of our lattices Eqs.(4) may be solved via a decimation procedure which inverts the iterative construction. Only fractals with connected generators will be considered here. For these every point of the generator can be reached from any other point through a succession of bonds in which two consecutive bonds have one point in common.

To be specific we write Eqs.(4) restricted to a single generator whose corners $\vec{s}_0,\vec{s}_1,\ldots,\vec{s}_d$ survive the decimation step. The deleted interior sites are labelled $\vec{d}_1,\ldots,\vec{d}_M$ (M=N(1)-d-1) beginning with the d nearest neighbours of $\vec{s}_0$ such that $\vec{s}_0,\vec{d}_i$ and $\vec{s}_i$ are collinear ($1\le i\le d$). Introducing the vectors $\vec{Q}_1=(Q(\vec{d}_1,u),\ldots,Q(\vec{d}_M,u))$ and $\vec{Q}_2=(Q(\vec{s}_0,u),\ldots,Q(\vec{s}_d,u))$ with $Q(\vec{r}_i,u)=P(\vec{r}_i,u)/z(\vec{r}_i)$ we get for the single generator

$$[(1-\alpha)D - A_1]\vec{Q}_1 = A_2\vec{Q}_2 \quad . \qquad (5)$$

Here D is the diagonal M×M-matrix given by $(D)_{ii} = z(\vec{d}_i)$. The matrices A_1 and A_2 are submatrices of the adjacency matrix A of the generator. The M×M-matrix A_1 is obtained from A by eliminating the rows and columns corresponding to the corners, while the M×(d+1)-matrix A_2 results from elimination of d+1 rows corresponding to the corners and M columns corresponding to interior sites.

Before proceeding with the decimation we have to analyse the invertibility of the matrix $(1-\alpha)D-A_1$. First we note that the matrix $D-A_1$ is diagonally dominant, i.e. that we have $|(D-A_1)_{ii}| \ge \sum_{j\neq i}^{M} |(D-A_1)_{ij}|$ for all $1\le i\le M$. This can be seen as follows: The element $(A_1)_{ij}$ equals 1 or 0 depending on whether the point $\vec{d}_i$ is connected to $\vec{d}_j$ or not. Thus $|(D-A_1)_{ii}| = z(\vec{d}_i)$ and $\sum_{j\neq i}^{M} |(D-A_1)_{ij}| = \sum_{j\neq i}^{M} (A_1)_{ij}$. Moreover the row sums $\sum_{j\neq i}(A_1)_{ij}$ give the number of nearest neighbours of $\vec{d}_i$ which are also interior points of the generator and hence equal at most $z(\vec{d}_i)$. In addition $D-A_1$ is irreducible which means that there is no permutation matrix B such that $B(D-A_1)B^{-1}$ reduces to block form. This is equivalent to the connectedness of the generator.[3] Knowing that $(D-A_1)$ is an irreducibly diagonally dominant matrix with positive diagonal elements and nonpositive off-diagonal elements, we infer from the theory of nonnegative matrices that its inverse $(D-A_1)^{-1}$ exists and is elementwise positive.[3] Furthermore, since for $u\ge 0$ we have $\alpha\le 0$, it follows that $1-\alpha \ge 1$ and therefore also $[(1-\alpha)D-A_1]^{-1}$ exists and fulfills

$$([(1-\alpha)D-A_1]^{-1})_{ij} > 0 \qquad (6)$$

for all i,j and $\alpha \le 0$.

We can now solve Eq. (5) for one of the nearest neighbours of $\vec{s}_0$, say $\vec{d}_1$, to obtain

$$Q(\vec{d}_1,u) = g(\alpha)Q(\vec{s}_0,u)+\sum_{i=1}^{d} h_i(\alpha)Q(\vec{s}_i,u) \qquad (7a)$$

with

$$g(\alpha) = ([(1-\alpha)D-A_1]^{-1}A_2)_{11} \qquad (7b)$$

and

$$h_i(\alpha) = ([(1-\alpha)D-A_1]^{-1}A_2)_{1,i+1} \qquad (7c)$$

Using the rotational symmetry of the tetrahedral generator the same result obtains for all nearest neighbours of $\vec{s}_0$. The calculation is then repeated for the l generators to which $\vec{s}_0$ belongs. We now write Eq.(4) for $P(\vec{s}_0,u)$ as

$$-\delta_{\vec{r}_0,\vec{s}_0}/w+ld(1-\alpha)Q(\vec{s}_0,u) = \sum_{k=1}^{l}\sum_{i=1}^{d} Q(\vec{d}_i^k,u) \qquad (8)$$

where the different generators are distinguished by upper indices k, $1\le k\le l$. For the $Q(\vec{d}_i^k,u)$ we insert the results from Eq.(7) and get

$$-\delta_{\vec{r}_0,\vec{s}_0}/[wh(\alpha)]+ld(1-\phi(\alpha))Q(\vec{s}_0,u) = \sum_{k=1}^{l}\sum_{i=1}^{d} Q(\vec{s}_i^k,u) \qquad (9)$$

with $\phi(\alpha)=1-\{[1-\alpha-g(\alpha)]/h(\alpha)\}$ and $h(\alpha)=\Sigma_{i=1}^{d} h_i(\alpha)$. Eq.(9) involves only $\vec{s}_0$ and the corners of adjacent generators, thus completing the decimation.

We proceed to show that Eqs.(8) and (9) for $\vec{s}_0$ in the full resp. decimated lattice are indeed identical in the limit $\alpha\to 0$. Consider a stage-n-structure for large n. The stationary solution ($t\to\infty$) is then

$$Q(\vec{r}_i) = P(\vec{r}_i)/z(\vec{r}_i) = \text{const} \qquad (10a)$$

as can be seen from Eq.(3). The constant follows from conservation of probability $\Sigma_{i=1}^{N(n)}P(\vec{r}_i,t) = 1$,

$$\text{const} = 1/\Sigma_{i=1}^{N(n)}z(\vec{r}_i). \qquad (10b)$$

Inserting (10a) into Eq.(7) gives $g(0)+h(0) = 1$ which in turn means that $\phi(0)=0$ via Eq.(9). With $\Sigma_{i=1}^{N(n)}z(\vec{r}_i) = 2\,E(n)$ where E(n) is the number of edges in the stage-n-structure we insert Eq.(10) into Eq.(8) for $\vec{r}_0=\vec{s}_0$ to obtain

$$-1/w+ld(1-\alpha)/2E(n) = ld/2E(n)$$

valid for $n\to\infty$ and $\alpha\to 0$. After one decimation step this becomes Eq.(9)

$$-1/[wh(\alpha)]+ld[1-\phi(\alpha)]/2E(n-1) = ld/2E(n-1)$$

Eliminating w from these equations we get $h(\alpha) = \alpha N/\phi(\alpha)$ and after taking the limit $\alpha\to 0$ we are left with

$$h(0) = N/\kappa \qquad (11)$$

where $\kappa=\phi'(0)$. This equality has been called fractal Einstein relation.[4] Consider now Eq.(9) for α around 0 where we have $\phi(\alpha)\cong\kappa\alpha$. In this region Eq.(9) for the renormalized quantities $NQ(\vec{s}_i,u)$ and rates w/κ is identical to Eq.(8) for the original quantities Q and w.

We now determine the spectral (fracton) dimension $\tilde{d}$ which for random walks follows from the probability to be at the origin $P(\vec{r}_0,t)$. For longer times one has[5]

$$P(\vec{r}_0,t) \sim (1/wt)^{\tilde{d}/2}$$

which after decimation reads

$$N\,P(\vec{r}_0,t) \sim (\kappa/wt)^{\tilde{d}/2}$$

and thus

$$\tilde{d} = 2\ln N/\ln\kappa = 2(1-\ln h(0)/\ln N)^{-1} \qquad (12)$$

where Eq.(11) was used for the second equality. In previous works we have evaluated $\tilde{d}$ explicitly for many structures[6,7] and we have also shown how to construct fractals for prescribed $\tilde{d}$-values,[7] $1\le\tilde{d}\le 2$, densely filling the interval [1,2].

Here we note that for the above fractals $\tilde{d}<2$. From inequality (6) plus the fact that A_2 is nonnegative we have, using Eq.(7), $g(\alpha) > 0$ and $h(\alpha) > 0$ for all α. Since $g(0)+h(0)=1$ it follows that $g(0),h(0)<1$. Thus $-\infty < \ln h(0) < 0$, while $\ln N > 0$ because of $N > 1$. This implies $(1-\ln h(0)/\ln N)^{-1} < 1$ and from Eq.(12) therefore $\tilde{d} < 2$.

Finally we extend the types of fractals by direct multiplication and thereby provide a dense set of $\tilde{d}$-values in $[1,\infty[$. As an example we show in Figure 3 the stage 2 result of multiplying a Sierpinski-gasket with a one-dimensional lattice. We call this the "Toblerone"-lattice.[8] Its spectral dimension is obtained from the low-frequency behaviour of its eigenmodes. In this model one envisages the lattice

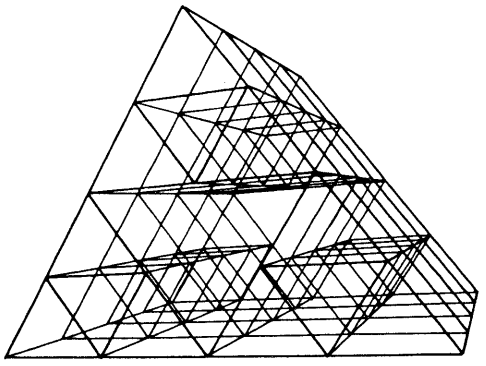

FIGURE 3
The Toblerone lattice: direct product of a Sierpinski-gasket with a linear chain

sites as occupied by masses $mz(\vec{r}_i)$ connected along the bonds through springs of strength f. For low frequencies ω the density of states obeys[5,9] $\rho(\omega) \sim \omega^{\tilde{d}-1}$. The equations of motion for the Fourier-transformed displacements $P(\vec{r}_i,\omega)$ are just Eqs.(4) without the inhomogeneity. Furthermore, α corresponds to $\alpha = m\omega^2/f$. Writing explicitly $\vec{r}_i=(x_i,y_i,z_i)$ we have

$$-\alpha P(x_i,y_i,z_i,\omega) = \sum_{j(i)} [P(x_j,y_j,z_j,\omega)-P(x_i,y_i,z_i,\omega)] \quad (13)$$

where we note that there are six nearest neighbours, $z(\vec{r}_i)=6$, on the Toblerone lattice. We can split the summation over nearest neighbours in Eq.(13) into a linear and a Sierpinski part as

$$\alpha P(x_i,y_i,z_i) = 2P(x_i,y_i,z_i) - P(x_i,y_i,z_i+1) - P(x_i,y_i,z_i-1) + \sum_{j(i)} [P(x_i,y_i,z_i) - P(x_j,y_j,z_j)] \quad (14)$$

where we assumed the lattice spacing to be one and have suppressed the ω-dependence for notational ease. Fourier-Transformation with respect to z, $P(x,y,k) = \Sigma_z e^{ikz} P(x,y,z)$, yields

$$(\alpha-2+2\cos k)\, P(x_i,y_i,k) = \sum_{j(i)} [P(x_i,y_i,k)-P(x_j,y_j,k)] \quad (15)$$

For fixed k Eqs.(15) are exactly the equations for the Sierpinski-gasket if we take as spectral parameter $\alpha_k = \alpha-2(1-\cos k)$. Since we are interested in the long wavelength limit $k\to 0$ we expand $\alpha_k \cong \alpha-k^2$. Thus from $\alpha = m\omega_T^2/f$, $\alpha_k = m\omega_S^2/f$ and $k^2 = m\omega_C^2/f$ we have $\omega_T^2 = \omega_S^2 + \omega_C^2$ where subscripts T,C or S refer to the Toblerone lattice, the chain or the Sierpinski gasket. To compute $\rho_T(\omega)$ we now count the number of modes with frequencies less than ω, $N_T(\omega) = \int_0^\omega \rho_T(\omega')d\omega'$. Since Eqs.(15) uncouple with respect to k we obtain $N_T(\omega)$ by summing $\rho_S(\omega')d\omega'\rho_C(\omega'')d\omega''$ subject to the condition $\omega'^2 + \omega''^2 \leq \omega^2$. Using $\rho_S(\omega) \sim \omega^{\tilde{d}_S-1}$ and $\rho_C(\omega) \sim$ const we get

$$N_T(\omega) = \iint_{\omega'^2+\omega''^2\leq\omega^2} \rho_S(\omega')\rho_C(\omega'')d\omega'd\omega'' \sim \omega^{\tilde{d}_S+1}$$

Differentiation yields $\rho_T(\omega) \sim \omega^{\tilde{d}_S}$ and therefore $\tilde{d}_T = \tilde{d}_S+1$ for the spectral dimension of the Toblerone lattice. Evidently one obtains higher dimensions $\tilde{d}$ by multiplying our fractals with higher dimensional regular lattices. The obtainable values are dense in $[1,\infty[$.

Summarizing we have concentrated here on general properties of finitely ramified fractals. For a general class we have shown that $\tilde{d} < 2$. In addition we have indicated how to build infinitely ramified fractals whose $\tilde{d}$-values are larger than 2 and form a dense set. -amdgl-

ACKNOWLEDGEMENT

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