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Old Problems and New Solutions for Multiphase Flow in Porous Media

by

Helge Besserer [1] and Rudolf Hilfer [1,2]

[1] *Institut für Computeranwendungen, ICA-1, Universität Stuttgart, Pfaffenwaldring 27, D-70569 Stuttgart, Germany*

[2] *Institut für Physik, Universität Mainz, D-55099 Mainz, Germany*

Abstract

The existing macroscopic equations of motion for multiphase flow in porous media are unsatisfactory in two general respects. On the one hand characteristic experimental features, such as relationships between capillary pressure and saturations, cannot be predicted. On the other hand the theoretical derivation of the equations from the well-known laws of hydrodynamics has not yet been accomplished. In this paper we discuss these deficiencies and present an alternative description which is based on energy balances. Our description includes surface tensions as parameters and interface areas as a new macroscopic state variable. The equations are obtained from general multiphase mixture theory by explicitly accounting for the pairwise character of interfacial energies. For the special case of two immiscible fluids in a porous medium the most important ingredient is the distinction between a connected and a disconnected subphase of each fluid phase. In this way it becomes possible to handle also the spatiotemporal variation of residual saturations. The connection between the new approach and the established formulation is given by identifying a generalized Darcy Law with generalized relative permeabilities. The new equations reproduce qualitatively the saturation dependent behaviour of capillary pressure in gravitational equilibrium.

1 Introduction

The theoretical understanding of displacement processes between incompressible fluids in porous rock is of great interest in fields such as oil reservoir engineering or environmental physics [15], [14], [18]. The established multiphase flow equations based on generalizations of Darcy's Law are known to be incomplete [11], [20]. In Section 2 we discuss the main approaches within which macroscopic equations of motion can be derived and we exhibit some of their problems.

In addition to their theoretical difficulties the known two-phase flow equations describe only inadequately some of the essential features of experiments on fluid displacements in porous rock [10], [2], [15], [16]. We mention relative permeabilities and capillary pressure which are treated as material parameter and parameter functions respectively although they are nonunique and hysteretic functions of the state variables [4], [25]. Within the traditional approach the residual saturations are constant parameters although experiments have shown that they vary in space and time and depend on the flow velocity [3],[19], [30]. Surface areas and surface tensions which are known to control capillary action and wetting properties do not appear in the mathematical formulation of the traditional theory [7].

In Section 3 we explain an alternative approach [16], [17], [7] which is based on energy balances within the theory of multiphase mixtures with volume fractions. The traditional theory includes only mass and momentum balance but fails to incorporate the energy balance. The interfacial energy depends on the interfacial area and interface tension and hence the set of macroscopic observables describing the state of the mixture must be enlarged to include surface areas and surface tensions. Hereby the elementary driving forces of capillary action are explicitly taken into account. This also means that the dependence of the results on wetting properties can be expressed through interface areas instead of phenomenological relative permeabilities.

In Section 4 we show that with some simple assumptions the traditional theory can be recovered from our new approach. For gravitational equilibrium the new equations predict relationships between capillary pressure and fluid saturations which are in qualitative agreement with experimental results.

2 Problems with Macroscopic Approaches

A tool for axiomatically deriving flow equations in porous media is furnished by the theory of immiscible mixtures (see for example [5]). Each phase α of the mixture is attributed a mass density $\rho_\alpha(\boldsymbol{x},t)$ and velocity field $\boldsymbol{v}_\alpha(\boldsymbol{x},t)$, where $\boldsymbol{x}$ and t are macroscopic space and time variables respectively. The general global form of the mass balance for phase α is then given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_\alpha} \rho_\alpha \mathrm{d}V_\alpha = \int\limits_{V} M_\alpha \mathrm{d}V \quad , \tag{1}$$

where M_α denotes mass transfer of all phases into phase α. The integral on the left-hand side is defined over the domain exclusively occupied by phase α with partial volume V_α. Introducing volume fractions $\phi_\alpha := \mathrm{d}V_\alpha/\mathrm{d}V$ with $\sum\limits_\alpha \phi_\alpha = 1$, the partial volume integral can be changed to an integral over the macroscopic mixture volume V. Then from (1) we get the local form

$$\frac{\mathrm{D}^\alpha}{\mathrm{D}t}(\phi_\alpha\rho_\alpha) + \phi_\alpha\rho_\alpha\nabla\cdot\boldsymbol{v}_\alpha = M_\alpha \quad , \tag{2}$$

where $\mathrm{D}^\alpha/\mathrm{D}t := \partial/\partial\, t + \boldsymbol{v}_\alpha\cdot\nabla$ denotes the material derivative following phase α.

The general global form of the momentum balance for phase α is written

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{V_\alpha} \rho_\alpha\boldsymbol{v}_\alpha \mathrm{d}V_\alpha - \int\limits_{S_\alpha} \mathbf{T}_\alpha\cdot\boldsymbol{n}_\alpha \mathrm{d}S_\alpha - \int\limits_{V_\alpha} \rho_\alpha\boldsymbol{b}_\alpha \mathrm{d}V_\alpha = \int\limits_{V} \boldsymbol{m}_\alpha \mathrm{d}V \quad , \tag{3}$$

where $\boldsymbol{b}_\alpha$ denotes an external momentum supply and the internal contact forces of phase α are expressed by a Cauchy stress tensor $\mathbf{T}_\alpha$. The term $\boldsymbol{m}_\alpha$ on the right side stands for the momentum transfer into phase α from all other phases. The left side of the equation contains partial volume integrals and a partial surface integral which is defined over the boundary S_α of the α-domain, with normal unit vector $\boldsymbol{n}_\alpha$. Using (2) we can write

$$\int\limits_V \left(\phi_\alpha \rho_\alpha \frac{\mathrm{D}^\alpha}{\mathrm{D}t} \boldsymbol{v}_\alpha\right) \mathrm{d}V - \int\limits_{S_\alpha} \mathbf{T}_\alpha \cdot \boldsymbol{n}_\alpha \mathrm{d}S_\alpha - \int\limits_V \phi_\alpha \rho_\alpha \boldsymbol{b}_\alpha \mathrm{d}V = \int\limits_V \left(\boldsymbol{m}_\alpha - M_\alpha \boldsymbol{v}_\alpha\right) \mathrm{d}V \quad . \tag{4}$$

Transformation of the momentum balance (4) into a local equation needs supplementary considerations because the surface integral is related to the partial surface S_α, whereas the only microinformation entering the macroscopic theory are the volume fractions. It is continuingly debated in volume fraction theories on how to meet that integral. Actually there are two competing ideas. One of them is to apply the Gaussian theorem independently in each partial domain and then to continue with volume fractions,

$$\int\limits_{S_\alpha} \mathbf{T}_\alpha \cdot \boldsymbol{n}_\alpha \mathrm{d}S_\alpha = \int\limits_{V_\alpha} \left(\nabla \cdot \mathbf{T}_\alpha\right) \mathrm{d}V_\alpha = \int\limits_V \phi_\alpha \left(\nabla \cdot \mathbf{T}_\alpha\right) \mathrm{d}V \quad . \tag{5}$$

Using (5) all integration domains in Eqn. (4) become equal and we can go over to the local form

$$\phi_\alpha \rho_\alpha \frac{\mathrm{D}^\alpha}{\mathrm{D}t} \boldsymbol{v}_\alpha - \phi_\alpha \nabla \cdot \mathbf{T}_\alpha - \phi_\alpha \rho_\alpha \boldsymbol{b}_\alpha = \boldsymbol{m}_\alpha - M_\alpha \boldsymbol{v}_\alpha \quad . \tag{6}$$

Decomposing the stress tensor into spherical and deviatoric parts,

$$\mathbf{T}_\alpha = -P_\alpha \mathbf{1} + \mathbf{D}(\mathbf{T}_\alpha) \quad , \tag{7}$$

the pressure term $-\phi_\alpha \nabla P_\alpha$ appears in the macroscopic momentum balance. (See for example the works of Murray on a theory of fluidized beds with compressible fluid and elastic particles [22], [23] or the works of Soo on two-phase flow with chemically reacting fluids [28], [29].)

The other community in this controversy defines an "effective" stress tensor $\mathbf{T}_\alpha^E$ by assuming

$$\int\limits_{S_\alpha} \mathbf{T}_\alpha \cdot \boldsymbol{n}_\alpha \mathrm{d}S_\alpha = \int\limits_{S} \mathbf{T}_\alpha^E \cdot \boldsymbol{n} \mathrm{d}S = \int\limits_{V} \nabla \cdot \mathbf{T}_\alpha^E \mathrm{d}V \quad . \tag{8}$$

This results in a different local momentum equation,

$$\phi_\alpha \rho_\alpha \frac{\mathrm{D}^\alpha}{\mathrm{D}t} \boldsymbol{v}_\alpha - \nabla \cdot \mathbf{T}_\alpha^E - \phi_\alpha \rho_\alpha \boldsymbol{b}_\alpha = \boldsymbol{m}_\alpha - M_\alpha \boldsymbol{v}_\alpha \quad . \tag{9}$$

(Compare for example investigations [21] on fluid saturated porous media with compressible fluid and elastic solid by Morland.) The pressure term introduced by Eqn. (9) has caused heated discussions, see for example [5], [8].

Finally there are authors who try to find a compromise as for example in [26] and [27]. They assumed a pressure term $-\phi_\alpha \nabla P + (1 - \beta_\alpha) P \nabla \phi_\alpha$ and introduced β_α as a constitutive function.

Let us turn to the more special problem of incompressible Newtonian fluids flowing in rigid porous rock. To present the established equations we give the ideas of Trangenstein [1]. He decides in favour of momentum balance (6). For single-phase flow he ignores the fluid's inertial force $\rho(\mathrm{D}\boldsymbol{v}/\mathrm{D}t)$ and shear stresses $\mathbf{D}(\mathbf{T})$ in the momentum balance and assumes a simple drag force for momentum transfer between rock and fluid,

$$\boldsymbol{m} = -\frac{\mu\phi}{K} \boldsymbol{v} \quad , \tag{10}$$

where μ is the viscosity of the fluid, ϕ the porosity and K the absolute permeability of the rock. Further he excludes chemical reactions between rock and fluid setting $M = 0$ and identifies the external body force with gravity, $\boldsymbol{b} = \boldsymbol{g}$. Altogether this [1] yields the famous Darcy Law

$$\boldsymbol{v} = -\frac{K}{\mu} \left(\nabla P - \rho \boldsymbol{g} \right) \quad . \tag{11}$$

Multiphase flow equations are then obtained by extending the single-phase formula. To generalize Eqn. (11) to multiphase systems one must assume that the flow of either fluid is hydrodynamically independent of the other fluids [11]. So-called relative permeabilities $k_{r\alpha}$ account for the fact that the flow medium and its permeability for each fluid phase α is altered by the presence of all other phases. Based on this idea Eqn. (11) is generalized to the velocity field $\boldsymbol{v}_\alpha$ of each phase α giving

$$\boldsymbol{v}_\alpha = -\frac{k_{r\alpha} K}{\mu_\alpha} \left(\nabla P_\alpha - \rho_\alpha \boldsymbol{g} \right) \quad . \tag{12}$$

In most applications the relative permeabilities are treated as functions of saturations $S_\alpha := \phi_\alpha / \phi$. In principle a more complicated dependence on physical quantities like surface tensions, contact angles, viscosities, mass densities, the pore structure, and flow conditions has to be considered [20]. Experiments

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[1] Actually the derivation in [1] contains two mistakes. The volume fraction ϕ_α in the pressure term arising through the transition from global to local equations is missed and the volume density of gravity is incorrectly introduced as $\boldsymbol{g}$ instead of $\phi \boldsymbol{g}$. This leads to the erroneous form $\boldsymbol{v} = -K \phi^{-1} \mu^{-1} \left(\nabla P - \rho \boldsymbol{g} \right)$. In consequence the phase velocity $\boldsymbol{v}$ and the so-called Darcy velocity $\overline{\boldsymbol{v}} = \phi \boldsymbol{v}$ are exchanged by mistake.

display hysteretic saturation dependence [11] and a strong influence of characteristic flow regimes [3], which is in clear contradiction to the assumed dependence on saturation alone.

In summary we have seen that the existing axiomatic approaches contain many ambiguities. Recent work [16], [17], [7] on two-phase flow of incompressible fluids through porous solids favour an approach after Eqn. (8), that we shall explain in the next section.

3 Energy Balance Approach

Let us consider two immiscible fluids, such as water and oil, flowing in a solid matrix, such as porous rock. We use the indices $\mathcal{W}$, $\mathcal{O}$, $\mathcal{R}$ for the three phases and denote the surface tension between phases α, $\beta \in \{\mathcal{W}, \mathcal{O}, \mathcal{R}\}$ by $\sigma_{\alpha\beta}$ and their respective interface area per unit volume by $A_{\alpha\beta}$. As another ingredient we distinguish between connected and disconnected fluid phase by dividing each fluid phase into a percolating (connected) and trapped (disconnected) subphase. A subphase is called connected if its interior can be connected to the boundaries of the sample by a path within the subphase. The disconnected subphase may be viewed as immobile droplets and ganglia. It is possible to handle the dynamics of residual phase saturations through the immobile subphases. For further kinematic details see [17].

The subdivision of $\mathcal{W}$ and $\mathcal{O}$ into connected and disconnected subphases results in a total of five phases (four fluid, one solid) with indices as shown in Table 1. To each phase we attribute a velocity $\boldsymbol{v}_\alpha(\boldsymbol{x}, t)$, $\alpha \in \{1, 2, 3, 4, 5\}$, and each phase occupies a volume fraction $\phi_\alpha(\boldsymbol{x}, t)$ with

$$\sum_{\alpha=1}^{4} \phi_\alpha = 1 - \phi_5 = \phi \quad , \tag{13}$$

where ϕ is the porosity of the rock. The velocities of the immobile subphases and of the rock phase are set to zero, $\boldsymbol{v}_2 = \boldsymbol{v}_4 = \boldsymbol{v}_5 \equiv 0$. Mass exchange occurs only between trapped and percolating subphases of the same fluid substance. Chemical reactions are excluded. Hence we specify $M_1 = -M_2 =: M_{\mathcal{W}}$ and $M_3 = -M_4 =: M_{\mathcal{O}}$. The local form of mass conservation (2) with given constant mass densities $\rho_{\mathcal{W}}$, $\rho_{\mathcal{O}}$ and $\rho_{\mathcal{R}}$ yields four equations

$$\rho_{\mathcal{W}} \frac{\partial}{\partial t} \phi_1 + \rho_{\mathcal{W}} \boldsymbol{\nabla} \cdot (\phi_1 \boldsymbol{v}_1) = M_{\mathcal{W}} \tag{14}$$

$$\rho_{\mathcal{W}} \frac{\partial}{\partial t} \phi_2 = -M_{\mathcal{W}} \tag{15}$$

$$\rho_{\mathcal{O}} \frac{\partial}{\partial t} \phi_3 + \rho_{\mathcal{O}} \boldsymbol{\nabla} \cdot (\phi_3 \boldsymbol{v}_3) = M_{\mathcal{O}} \tag{16}$$

$$\rho_{\mathcal{O}} \frac{\partial}{\partial t} \phi_4 = -M_{\mathcal{O}} \quad , \tag{17}$$

while the corresponding equation for the rock phase turns out to be trivial.

Let us continue with the momentum balance. The stress tensor is specified by two assumptions. Shear stresses are neglected which leads to a spherical tensor, and we follow the idea of an effective stress tensor as discussed in section 2. Thus we specify the stress tensor as

$$\mathbf{T}_\alpha^E = -\phi_\alpha P_\alpha \mathbf{1} \quad . \tag{18}$$

The macroscopic pressure fields of the disconnected fluid phases are not exptected to be continuous when obtained by averaging or homogenization techniques. Hence only the momentum equations for the connected phases

$\alpha = 1$ and $\alpha = 3$ are considered. Moreover inertial forces are neglected. Eqn. (9) together with (18) then gives the equations

$$\boldsymbol{\nabla}(\phi_1 P_1) - \phi_1 \rho_{\mathcal{W}} \boldsymbol{g} = \boldsymbol{m}_1 - M_{\mathcal{W}} \boldsymbol{v}_1 \quad , \tag{19}$$

$$\boldsymbol{\nabla}(\phi_3 P_3) - \phi_3 \rho_{\mathcal{O}} \boldsymbol{g} = \boldsymbol{m}_3 - M_{\mathcal{O}} \boldsymbol{v}_3 \quad . \tag{20}$$

Our energy balance contains kinetic and interfacial energies while thermal effects are ignored. The internal interfacial energy densities are given as $\sigma_{\alpha\beta} A_{\alpha\beta}$. The total energy content for the phase α and all adjacent phases β with $A_{\alpha\beta} \neq 0$ reads

$$\Psi_\alpha = \frac{1}{2}\phi_\alpha \rho_\alpha \boldsymbol{v}_\alpha^2 + \frac{1}{2} \sum_{\substack{\beta \\ A_{\alpha\beta}\neq 0}} \phi_\beta \rho_\beta \boldsymbol{v}_\beta^2 + \sum_{\substack{\beta \\ A_{\alpha\beta}\neq 0}} \sigma_{\alpha\beta} A_{\alpha\beta} \quad . \tag{21}$$

We allow for energy exchanges between the phases with E_α denoting the energy transfer into phase α from the nearest surrounding phases. A flux of surface energy can be ignored on the macroscopic scale and external energy supply is excluded. Hence the balance equation for the general energy sum (21) reads

$$\begin{aligned} & \boldsymbol{\nabla} \cdot \Big(\sum_{\substack{\beta \\ A_{\alpha\beta}\neq 0}} \sigma_{\alpha\beta} A_{\alpha\beta} \boldsymbol{v}_\beta \Big) + \frac{\partial}{\partial t} \sum_{\substack{\beta \\ A_{\alpha\beta}\neq 0}} \sigma_{\alpha\beta} A_{\alpha\beta} \\ = \quad & \frac{1}{2} M_\alpha \boldsymbol{v}_\alpha^2 + \frac{1}{2} \sum_{\substack{\beta \\ A_{\alpha\beta}\neq 0}} M_\beta \boldsymbol{v}_\beta^2 - \boldsymbol{m}_\alpha \cdot \boldsymbol{v}_\alpha - \sum_{\substack{\beta \\ A_{\alpha\beta}\neq 0}} \boldsymbol{m}_\beta \cdot \boldsymbol{v}_\beta \\ & -\phi_\alpha P_\alpha \boldsymbol{\nabla} \cdot \boldsymbol{v}_\alpha - \sum_{\substack{\beta \\ A_{\alpha\beta}\neq 0}} \phi_\beta P_\beta \boldsymbol{\nabla} \cdot \boldsymbol{v}_\beta + E_\alpha + \sum_{\substack{\beta \\ A_{\alpha\beta}\neq 0}} E_\beta \quad , \end{aligned} \tag{22}$$

where the equations of continuity and momentum have been inserted. A detailed derivation of Eqn. (22) within the fundamental postulates of mixture theory is given in [7]. Before continuing with the energy balance for the individual phases we want to give some further specifications. The interfacial area between a disconnected subphase and its superphase vanishes and the same holds for the contact between two disconnected subphases. Hence $A_{12} = A_{34} = A_{24} \equiv 0$. Of course $A_{\alpha\alpha} = 0$ and $A_{\alpha\beta} = A_{\beta\alpha}$ also holds. Moreover we assume constant wetting properties which means that the individual interface areas with the rock are constant,

$$A_{15}(\boldsymbol{x}, t) = A_{1\mathcal{R}} = \text{const.} \tag{23}$$

$$A_{25}(\boldsymbol{x}, t) = A_{2\mathcal{R}} = \text{const.} \tag{24}$$

$$A_{35}(\boldsymbol{x}, t) = A_{3\mathcal{R}} = \text{const.} \tag{25}$$

$$A_{45}(\boldsymbol{x}, t) = A_{4\mathcal{R}} = \text{const.} \qquad . \tag{26}$$

The rock phase is assumed to be a perfectly rigid body and it will dissipate a certain amount of energy. Hence we evaluate the general energy balance (22) for the four fluid phases leading to the following equations:

Phase 1 – Connected Water ($\alpha = 1$, $\beta = 3, 4, 5$)

$$\begin{aligned} & \boldsymbol{\nabla} \cdot (\sigma_{\mathcal{WO}} A_{13} \boldsymbol{v}_3) + \frac{\partial}{\partial t} (\sigma_{\mathcal{WO}} A_{13} + \sigma_{\mathcal{WO}} A_{14} + \sigma_{\mathcal{WR}} A_{15}) \\ = \; & \frac{1}{2} M_{\mathcal{W}} \boldsymbol{v}_1^2 + \frac{1}{2} M_{\mathcal{O}} \boldsymbol{v}_3^2 - \boldsymbol{m}_1 \cdot \boldsymbol{v}_1 - \boldsymbol{m}_3 \cdot \boldsymbol{v}_3 - \phi_1 P_1 \boldsymbol{\nabla} \cdot \boldsymbol{v}_1 - \phi_3 P_3 \boldsymbol{\nabla} \cdot \boldsymbol{v}_3 \\ & + E_1 + E_3 + E_4 + E_5 \end{aligned} \tag{27}$$

Phase 2 – Disconnected Water ($\alpha = 2$, $\beta = 3, 5$)

$$\boldsymbol{\nabla} \cdot (\sigma_{\mathcal{WO}} A_{23} \boldsymbol{v}_3) + \frac{\partial}{\partial t} (\sigma_{\mathcal{WO}} A_{23} + \sigma_{\mathcal{WR}} A_{25})$$

$$
\begin{aligned}
= \quad & \frac{1}{2}M_{\mathcal{O}}\boldsymbol{v}_3^2 - \boldsymbol{m}_3 \cdot \boldsymbol{v}_3 - \phi_3 P_3 \boldsymbol{\nabla} \cdot \boldsymbol{v}_3 \\
& +E_2 + E_3 + E_5 \qquad (28)
\end{aligned}
$$

Phase 3 – Connected Oil ($\alpha = 3$, $\beta = 1, 2, 5$)

$$
\begin{aligned}
& \boldsymbol{\nabla} \cdot (\sigma_{\mathcal{WO}} A_{13} \boldsymbol{v}_1) + \frac{\partial}{\partial t}(\sigma_{\mathcal{WO}} A_{13} + \sigma_{\mathcal{WO}} A_{23} + \sigma_{\mathcal{OR}} A_{35}) \\
= \quad & \frac{1}{2}M_{\mathcal{W}}\boldsymbol{v}_1^2 + \frac{1}{2}M_{\mathcal{O}}\boldsymbol{v}_3^2 - \boldsymbol{m}_1 \cdot \boldsymbol{v}_1 - \boldsymbol{m}_3 \cdot \boldsymbol{v}_3 - \phi_1 P_1 \boldsymbol{\nabla} \cdot \boldsymbol{v}_1 - \phi_3 P_3 \boldsymbol{\nabla} \cdot \boldsymbol{v}_3 \\
& +E_1 + E_2 + E_3 + E_5 \qquad (29)
\end{aligned}
$$

Phase 4 – Disconnected Oil ($\alpha = 4$, $\beta = 1, 5$)

$$
\begin{aligned}
& \boldsymbol{\nabla} \cdot (\sigma_{\mathcal{WO}} A_{14} \boldsymbol{v}_1) + \frac{\partial}{\partial t}(\sigma_{\mathcal{WO}} A_{14} + \sigma_{\mathcal{OR}} A_{45}) \\
= \quad & \frac{1}{2}M_{\mathcal{W}}\boldsymbol{v}_1^2 - \boldsymbol{m}_1 \cdot \boldsymbol{v}_1 - \phi_1 P_1 \boldsymbol{\nabla} \cdot \boldsymbol{v}_1 \\
& +E_1 + E_4 + E_5 \quad . \qquad (30)
\end{aligned}
$$

4 Conclusions

We have obtained a closed system of 11 equations (13), (14), (15), (16), (17), (19), (20), (27), (28), (29), (30) for the 11 variables ϕ_1, ϕ_2, ϕ_3, ϕ_4, P_1, P_3, $\boldsymbol{v}_1$, $\boldsymbol{v}_3$, A_{13}, A_{14}, A_{23}. The parameters $\rho_{\mathcal{W}}$, $\rho_{\mathcal{O}}$, $\rho_{\mathcal{R}}$, $\sigma_{\mathcal{WO}}$, $\sigma_{\mathcal{WR}}$, $\sigma_{\mathcal{OR}}$, ϕ, $A_{1\mathcal{R}}$, $A_{2\mathcal{R}}$, $A_{3\mathcal{R}}$, $A_{4\mathcal{R}}$ describe the properties of the fluid, the rock and the wetting. The constitutive functions $\boldsymbol{m}_1$, $\boldsymbol{m}_3$, $M_{\mathcal{W}}$, $M_{\mathcal{O}}$ and E_1, E_2, E_3, E_4, E_5 may in general depend on the variables, parameters and on the other functions.

A crucial assumption is that the exchange of interfacial energies between the fluid phases is dominated by breakup and coalescence of disconnected subphase droplets from and with their connected parent phase, i. e. through mass transfer [2]. All other mechanisms of energy redistribution between the phases should be negligible, except for the energy exchange between a fluid phase and the rock phase. Different functional forms for mass transfer $M_{\mathcal{W}}$, $M_{\mathcal{O}}$ and energy exchange E_1, E_2, E_3, E_4, E_5 are currently investigated numerically and analytically.

The momentum exchange $\boldsymbol{m}_\alpha$ also appears in the traditional theories [20], [1]. Usually one invokes a naïve generalization of the Stokesian formula for the drag acting on a sphere in a lineal steady flow of a Newtonian fluid. We proceed here in the same manner and define the momentum transfer $\boldsymbol{m}_{\alpha\beta}$ from phase β to phase α by

$$\boldsymbol{m}_{\alpha\beta} := R_{\alpha\beta}(\boldsymbol{v}_\beta - \boldsymbol{v}_\alpha) \quad , \tag{31}$$

with the resistivity coefficients $R_{\alpha\beta}$. Summing up the contributions from the surrounding phases yields

$$\boldsymbol{m}_\alpha =: \sum_{\substack{\beta \\ A_{\alpha\beta}\neq 0}} \boldsymbol{m}_{\alpha\beta} \quad , \tag{32}$$

and thus we have

$$\begin{aligned} \boldsymbol{m}_1 &= R_{13}\boldsymbol{v}_3 - (R_{13} + R_{14} + R_{15})\boldsymbol{v}_1 \\ \boldsymbol{m}_3 &= R_{31}\boldsymbol{v}_1 - (R_{31} + R_{32} + R_{35})\boldsymbol{v}_3 \qquad . \end{aligned}$$

The momentum balance (19), (20) becomes now

$$\boldsymbol{\nabla}(\phi_1 P_1) - \phi_1 \rho_{\mathcal{W}} \boldsymbol{g} = R_{13}\boldsymbol{v}_3 - (R_{13} + R_{14} + R_{15} + M_{\mathcal{W}})\boldsymbol{v}_1 \tag{33}$$

$$\boldsymbol{\nabla}(\phi_3 P_3) - \phi_3 \rho_{\mathcal{O}} \boldsymbol{g} = R_{31}\boldsymbol{v}_1 - (R_{31} + R_{32} + R_{35} + M_{\mathcal{O}})\boldsymbol{v}_3 \quad . \tag{34}$$

Solving (33) and (34) for $\boldsymbol{v}_1$ and $\boldsymbol{v}_3$ leads to a generalized form of Darcy's Law,

$$\boldsymbol{v}_1 = k^r_{\mathcal{WW}} \frac{K}{\mu_{\mathcal{W}}} (\boldsymbol{\nabla}(\phi_1 P_1) - \phi_1 \rho_{\mathcal{W}} \boldsymbol{g}) + k^r_{\mathcal{WO}} \frac{K}{\mu_{\mathcal{O}}} (\boldsymbol{\nabla}(\phi_3 P_3) - \phi_3 \rho_{\mathcal{O}} \boldsymbol{g}) \tag{35}$$

$$\boldsymbol{v}_3 = k^r_{\mathcal{OW}} \frac{K}{\mu_{\mathcal{W}}} (\boldsymbol{\nabla}(\phi_1 P_1) - \phi_1 \rho_{\mathcal{W}} \boldsymbol{g}) + k^r_{\mathcal{OO}} \frac{K}{\mu_{\mathcal{O}}} (\boldsymbol{\nabla}(\phi_3 P_3) - \phi_3 \rho_{\mathcal{O}} \boldsymbol{g}) \tag{36}$$

with K being the absolute or Darcy permeability of the rock as defined in (11). The coefficients k^r_{ij} may be recognized as generalizations of the relative permeabilities introduced in (12). With the abbreviations

$$\begin{aligned} \Theta_{\mathcal{W}} &:= R_{13} + R_{14} + R_{15} + M_{\mathcal{W}} \quad , \\ \Theta_{\mathcal{O}} &:= R_{31} + R_{32} + R_{35} + M_{\mathcal{O}} \quad , \\ \Theta_{13} &:= R_{13} R_{31} \quad , \end{aligned}$$

they are identified here as

$$k^r_{\mathcal{WW}} = \frac{\mu_{\mathcal{W}}}{K} \frac{\Theta_{\mathcal{O}}}{\Theta_{13} - \Theta_{\mathcal{O}} \Theta_{\mathcal{W}}} \quad , \tag{37}$$

$$k^r_{\mathcal{WO}} = \frac{\mu_{\mathcal{O}}}{K} \frac{R_{13}}{\Theta_{13} - \Theta_{\mathcal{O}} \Theta_{\mathcal{W}}} \quad , \tag{38}$$

$$k^r_{\mathcal{OW}} = \frac{\mu_{\mathcal{W}}}{K} \frac{R_{31}}{\Theta_{13} - \Theta_{\mathcal{O}} \Theta_{\mathcal{W}}} \quad , \tag{39}$$

$$k^r_{\mathcal{OO}} = \frac{\mu_{\mathcal{O}}}{K} \frac{\Theta_{\mathcal{W}}}{\Theta_{13} - \Theta_{\mathcal{O}} \Theta_{\mathcal{W}}} \quad . \tag{40}$$

The non-diagonal coupling coefficients $k^r_{\mathcal{WO}}$, $k^r_{\mathcal{OW}}$ are frequently neglected in the traditional theory ([11]). The functional form of the relative permeabilities are determined through the resistivities $R_{\alpha\beta}$. They may depend on the variable saturations and specific surfaces as well as on parameters such as the wetting conditions $A_{\alpha\mathcal{R}}$.

Turning to the capillary pressure defined as $P_c := P_3 - P_1$ we consider complete gravitational equilibrium with $\boldsymbol{v}_\alpha = 0$ and all terms involving time derivatives set to zero. In this special case only eqs. (19), (20) remain nontrivial, but they nevertheless contain essential features of capillary pressure relationships that are known from experiment and that cannot be expressed through traditional descriptions. For the sake of simplicity let us consider one spatial direction which should be the direction of gravity, denoted by the positive z-axis. Introducing saturations $S_\alpha := \phi_\alpha/\phi$, $\alpha = 1, 3$, eqs. (19), (20) read

$$\frac{\mathrm{d}(P_\alpha S_\alpha)}{\mathrm{d}z} - \rho_\alpha g S_\alpha = 0 \quad . \tag{41}$$

Let us further assume the depth-dependent saturation profiles $S_\alpha = S_\alpha(z)$ as being given. With the primitives $\Sigma_\alpha(z) := \int^z S_\alpha(\zeta)\mathrm{d}\zeta$, solutions of eq. (41) are readily written down. Denoting the water saturation by $S_\mathcal{W} = S_1 + S_2$ we find for the capillary pressure

$$P_c(S_\mathcal{W}) = \frac{\rho_\mathcal{O} g \Sigma_3}{1 - S_4 - S_\mathcal{W}} - \frac{\rho_\mathcal{W} g \Sigma_1}{S_\mathcal{W} - S_2} \quad . \tag{42}$$

By virtue of the primitives the capillary pressure depends on the whole saturation profile $S_\mathcal{W}(z)$. Since a multitude of profiles is possible in equilibrium, $P_c(S_\mathcal{W})$ is expected to be multivalued. Assuming further that S_2, S_4 are constant the capillary pressure function (42) exhibits the characteristic shape of an S-curve with poles at $S_\mathcal{W} = S_2$ an $S_\mathcal{W} = 1 - S_4$. This together with the multivaluedness of $P_c(S_\mathcal{W})$ is in qualitative agreement with experiment. Moreover it suggests to identify S_2 and S_4 as the irreducible water and residual oil saturations known from capillary pressure measurements. These qualitative results corroborate the specification of the stress tensor given by (18).

In summary we have presented here novel macroscopic equations of motion for two-phase flow of immiscible and incompressible fluids through a rigid porous medium. Compared to the traditional theory the set of macroscopic state variables is enlarged to include specific internal surface. Simultaneously the set of balance laws is enlarged to include interfacial energy balance in a way that respects the pairwise character of interfacial energies. The new equations include wetting properties through the specific internal surface areas of the fluids with the rock. The equations can potentially describe also the dynamics of the disconnected fluid phases and of residual saturations. The theory predicts a capillary pressure behaviour whose dependence on saturation appears to be in rough agreement with experiment.

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Table Captions

Table 1:

Survey of the nomenclature and indices used for phases and subphases.

TABLE 1

Phase Index		Phase	Substance
$\mathcal{W}$	1	connected (percolating) fluid	water
	2	disconnected (trapped) fluid	
$\mathcal{O}$	3	connected (percolating) fluid	oil
	4	disconnected (trapped) fluid	
$\mathcal{R}$	5	rigid porous solid	rock minerals

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