Chapter 9

Foundations of Fractional Dynamics: A Short Account

R. Hilfer*
Institut für Computerphysik, Universität Stuttgart, Pfaffenwaldring 27, 70569 Stuttgart, Germany

Applications of fractional dynamics have received a steadily increasing amount of attention during the past decade. Its foundations have found less interest. This chapter briefly reviews the physical foundations of fractional dynamics.

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1. Introduction

A *fractional dynamical system* has been defined [1] as a dynamical system involving fractional (i.e. noninteger order) time derivatives instead of integer order time derivatives. Despite the long history of fractional calculus in mathematics (see [2–6] for reviews), despite numerous publications on

*Also at Institut für Physik, Universität Mainz, 55099 Mainz, Germany.

fractional powers of infinitesimal generators [7–16], and despite a rapidly growing literature on possible applications of fractional dynamical systems to physical phenomena (see [17–20] and the present volume for reviews), there seem to exist only few publications discussing the physical foundations of fractional dynamics.[a]

My objective in this chapter is to call attention to the foundations of fractional dynamics and fractional time evolution by reformulating the problem stated originally in [1, 21] and briefly summarizing some known results. As everyone knows, fractional time derivatives do not appear in any established fundamental theory of physics such as classical mechanics, electrodynamics, or quantum mechanics. Instead, integer (first and second) order time derivatives occur in all fundamental theories of physics. Obviously, time is a primordial and fundamental concept from the foundations of physics. Replacing integer order with fractional order time derivatives therefore changes the fundamental concept of time and with it the concept of evolution in the foundations of physics. Evolution equations in physics do not contain fractional time derivatives, because this would contradict the deep and fundamental principle, that time evolution is time translation. Most publications on fractional dynamics proceed directly to applied problems, but do not justify, discuss or even mention, that they remove the fundamental concept of time evolution (= time translation) from the foundations of physics.

Difficulties with fractional dynamics arise also, because fractional derivative operators can be defined in numerous ways [4–6]. Embedding a conventional dynamical system into a family of fractional dynamical systems is not unique. In fact, an infinite number of choices are possible and many publications fail to justify or discuss their particular choice.

Given the need for a fundamental justification of fractional dynamics, this chapter is structured as follows. Let me first recall some basic ideas about time. Observables, states and their time evolution are discussed next. Restricting attention to conservative dynamical systems raises a fundamental problem for the time evolution of macroscopic states. Induced measure preserving transformations are then introduced to solve this problem. Averaging them shows that macroscopic states evolve in time by convolution rather than translation. My short account

[a]The term "fractional dynamics" is used synonymously with "fractional dynamical system."

of the foundations of fractional dynamics concludes with remarks about irreversibility, experimental evidence, and dissipative systems.

2. The Aristotelian Concept of Time

The concepts of time and time evolution are fundamental for physics (and other sciences). Aristotle [22, Δ 11] defined time as ἀριθμὸς κινήσεως (i.e. as the integer or rational number of motion),[b] and formulates the idea, that past and future are separated by a mathematical point, that he calls τὸ νῦν (the Now). Newton [24, p. 5] formulates and postulates "*Tempus absolutum verum et Mathematicum, in se et natura sua absque relatione ad externum quodvis, aequabiliter fluit, alioque nomine dicitur Duratio*".[c] The concept of time in modern physics is based on the ideas of Aristotle in their Newtonian formulation. Time is viewed as a *flux aequabilis* (uniform flow) or succession of Aristotelian time instants.

The theoretical and mathematical abstraction of this concept of time from general mathematical theories of physical phenomena has led to the fundamental principle of time translation invariance and energy conservation in modern physics. All fundamental theories of contemporary physics postulate time translation invariance as a basic symmetry of nature.

3. Time Evolution of Observables

Time is commonly considered as the set of Aristotelian time instants $\dot{t}$. The set of all time instants is represented mathematically by the set of real numbers $\mathbb{R}$. Time is "measured" by observing clocks. Clocks are physical systems. Let a be an observable quantity (e.g., the position of the sun, the moon or some hand on a watch), and let $\mathcal{A}$ be the set of observables of such a physical system. A *dynamical system* is a triple $(\mathcal{A}, \mathbb{R}, T)$ where $\mathcal{A}$ is the set of observables of a physical system, $\mathbb{R}$ represents time, and the mapping

$$\begin{aligned} T : \mathcal{A} \times \mathbb{R} &\to \mathcal{A} \\ (a, \dot{t}) &\mapsto T(a, \dot{t}) \end{aligned} \tag{1}$$

[b]While Aristotle was perhaps counting heart beats, days, months, years, or time intervals determined with a κλεψύδρα, the idea to count periods has persisted. Today the unit of time corresponds to counting 9 192 631 770 periods of oscillation of a certain form of radiation emitted from ^{133}Cs-atoms [23].

[c]Transl.: "Absolute, true and mathematical time flows uniformly, in itself, according to its own nature, and without relation to anything outside itself; it is also called by the name duration."

is its dynamical rule [25]. It describes the change of observable quantities with time. For the dynamical rule T the following properties are postulated:

(1) For all time instants $\dot{s}, \dot{t} \in \mathbb{R}$ the dynamical rule obeys

$$T(T(a,\dot{s}),\dot{t}) = T(a,\dot{s}+\dot{t}) \tag{2}$$

for all $a \in \mathcal{A}$.

(2) There exists a time instant $\dot{t}_* \in \mathbb{R}$, called *beginning*, such that

$$T(a,\dot{t}_*) = a \tag{3}$$

holds for all $a \in \mathcal{A}$.

(3) The map T is continuous in time in a suitable topology.

The set of observables reflects the kinematical structure of the physical system. The dynamical rule prescribes the *time evolution* of the system. Setting $\dot{s} = \dot{t}_*$ in Eq. (2) and using Eq. (3) shows that either $\dot{t}_* = 0$ must hold, or else the observable must be time independent. The time evolution of observables is the one-parameter family $\{{}_{\mathcal{A}}T^{\dot{t}}\}_{\dot{t}\in\mathbb{R}}$ of maps ${}_{\mathcal{A}}T^{\dot{t}} : \mathcal{A} \to \mathcal{A}$ defined by

$${}_{\mathcal{A}}T^{\dot{t}} a := T(a,\dot{t}) \tag{4}$$

for $\dot{t} \in \mathbb{R}$. The time evolution obeys the group law

$${}_{\mathcal{A}}T^{\dot{t}}\,{}_{\mathcal{A}}T^{\dot{s}} = {}_{\mathcal{A}}T^{\dot{s}+\dot{t}} \tag{5}$$

for all $\dot{t}, \dot{s} \in \mathbb{R}$, and the identity law

$${}_{\mathcal{A}}T(0) = \mathbf{1}, \tag{6}$$

where $\mathbf{1}$ is the identity on $\mathcal{A}$. The continuity law requires a topology. It is usually assumed, that $\mathcal{A}$ is a Banach space with norm $\|\cdot\|$, and that

$$\lim_{\dot{t}\to 0^+} \|{}_{\mathcal{A}}T^{\dot{t}} a - a\| = 0 \tag{7}$$

holds for all $a \in \mathcal{A}$. Equations (5), (6) and (7) define a strongly continuous one parameter group of operators $\{{}_{\mathcal{A}}T^{\dot{t}}\}_{\dot{t}\in\mathbb{R}}$ on $\mathcal{A}$, called a *flow* [26, 27]. For bounded linear operators strong and weak continuity are equivalent [28].

Identifying $a = a(0)$ and writing $T(a, \dot{t}) = a(\dot{t})$, the time evolution becomes time translation to the left, i.e.

$$ {}_{\mathcal{A}}T^{\dot{t}} a(\dot{s}) = a(\dot{s} + \dot{t}) \tag{8}$$

for all $\dot{t}, \dot{s} \in \mathbb{R}$. If the arrow of time is taken into account, then the flow of time is directed, and only the time instants $\dot{t} \geq 0$ after the beginning can occur. In that case, inverse elements do not exist, and the family $\{{}_{\mathcal{A}}T^{\dot{t}}\}_{\dot{t}\geq 0}$ of operators forms only a semigroup [28, 29] instead of a group.

4. Time Evolution of States

In general, the set of observables $\mathcal{A}$ of a physical system is not only a Banach space, but forms an algebra, more specifically, a C*-algebra [30]. In classical physics this algebra is commutative. The states μ of a physical system are normalized, positive linear functionals on its algebra of observables [30]. As such they are elements of the dual space $\mathcal{A}^*$. The notation $\langle \mu, a\rangle$ is used for the value $\mu(a)$ of the observable a in the state μ. Convex combinations of states are again states. If a state cannot be written as a convex combination of other states, it is called pure. Because the observable algebra $\mathcal{A}$ is a subset of its bidual, $\mathcal{A} \subset \mathcal{A}^{**}$, its elements can be considered as functions on the set $X(\mathcal{A})$ of its characters,[d] i.e. $a(\chi) = \langle \chi, a\rangle$ for $a \in \mathcal{A}, \chi \in X(\mathcal{A})$. By virtue of this correspondence, known as the Gelfand isomorphism [30, 31], a commutative C^*-algebra is isomorphic to the algebra $C_0(X(\mathcal{A}))$ of continuous functions on the set $X(\mathcal{A})$ of its characters equipped with the weak* topology. Characters are pure states.

The time evolution of states is obtained from the time evolution of observables by passing to adjoints [28,32]. The adjoint time evolution ${}_{\mathcal{A}^*}T^{\dot{t}}$: $\mathcal{A}^* \to \mathcal{A}^*$ with $\dot{t} \in \mathbb{R}$ consists of all adjoint operators $({}_{\mathcal{A}}T^{\dot{t}})^*$ on the dual space $\mathcal{A}^*$ [28, 33]. If $\mu(\dot{t}_1)$ denotes the state at time $\dot{t}_1 \in \mathbb{R}$, then Eq. (8) implies

$$\begin{aligned}\langle \mu(\dot{t}_1), {}_{\mathcal{A}}T^{\dot{t}} a(\dot{t}_1)\rangle &= \langle \mu(\dot{t}_1), a(\dot{t}_1 + \dot{t})\rangle \\ &= \langle \mu(\dot{t}_2 - \dot{t}), a(\dot{t}_2)\rangle = \langle {}_{\mathcal{A}^*}T^{\dot{t}} \mu(\dot{t}_2), a(\dot{t}_2)\rangle,\end{aligned} \tag{9}$$

where $\dot{t}_2 = \dot{t}_1 + \dot{t} \in \mathbb{R}$ is arbitrary. For left translations the adjoint group

$$ {}_{\mathcal{A}^*}T^{\dot{t}} \mu(\dot{s}) = \mu(\dot{s} - \dot{t}) \tag{10}$$

[d] A character is an algebraic *-homomorphism from a commutative C*-algebra to $\mathbb{C}$.

is the group of right translations with $\dot{s}, \dot{t} \in \mathbb{R}$. The adjoint semigroup is weak* continuous, but in general not strongly continuous, unless the Banach space $\mathcal{A}$ is reflexive [32].

5. Conservative Systems

In classical mechanics, the commutative algebra of observables $\mathcal{A} = C_0(\Gamma)$ is the algebra of continous functions on phase space Γ, that vanish at infinity.[e] The characters (pure states) are point measures on phase space Γ, and one has the isomorphism $\Gamma \equiv X(\mathcal{A})$. By the Riesz representation theorem the states $\mu \in \mathcal{A}^* \equiv C_0(X(\mathcal{A}))^* \equiv C_0(\Gamma)^*$ in classical mechanics are probability measures on phase space $\Gamma \equiv X(\mathcal{A})$. Every state $\mu \in C_0(\Gamma)^*$ gives rise to a probability measure space $(\Gamma, \mathfrak{G}, \mu)$ where $\mathfrak{G}$ is the σ-algebra of measurable subsets of phase space Γ.

Let $S : \Gamma \to \Gamma$ be an invertible map such that S and S^{-1} are both measurable, i.e. such that $S^{-1}\mathfrak{G} = S\mathfrak{G} = \mathfrak{G}$ where $SG := \{Sx : x \in G\}$ for $G \in \mathfrak{G}$. The map S is called a *measure preserving transformation* and the measure μ on Γ is called *invariant under* S, if $\mu(G) = \mu(SG) = \mu(S^{-1}G)$ for all $G \in \mathfrak{G}$. An invariant measure is called *ergodic* with respect to S, if it cannot be decomposed into a convex combination of S-invariant measures.

Here and in the following the measure preserving transformation is the adjoint time evolution ${}_{\mathcal{A}^*}T^{\dot{t}}$ which is denoted more briefly as ${}_{\Gamma}T^{\dot{t}} = {}_{\mathcal{A}^*}T^{\dot{t}}$. Pure states (characters) are not invariant under ${}_{\Gamma}T^{\dot{t}}$. Examples of invariant probability measures are furnished by the set of equilibrium states of a conservative system with Hamiltonian dynamics. If μ is an equilibrium state of a conservative system, then $(\Gamma, \mathfrak{G}, \mu, {}_{\Gamma}T^{\dot{t}})$ is a measure preserving system.

6. Statement of the Problem

Let $(\Gamma, \mathfrak{G}, \mu, {}_{\Gamma}T^{\dot{t}})$ be a measure preserving many-body system. The detailed microscopic time evolution ${}_{\Gamma}T^{\dot{t}} : \Gamma \to \Gamma$ is frequently not of interest in applications, because it is much too detailed to be computable. Instead one is interested in a reduced, coarse grained or averaged time evolution of macroscopic states where the system is locally or globally in equilibrium. Examples are isolated systems at phase coexistence or in metastable

[e] This means that for each $a \in C_0(\Gamma)$ and $\varepsilon > 0$ there is a compact subset $K \subseteq \Gamma$ such that $|a(x)| < \varepsilon$ for all $x \in \Gamma \setminus K$.

states.[f] This gives rise to the problem of finding the time evolution ${}_G T^{\dot{t}}$: $G \to G$ on subsets $G \subset \Gamma$ of phase space.

It is not possible to define ${}_G T^{\dot{t}} = {}_\Gamma T^{\dot{t}}|_G$ as the restriction of ${}_\Gamma T^{\dot{t}}$ to G, because for fixed initial state $x(0) \in G \subset \Gamma$ the time evolution ${}_\Gamma T^{\dot{t}}$ produces states ${}_\Gamma T^{\dot{t}} x(0) = x(\dot{t}) \notin G$. Equivalently, for fixed time $\dot{t}$ the map ${}_\Gamma T^{\dot{t}}$ maps states $x \in G$ to states not in G. The restriction ${}_\Gamma T^{\dot{t}}|_G$ is not defined for all $\dot{t} \in \mathbb{R}$. This seems to preclude a sensible definition of ${}_G T^{\dot{t}}$.

This problem of defining an induced continuous time evolution for mixed states on subsets of small measure was introduced and solved in [1, 21]. It originated from the general classification theory for phase transitions [35–39]. The solution involves discretization of ${}_\Gamma T^{\dot{t}}$, averaging Kakutani's induced measure preserving transformation [26,40] and Kac's theorem for recurrence times [1, 21].

7. Induced Measure Preserving Transformations

Consider a subset $G \subset \Gamma$ with small but positive measure $\mu(G) > 0$ of a measure preserving many body system $(\Gamma, \mathfrak{G}, \mu, {}_\Gamma T^{\dot{t}})$. Because $\mu(G) > 0$, the subset G becomes a probability measure space $(G, \mathfrak{S}, \nu)$. The induced probability measure is $\nu = \mu/\mu(G)$ and $\mathfrak{S} = \mathfrak{G} \cap G$ is the trace of $\mathfrak{G}$ in G [41].

The measure preserving continuous time evolution ${}_\Gamma T^{\dot{t}}$ is discretized by setting

$$\dot{t} = k\tau \tag{11}$$

with $k \in \mathbb{Z}$ and $\tau > 0$ the discretization time step. A character $x \in G$ is called *recurrent*, if there exists an integer $k \geq 1$ such that ${}_\Gamma T^{k\tau} x \in G$. If $G \in \mathfrak{G}$ and μ is invariant under ${}_\Gamma T$, then almost every character in G is recurrent by virtue of the Poincaré recurrence theorem. A subset G is called recurrent, if μ-almost every point $x \in G$ is recurrent. By Poincaré's recurrence theorem the recurrence time $t_G(x)$ of the character $x \in G$, defined as

$$t_G(x) = \tau \min\{k \geq 1 : {}_\Gamma T^{k\tau}(x) \in G\}, \tag{12}$$

is positive and finite for almost every $x \in G$. For every $k \geq 1$ let

$$G_k = \{x \in G : t_G(x) = k\tau\} \tag{13}$$

[f] This differs from relaxation to equilibrium discussed in [34].

denote the set of characters with recurrence time $k\tau$. Then the number

$$p(k) = \nu(G_k) \tag{14}$$

is the probability to find a recurrence time $k\tau$. The numbers $p(k)$ define a discrete probability density $p(k)\delta(\dot{s} - k\tau)$ on the arithmetic progression $\dot{s} - k\tau, k \in \mathbb{N}, \dot{s} \in \mathbb{R}$. Every probability measure $\varrho(\dot{s})$ on $(G, \mathfrak{G})$ at time instant $\dot{s}$ is then defined on the same arithmetic progression through

$$\varrho(B, \dot{s} - k\tau) = \varrho(B \cap G_k, \dot{s}) \tag{15}$$

for all $B \in \mathfrak{G}$ and $\dot{s} \in \mathbb{R}$. The induced time evolution ${}_G T$ on the subset G is defined for every $B \in \mathfrak{G}$ as the average [1, 21]

$${}_G T \varrho(B, \dot{s}) = \sum_{k=1}^{\infty} p(k) \varrho(B, \dot{s} - k\tau), \tag{16}$$

where $\dot{s} \in \mathbb{R}$. For characters $\varrho = x \in G$, one recovers the first step in the discretized microscopic time evolution ${}_G T x(\dot{s}) = x(\dot{s} - t_G(x))$ as expected. For mixed states ϱ this formula allows the transition from the microscopic to the macroscopic time evolution. It assigns an averaged translation to the first step in the induced time evolution of mixed states.

8. Fractional Time Evolution

The induced time evolution is obtained from ${}_G T$ by iteration. According to its definition in Eq. (16), the induced measure preserving transformation ${}_G T$ acts as a convolution in time,

$${}_G T \varrho = \varrho * p, \tag{17}$$

where ϱ is a mixed state on $(G, \mathfrak{G})$. Iterating N times gives

$${}_G T^N \varrho = ({}_G T^{N-1} \varrho) * p = \varrho * \underbrace{p \cdots * p}_{N \text{ factors}} = \varrho * p_N, \tag{18}$$

where $p_N(k) = p(k) * \cdots * p(k)$ is the probability density of the sum

$$\mathcal{T}_N = \tau_1 + \cdots + \tau_N \tag{19}$$

of N independent and identically with $p(k)$ distributed random recurrence times τ_i. Then the long time limit $N \to \infty$ for induced measure preserving transformations on subsets of small measure is generally governed by well-known local limit theorems for convolutions [42–45]. Application to the case at hand yields the following fundamental theorem of fractional dynamics [1, 21].

Theorem 1. *Assume that $\tau > 0$ is maximal in the sense that there is no larger τ for which all recurrence times lie in $\tau\mathbb{N}$. Then the following conditions are equivalent:*

(1) *Either $\sum_{k=1}^{\infty} kp(k) < \infty$ or there exists a number $0 < \gamma < 1$ such that*

$$\gamma = \sup\left\{0 < \beta < 1 : \sum_{k=1}^{\infty} k^{\beta} p(k) < \infty\right\}. \tag{20}$$

(2) *There exist constants $D_N \geq 0, D \geq 0$ and $0 < \alpha \leq 1$ such that*

$$\lim_{N\to\infty} \sup_k \left| \frac{D_N}{\tau} p(k) - \frac{1}{D^{1/\alpha}} h_\alpha\left(\frac{k\tau}{D_N D^{1/\alpha}}\right)\right| = 0, \tag{21}$$

where $\alpha = 1$, if $\sum_{k=1}^{\infty} kp(k) < \infty$, and $\alpha = \gamma$ otherwise. The function $h_\alpha(x)$ vanishes for $x \leq 0$, and is

$$h_\alpha(x) = \frac{1}{x}\sum_{j=0}^{\infty} \frac{(-1)^j x^{-\alpha j}}{j!\,\Gamma(-\alpha j)} \tag{22}$$

for $x > 0$.

If the limit exists, and is nondegenerate, i.e. $D \neq 0$, then the rescaling constants D_N have the form

$$D_N = (N\Lambda(N))^{1/\alpha}, \tag{23}$$

where $\Lambda(N)$ is a slowly varying function [46], i.e.

$$\lim_{x\to\infty} \frac{\Lambda(bx)}{\Lambda(x)} = 1 \tag{24}$$

for all $b > 0$.

The theorem shows that

$$p_N(k) \approx \frac{\tau}{D_N D^{1/\alpha}} h_\alpha\left(\frac{k\tau}{D_N D^{1/\alpha}}\right) \tag{25}$$

holds for sufficiently large N. The asymptotic behavior of the iterated induced measure preserving transformation ${}_G T^N$ for $N \to \infty$ allows to remove the discretization, and to find the induced continuous time evolution on subsets $G \subset \Gamma$. First, the definition (15) is extended from the arithmetic progression $\dot{s} - \tau\mathbb{N}$ to $\dot{t} \leq \dot{s}$ by linear interpolation. Let $\widetilde{\varrho}(\dot{t})$ denote the extended measure defined for $\dot{t} \leq \dot{s}$. Using Eq. (11) and setting

$$t = D_N D^{1/\alpha} \tag{26}$$

the summation in Eq. (16) can be approximated for sufficently large $N \to \infty$ by an integral. Then ${}_{G}T^{N}\widetilde{\varrho}(\dot{s}) \approx {}_{G}T^{t}_{\alpha}\widetilde{\varrho}(\dot{s})$, where

$${}_{G}T^{t}_{\alpha}\widetilde{\varrho}(\dot{s}) = \int_0^\infty \widetilde{\varrho}(\dot{s} - \dot{t}) h_\alpha\left(\frac{\dot{t}}{t}\right)\frac{\mathrm{d}\dot{t}}{t} \tag{27}$$

is the induced continuous time evolution. ${}_{G}T^{t}_{\alpha}$ is also called *fractional time evolution*. Laplace tranformation shows that ${}_{G}T^{t}_{\alpha}$ fulfills Eq. (5). It is an example of subordination of semigroups [7, 33, 47, 48]. Indeed

$${}_{G}T^{t}_{\alpha} = \frac{1}{t}\int_0^\infty T^{\dot{t}} h_\alpha\left(\frac{\dot{t}}{t}\right)\mathrm{d}\dot{t}, \tag{28}$$

where $T^{\dot{t}}$ denotes right translations on the interpolated measure. As $D_N \geq 0$ and $D \geq 0$, Eq. (26) implies $t \geq 0$. As remarked in the introduction, the induced time evolution is in general not a translation (group or semigroup), but a convolution semigroup. The fundamental classification parameter

$$\alpha = \alpha(T, G, \tau) \tag{29}$$

depends not only on the dynamical rule $T(\cdot, \dot{t})$ and the subset G, but also on the discretization time step τ, i.e. on the time scale of interest.

9. Irreversibility

The result in Eq. (27) has provided new insight into the irreversibility paradox [21, p. 554]. For $\alpha \to 0^-$, one finds

$$h_1(x) = \lim_{\alpha \to 0^-} h_\alpha(x) = \delta(x - 1) \tag{30}$$

and therefore

$${}_{G}T^{t}_{1}\widetilde{\varrho}(\dot{s}) = \int_0^\infty \widetilde{\varrho}(\dot{s} - \dot{t})\delta\left(\frac{\dot{t}}{t} - 1\right)\frac{\mathrm{d}\dot{t}}{t} = \widetilde{\varrho}(\dot{s} - t) \tag{31}$$

is a right translation. Here $\dot{t} \in \mathbb{R}$ denotes a time instant, while $t \geq 0$ is a time duration. This shows, that induced right translations do not form a group, but only a semigroup.

These observations suggest a reformulation of the controversial irreversibility problem [6, 49]. The problem of irreversibility is normally formulated as:

Definition 1 (The normal irreversibility problem). Assume that time is reversible. Explain how and why time irreversible equations arise in physics.

The assumption that time is reversible, i.e. $\dot{t} \in \mathbb{R}$, is made in all fundamental theories of modern physics. The explanation of macroscopically irreversible behavior for macroscopic nonequilibrium states of subsystems is due to Boltzmann. It is based on the applicability of statistical mechanics and thermodynamics, the large separation of scales, the importance of low entropy initial conditions, and probabilistic reasoning [34].

The problem with assuming $\dot{t} \in \mathbb{R}$ is not the second law of thermodynamics, because the foundations of thermodynamics and statistical mechanics do not cover all dynamical systems in nature. The problem with the arrow of time is that an experiment (i.e. the preparation of certain intial conditions for a dynamical system) cannot be repeated yesterday, but only tomorrow [49]. While it is possible to translate the spatial position of a physical system, it is not possible to translate the temporal position of a physical system backwards in time.[g] This was emphasized in [6, 49]. These simple observations combined with Eqs. (30) and (31) suggest to reformulate the standard irreversibility problem:

Definition 2 (The reversed irreversibility problem). Assume that time is irreversible. Explain how and why time reversible equations arise in physics.

The reversed irreversibility problem was introduced in [49]. Its solution is given by Theorem 1 combined with (30) and (31). The impossibility of performing experiments in the past is fundamental and evident. Therefore, as emphasized in [49], it must be assumed that time is irreversible. The normal irreversibility problem starts from an assumption, that contradicts experiment, while the reversed problem starts from the correct assumption. Theorem 1 combined with (30) and (31) explains why time translations, i.e. the case $\alpha = 1$, arise in physics, and why it arises more frequently than the case $\alpha < 1$.

[g]Note that this is not the same as reversing the momenta of all particles in a physical system.

10. Infinitesimal Generators

The operators ${}_G T^t_\alpha$ form a family of strongly continous semigroups on $C_0(G)^*$ provided that the coarsened translations T^t in Eq. (28) are strongly continuous [33,48] and $\int_0^\infty \|T^s\| h_\alpha(s/t)/t \, \mathrm{d}s < \infty$. In this case the infinitesimal generators for $0 < \alpha \leq 1$ are defined by

$$\mathrm{A}_\alpha \widetilde{\varrho} = \operatorname*{s-lim}_{t \to 0} \frac{{}_G T^t_\alpha \widetilde{\varrho} - \widetilde{\varrho}}{t} \tag{32}$$

for all $\widetilde{\varrho} \in C_0(G)^*$ for which the strong limit s-lim exists. In general, the infinitesimal generators are unbounded operators. If $\mathrm{A} = -\mathrm{d}/\mathrm{d}t$ denotes the infinitesimal generator of the coarsened translation, T^t, then

$$\mathrm{A}_\alpha = -(-\mathrm{A})^\alpha = -\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^\alpha \tag{33}$$

are fractional time derivatives [16, 50]. The action of A_α on mixed states can be represented in different ways. Frequently an integral representation

$$\mathrm{A}_\alpha \widetilde{\varrho} = \lim_{\epsilon \to o} C \int_\epsilon^\infty t^{-\alpha-1} (\mathbf{1} - T^t) \widetilde{\varrho} \mathrm{d}t \tag{34}$$

of Marchaud type [8, 51] is used. The integral representation

$$\mathrm{A}_\alpha \widetilde{\varrho} = \lim_{\epsilon \to o} C \int_\epsilon^\infty t^{-\alpha} \mathrm{A} (\mathbf{1} - t\mathrm{A})^{-1} \widetilde{\varrho} \mathrm{d}t \tag{35}$$

in terms of the resolvent of A [12] defines the same fractional derivative operator [52]. Representations of Grünwald–Letnikov type are also well known [16].

In summary, fractional dynamical systems must be expected to appear generally in mathematical models of macroscopic phenomena. They arise as coarse grained macroscopic time evolutions from inducing a microscopic time evolution on the subsets $G \subset \Gamma$ of small measure in phase space, that are typically incurred in statistical mechanics [1, 21, 50].

11. Experimental Evidence

If fractional time evolutions from Eq. (27) with $\alpha < 1$ must be expected on general grounds, then they should be observable in experiment. Numerous experimental examples of anomalous dynamics or strange kinetics have been identified (see [17–20] and the present volume for reviews). Here the example of dielectric α-relaxation in glasses is briefly discussed [53, 54], because it concerns experimental data over up to 19 decades in time [55], and because the explanation of its excess wing has been a matter of debate.

For every induced time evolution on G with time scale $\tau > 0$ and fractional order $\beta(\tau)$

$$ {}_G T^{\tau t}_{\beta(\tau)} = {}_G T^{\tau_1 t}_{\beta(\tau)} {}_G T^{(\tau-\tau_1)t}_{\beta(\tau)} = {}_G T^{\tau_1 t}_{\beta(\tau)} {}_G T^{\tau_2 t}_{\beta(\tau_1+\tau_2)} \tag{36} $$

holds generally with $\tau = \tau_1 + \tau_2$. A physical system typically shows different physical phenomena on different time scales τ_i. In [53, 54] it was assumed that the second factor in Eq. (36) becomes approximately fractional in the sense that

$$ {}_G T^{\tau_2 t}_{\beta(\tau_1+\tau_2)} \approx {}_G T^{\tau_2 t}_{\alpha(\tau_2)} \tag{37} $$

holds in the weak* or strong topology with

$$ \lim_{\tau_2 \to 0} \alpha(\tau_2) = \beta(\tau). \tag{38} $$

The resulting composite time evolution ${}_G T^{\tau_1 t}_{\beta(\tau)} {}_G T^{\tau_2 t}_{\alpha(\tau_2)}$ was studied in [53,54] for the case $\beta(\tau) = 1$. Rescaling this composite operator as in the case of Debye relaxation and computing the infinitesimal generator yields the fractional differential equation [53, 54]

$$ \tau_1 \frac{\mathrm{d}f}{\mathrm{d}t} + \tau_2^{\alpha} \mathrm{A}_{\alpha} f = -f \tag{39} $$

with A_α from Eq. (33) and initial value $f(0) = 1$. Its solution is

$$ f(t) = \mathrm{E}_{(1,1-\alpha),1}\left(-\frac{t}{\tau_1}, -\frac{\tau_2^{\alpha}}{\tau_1} t^{1-\alpha}\right), \tag{40} $$

where

$$ \mathrm{E}_{(a_1,a_2),b}(z_1, z_2) = \sum_{k=0}^{\infty} \sum_{\substack{\ell_1 \geq 0 \\ \ell_1+\ell_2=k}} \sum_{\ell_2 \geq 0} \frac{k!}{\ell_1! \ell_2!} \frac{z_1^{\ell_1} z_2^{\ell_2}}{\Gamma(b + a_1\ell_1 + a_2\ell_2)} \tag{41} $$

with $a_1, a_2 > 0$ and $b, z_1, z_2 \in \mathbb{C}$ is the binomial Mittag-Leffler function [61].

The complex frequency dependent susceptibility is obtained from the the normalized relaxation function as $\chi(u) = 1 - u\hat{f}(u)$ where $\hat{f}(u)$ is the Laplace transform of $f(t)$ and $u = \mathrm{i}\omega$ is the imaginary circular frequency [54, p. 402, Eq. (18)]. The real part of the complex dielectric susceptibility for propylene carbonate at temperature $T = 193\,\mathrm{K}$ is plotted in Fig. 1, its imaginary part in Fig. 2. These figures are taken from [53]. Crosses

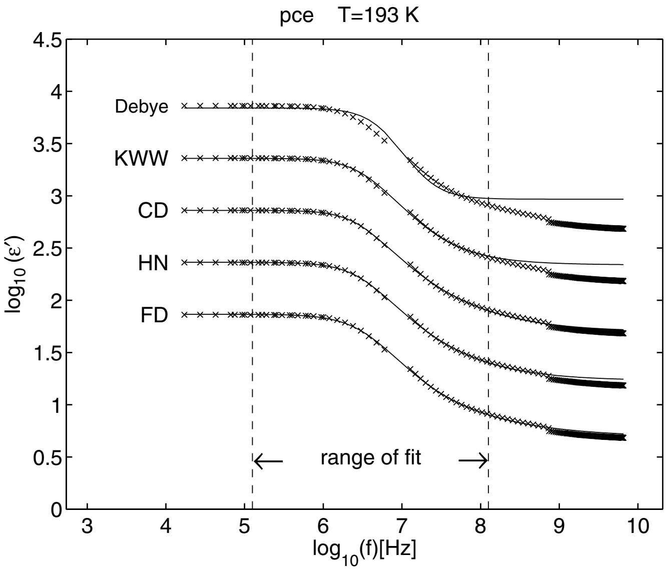

Fig. 1. Five different fits to the real part $\varepsilon'(f)$ of the complex dielectric function of propylene carbonate at $T = 193\,\mathrm{K}$ as a function of frequency f. Experimental data represented by crosses are from [56]. The fitting functions correspond to an exponential (Debye), stretched exponential (KWW), Cole–Davidson [57, 58] Havriliak–Negami [59, 60] and the fractional dynamics (FD) relaxation from (40). The range over which the data were fitted is indicated by dashed vertical lines in the figure. For clarity the data were displaced vertically by half a decade each. The original location of the data corresponds to the curve labelled FD.

represent experimental data. Different fit functions are shifted by half a decade for better visibility. The range over which the data were fitted is indicated by two dashed vertical lines. The curve labeled FD (fractional dynamics) is the susceptibility corresponding to the relaxation function in Eq. (40). It reproduces the high frequency wing even outside the range of its fit. This is not the case for the other four curves, labeled Debye, KWW, CD and HN. They correspond to four popular fit functions for dielectric relaxation [55, 62]. The curve Debye corresponds to a simple exponential function, KWW (Kohlrausch–Williams–Watts) is a stretched exponential relaxation function. The relaxation functions for the two remaining cases, CD (Cole–Davidson) and HN (Havriliak–Negami) were given for the first time in [58, 60].

Figure 3 from [54] shows the real and imaginary parts of the dielectric susceptibility for glycerol as its temperature varies over the glass transition

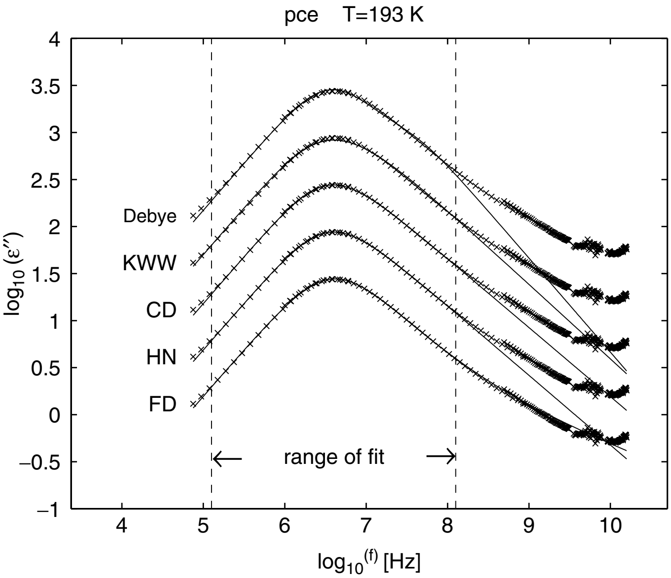

Fig. 2. Five different fits as in Fig. 1 for the imaginary part $\varepsilon''(f)$ of the complex dielectric function of propylene carbonate at $T = 193\,\mathrm{K}$ as a function of frequency f.

range from $T = 323\,\mathrm{K}$ to $T = 184\,\mathrm{K}$. The fits are based on a trinomial fractional relaxation function as detailed in [54, 61].

12. Dissipative Systems

The concept of time is the same for conservative and dissipative systems. For conservative dynamical systems a mathematically rigorous derivation of fractional dynamics from an underlying nonfractional dynamical system has remained elusive, although some authors have tried to relate α to invariant tori, strange attractors or other phase space structures [63, 64]. For dissipative systems the rigorous derivation has been possible for Bochner–Levy diffusion [7, 44, 47, 65] and Montroll–Weiss diffusion [66–70]. Due to restrictions on page number and preparation time only the latter case will be considered very briefly.

For diffusive dynamical systems a mathematically rigorous relation of fractional dynamics with microscopic Montroll–Weiss continuous time random walks was discovered in [71, 72]. It was shown that a diffusion (or master) equation with fractional time derivatives (i.e. a dissipative fractional dynamical system) can be related rigorously to the microscopic model

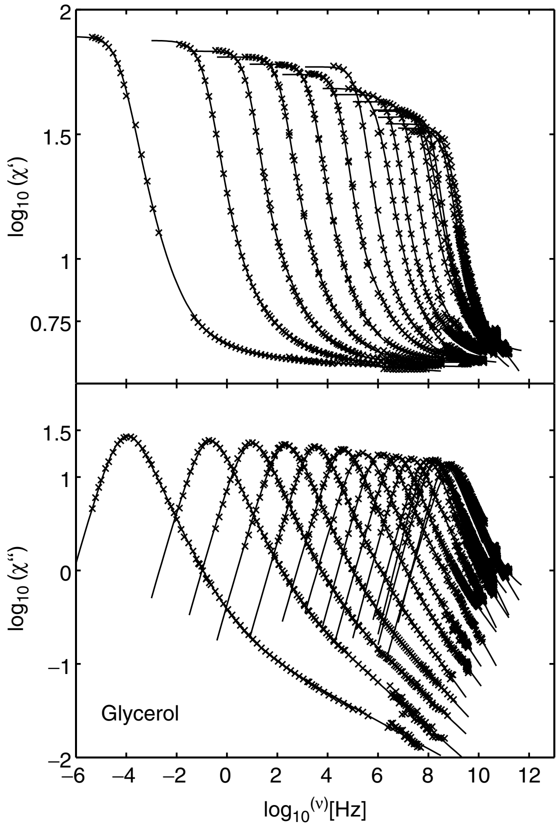

Fig. 3. Separate fits for real part (upper figure) and imaginary part (lower figure) of the complex dielectric susceptibility $\chi(\nu) = \chi'(\nu) + \mathrm{i}\chi''(\nu)$ of glycerol for temperatures $T = 323$, 303, 295, 289, 273, 263, 253, 243, 234, 223, 213, 204, 195, 184 K (from right to left) as function of frequency ν (from [54]). The experimental data are taken from [56], the fit uses a generalized composite fractional relaxation model. For details see [54].

of Montroll–Weiss continuous time random walks (CTRW's) [66, 70] in the same way as ordinary diffusion is related to random walks [44]. This discovery became decoupled from its source in the widely cited review [19], and was later incorrectly attributed to [74] in [73].[h]

The fractional order α can be identified and has a physical meaning related to the statistics of waiting times in the Montroll–Weiss theory. The relation was established in two steps. First, it was shown in [71] that

[h]Contrary to [73, p. 51] fractional derivatives are never mentioned in [74].

Montroll–Weiss continuous time random walks with a Mittag-Leffler waiting time density are rigorously equivalent to a fractional master equation.

Then, in [72] this underlying random walk model was connected to the fractional time diffusion equation in the usual asymptotic sense [75] of long times and large distances.[i] For additional results see also [50, 76–78].

The relation between fractional diffusion and continuous time random walks, established in [71,72] and elaborated in [50,76–78], has initiated many subsequent investigations of fractional dissipative systems, particularly into fractional Fokker–Planck equations with drift [17–19, 73, 79–86].

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[i] This is emphasized in Eqs. (1.8) and (2.1) in [72] that are, of course, asymptotic.

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