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Mathematical and Physical Interpretations of Fractional Derivatives and Integrals

Abstract: Brief descriptions of various mathematical and physical interpretations of fractional derivatives and integrals have been collected into this chapter as points of reference and departure for deeper studies. “Mathematical interpretation” in the title means a brief description of the basic mathematical idea underlying a precise definition. “Physical interpretation” means a brief description of the physical theory underlying an identification of the fractional order with a known physical quantity. Numerous interpretations had to be left out due to page limitations. Only a crude, rough and ready description is given for each interpretation. For precise theorems and proofs an extensive list of references can serve as a starting point.

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1 Prolegomena

1.1 A multitude of mathematical and physical interpretations for fractional derivatives and integrals have been developed since Gottfried Wilhelm Leibniz [123, p. 301] first noted and then asked *„...on le peut exprimer* PER SERIEM INFINITAM, SED QUID EST IN GEOMETRIA*?“*[1] Derivatives D^α and integrals I^α of fractional (non-integer) order arise from viewing the symbolic relations

$$I\,I^n = I^{n+1}, \qquad n \in \mathbb{N}, \tag{1a}$$
$$D\,D^n = D^{n+1}, \qquad n \in \mathbb{N}, \tag{1b}$$
$$D\,I^n = I^{n-1}, \qquad n \in \mathbb{N}, \tag{1c}$$

for iterated integrals I and derivatives D as almost representing the additive semigroup $(\mathbb{N},+)$ of natural numbers, and extending (1) to the semigroup $(\mathbb{R}_0^+,+)$ of non-negative reals[2] $\mathbb{R}_0^+ := [0,\infty) \subset \mathbb{R}$ or to the full field $(\mathbb{R},+,\cdot)$.

1.2 Many ideas for interpreting and extending the formal relations (1) from $n \in \mathbb{N}$ to $n \in \mathbb{Q},\mathbb{R}$ or $\mathbb{C}$ have been proposed over the centuries. All attempts are met squarely by some basic facts of calculus. Indeed, the geometric property $D^2 = 0$ of (exterior) derivatives [53, p.20] would restrict eq. (1b) to $n \leq 2$. Or the non-commutativity $I\,D \neq D\,I$ seems to prevent extension of eq. (1c) to $n = 0$. Recall that the power functions

$$P_n(x) := \frac{x^n}{n!}, \qquad x \geq 0, n \in \mathbb{N} \tag{2}$$

obey the rules $I\,P_n = P_{n+1}$ and $D\,P_n = P_{n-1}$ similar to I^n in (1a) and (1c). Extension to $n = 0$ (or $n < 0$), however, is fraught with a singularity at $x = 0$. Many mathematical impediments arise from these basic facts.

1.3 Derivatives and integrals of fractional (non-integer) order originate from the operational analogy (*„analogie merveilleuse“*) observed by Leibniz [122]

$$D^n(fg) = \sum_{k=0}^{n} \binom{n}{k} (D^k f)(D^{n-k} g) \tag{3}$$

$$\mathrm{P}^n(f+g) = \sum_{k=0}^{n} \binom{n}{k} (\mathrm{P}^k f)(\mathrm{P}^{n-k} g) \tag{4}$$

between n-th derivative of a product (3) of two real-valued functions f, g, and the n-th power of their sum (4), if powers $P_n(f) = f^n = \mathrm{P}^n f$ with $f^0 = 1$ are

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1 *„... one can express that“* difference (or sum) whose exponent is a fraction* *„by an infinite series, but what is it in geometry?“* *inserted from context [123, p. 300-301]

2 The notations $\mathbb{R}^+ := (0,\infty)$, $\mathbb{R}_0^+ := [0,\infty)$, and $\mathbb{R}^- := (-\infty,0)$ will be used.

written using an operational symbol P. Extension of eqs. (2)–(4) from $n \in \mathbb{N}$ to $\alpha \in \mathbb{C}$, $\operatorname{Re}\alpha > 0$ (or $\alpha \in \mathbb{C}$ [162, Sec. 2.3]) requires interpolation formulae

$$\text{from } \binom{n}{k} = \frac{n!}{k!(n-k)!} \text{ to } \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}, \qquad k \in \mathbb{N}, \quad \text{and} \tag{5a}$$

$$\text{from } n! = \prod_{k=1}^{n} k \text{ to } \Gamma(\alpha+1) = \int_0^1 (-\log x)^\alpha \,\mathrm{d}x = \int_0^\infty x^\alpha e^{-x} \mathrm{d}x \tag{5b}$$

for factorials and binomial coefficients. It was for this purpose that Euler solved the 'interpolation problem' and introduced the Γ-function in [50, §27-29].

1.4 Given the operational analogy between (3) and (4), Lagrange obtained the celebrated symbolic formula [61, p. 194-195]

$$\Delta_h^\alpha f = \left(e^{h\frac{\mathrm{d}f}{\mathrm{d}x}} - 1\right)^\alpha \tag{6}$$

for general α, where $(\Delta_h f)(x) = f(x+h) - f(x)$ is the finite difference with shift $h \geq 0$. Lagrange's formula results from treating the symbol of differentiation systematically as an algebraic quantity with the provision of replacing its n-th powers

$$\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)^n \longrightarrow \frac{\mathrm{d}^n f}{\mathrm{d}x^n} \tag{7}$$

with the n-th derivative at the end. Other authors have subsequently tried to build differential calculus on an algebraic or algorithmic basis [63, 7, 180, 31]. Recall that Leibniz' product rule

$$D(fg) = g(D\,f) + f(D\,g) \tag{8}$$

is the algebraic basis of D^n and eq. (3) reduces to eq. (8) for $n = 1$. Indeed eq. (8) is the axiomatic basis for differentiation (resp. derivation) on manifolds [33] and algebras [26, 119]. A full account of Leibniz' influential idea to generalize eq. (1) is not possible in these pages. Many books and specialized treatises [125, 157, 168, 82, 193] contain a section on history (see also [134]).

1.5 The objective of this chapter is to collect a list of mathematical and physical interpretations in the sense of Definitions 1 and 2 below. The scope is necessarily rather restricted due to the long history of fractional calculus. Only a selection of established and widely known mathematical interpretations is aimed at here. As to the physical interpretations, there will be even less examples, because, in the opinion of this author, most physical interpretations are still tentative.

2 Integrals and Derivatives

2.1 Integrals. Integrals are (weighted) sums with infinitely many terms or linear functionals on function spaces. More concretely, let $(\Omega, \mathscr{M}, \mu)$ be a measure space with σ-algebra $\mathscr{M}$ and measure (or weight) μ, and let $f : \Omega \to \mathbb{R}$ be a real-valued function on Ω. Then

$$I_{\mu,\Omega}(f) = \int\limits_{\Omega} f\mu = \int\limits_{\Omega} f(x)\,\mathrm{d}\mu(x) = \mu_\Omega(f) \in \mathbb{R} \tag{9a}$$

denotes the integral of f with respect to μ. When $\mu = \lambda$ is the Lebesgue measure and $\Omega \subset \mathbb{R}$ is an interval, notations such as

$$I_\Omega(f) = \int\limits_{\Omega} f = \int\limits_{\Omega} f(x)\,\mathrm{d}x = \lambda_\Omega(f) \tag{9b}$$

often suppress the dependence on the measure or weight μ.

2.2 Let $\mathbb{J} = (a, b)$ be an interval with $-\infty \le a < b \le \infty$ and $\Omega = (a, x)$ an interval with $a < x < b$. For a locally integrable function $f : \mathbb{J} \to \mathbb{R}$ the notation

$$(I_{a+} f)(x) := I_\Omega(f) = \int\limits_a^x f = \int\limits_a^x f(y)\mathrm{d}y \tag{10a}$$

$$(I_{b-} f)(x) := I_{\mathbb{J}\setminus\Omega}(f) = \int\limits_x^b f = \int\limits_x^b f(y)\mathrm{d}y \tag{10b}$$

is used to emphasize the dependence on the variable upper or lower limit x.

2.3 Derivatives. Let $\Omega \subset \mathbb{R}$ be open and let $\mathsf{F}(\Omega)$ be the algebra of real valued functions $f : \Omega \to \mathbb{R}$ with pointwise addition and pointwise multiplication[3]. Algebraically, the derivative (or derivation) $D\,f$ is defined as a linear operator $D : \mathsf{F} \to \mathsf{F}$ on F such that the product rule (8) holds true for all $f, g \in \mathsf{F}$. On commutative Banach algebras every such D has range contained in the Jacobson radical of F, i.e. in the kernel of the Gelfand isomorphism [183, 194].

2.4 Difference Quotients. Analytically, the derivative of f at $x \in \Omega$ is defined as the limit of finite difference quotients ($h \ge 0$)

$$(D\,f)(x) := \lim_{h\to 0} \frac{f(x \pm h) - f(x)}{h} = \lim_{h\to 0} \left(\frac{\mathrm{T}_\pm^h - 1}{h} \right) f(x) \tag{11a}$$

3 Sets of functions such as $\mathsf{F}(\Omega)$, or the standard Lebesgue spaces $\mathsf{L}^p(\Omega)$ are denoted with a *sans serif* font.

$$= \lim_{h \to 0} \left(\frac{\Delta_{\pm h}}{h} \right) f(x) \tag{11b}$$

if both limits exist, are well-defined, and equal. Here $\mathsf{1}$ is the identity (see eq. (100) in the appendix),

$$\Delta_{\pm h} = \mathrm{T}_{\pm}^{h} - \mathsf{1} \tag{12}$$

are the right/left-sided difference operators appearing in eq. (6), and

$$\left(\mathrm{T}_{\pm}^{h} f\right)(x) = f(x \pm h) \tag{13}$$

stands for translations of f to the left (+) or right (-) by h.

2.5 Iteration. The algebra of integer powers of D and I in eq. (1) is the starting point of fractional calculus. Let $A : \mathsf{X} \to \mathsf{X}$ be a linear operator on a linear space X with domain $\mathsf{D}(A) \subset \mathsf{X}$. For $n \in \mathbb{N}$ the n-th power of an operator

$$A^n := \underbrace{A \circ \cdots \circ A}_{n\text{-times}}. \tag{14}$$

is interpreted as its n-fold composition or iteration and defined recursively as

$$A^0 := \mathsf{1}, \qquad \mathsf{D}(A^0) = \mathsf{X} \tag{15a}$$

$$A^{n+1} f := A^n \circ (A\, f), \qquad \mathsf{D}(A^{n+1}) = \{f \in \mathsf{D}(A) |\, A\, f \in \mathsf{D}(A^n)\} \tag{15b}$$

where $\mathsf{1}$ is the identity operator on X (see eq. (99), Appendix). Then the law of exponents $A^n\, A^m = A^{n+m}$ holds for $n, m \in \mathbb{N}$. This shows that $A^{n+1} = A\, A^n$ and thus the domains are shrinking $\mathsf{D}(A^{n+1}) \subset \mathsf{D}(A^n)$. Moreover $A^n(\mathsf{D}(A^m)) \subset \mathsf{D}(A^{m-n})$ for $m \geq n$.

2.6 Iterated Integrals. Let $\mathbb{J} = (a, b) \subseteq \mathbb{R}$ with $-\infty \leq a < b \leq \infty$ as in paragraph 2.2, and let $f \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{J}) := \{f : \mathbb{J} \to \mathbb{R} \,|\, f$ is integrable on all compact subsets $K \subset \Omega\}$ be locally integrable. Iterating $I_{a+} : \mathsf{D}(I_{a+}) \to \mathsf{L}^1_{\mathrm{loc}}(\mathbb{J})$ from eq. (10) gives

$$\left[(I_{a+})^n f\right](x) = \int_a^x \int_a^{y_1} \dots \int_a^{y_{n-1}} f(y_n)\, \mathrm{d}y_n \dots \mathrm{d}y_2 \mathrm{d}y_1 \tag{16a}$$

$$= \frac{1}{(n-1)!} \int_a^x (x-y)^{n-1} f(y)\, \mathrm{d}y = \int_a^x f(y) \left[\mathrm{T}_{-}^{x}\, (\mathrm{S}\, K^n)\right](y)\, \mathrm{d}y \tag{16b}$$

where $n \in \mathbb{N}$ and $\mathsf{D}(I_{a+}) = \{f \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{J}) : I_{a+} f \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{J})\}$. The function $K^n : \mathbb{R} \to \mathbb{R}$ is defined as

$$K^n(x) := \Theta(x)\, P_{n-1}(x) = \Theta(x)\, \frac{x^{n-1}}{\Gamma(n)} \tag{17}$$

where P_n is from eq. (2), Γ is Euler's Gamma function in eq. (5b) and $\Theta : \mathbb{R} \to \mathbb{R}$

$$\Theta(x) := \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases} \tag{18}$$

is the Heaviside step[4] function. The reflection operator $\mathrm{S} : \mathsf{F}(\mathbb{R}) \to \mathsf{F}(\mathbb{R})$ is defined by $(\mathrm{S}\, f)(x) := f(-x)$ for $x \in \mathbb{R}$ and the translation operators $\mathrm{T}^h_\pm : \mathsf{F}(\mathbb{R}) \to \mathsf{F}(\mathbb{R})$ with $h \geq 0$ are given in eq. (13) on the set $\mathsf{F}(\mathbb{R})$ of real-valued functions $f : \mathbb{R} \to \mathbb{R}$. A similar formula $\left[(I_{b-})^n\, f\right](x) = \int_x^b f(y) \left(\mathrm{T}^x_-\, K^n\right)(y)\, \mathrm{d}y$ holds for I_{b-}.

2.7 Iterated Derivatives. Let $\mathbb{J} = (a, b) \subseteq \mathbb{R}$ with $-\infty \leq a < b \leq \infty$. Iterates of the translation operators for $f : \mathbb{J} \to \mathbb{R}$

$$\left[\left(\mathrm{T}^h_\pm\right)^n f\right](x) = \left[\left(\mathrm{T}^{nh}_\pm\right) f\right](x) = f(x \pm nh), \qquad n \in \mathbb{N}, a < x < b \tag{19}$$

are well defined for h with $|h| < \min\{x - a, b - x\}/n$. The iterated derivative

$$(D^n\, f)(x) = \lim_{h\to 0} \left(\frac{\Delta_{\pm h}}{h}\right)^n f(x) = \lim_{h\to 0} \left(\frac{\mathrm{T}^h_\pm - 1}{h}\right)^n f(x) \tag{20a}$$

$$= \lim_{h\to 0} \frac{1}{h^n} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (\mathrm{T}^{hk}_\pm\, f)(x) \tag{20b}$$

is defined, if the limits exist and are equal. A domain for $D : \mathsf{C}(\mathbb{J}) \to \mathsf{C}(\mathbb{J})$ is

$$\mathsf{D}(D) = \{f \in \mathsf{C}^1_0(\mathbb{J}) : D\, f \in \mathsf{C}_0(\mathbb{J})\} \tag{21}$$

where $\mathsf{C}(\mathbb{J}) := \{f : \mathbb{J} \to \mathbb{R} : f \text{ is continuous}\}$, $\mathsf{C}^k_0(\mathbb{J}) := \{f \in \mathsf{C}(\mathbb{J}) : f \text{ is } k\text{-times continuously differentiable}\}$, and $\mathsf{C}_0(\mathbb{J}) := \{f \in \mathsf{C}(\mathbb{J}) : f \text{ vanishes at } \infty\}$.

3 Mathematical Interpretations

3.1 It seems pertinent to specify what is meant by an interpretation.

Definition 1. A *mathematical interpretation* of fractional derivatives D^α or integrals I^α is an incomplete mathematical definition. Interpretations are abbreviated as (A) $\multimap$ (B), which is to be read as "(A) is interpreted as (B)".

3.2 Many mathematical interpretations of fractional derivatives and integrals are based on eqs. (16) or (20) above (see Sections 4 and 5 below). For more information on standard interpretations see [76] or the preceding chapter [111].

4 Other conventions for $\Theta(0)$ are sometimes used.

4 Standard Interpretations for Integrals

4.1 Riemann-Liouville Interpretation. A standard interpretation of fractional integration is

$$I^{\alpha}_{a+} \coloneqq (I_{a+})^{n} \text{ from eq.(16b)} \tag{22}$$

with $\alpha \coloneqq n \notin \mathbb{N}$ interpreted as a non-integer power of integration.

4.2 Riemann-Liouville Integrals. Let $\mathbb{J} = (a, b)$ and $-\infty < a < x < b < \infty$. *Riemann-Liouville fractional integrals of order* $\alpha > 0$ are defined as

$$(I^{\alpha}_{a+} f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_{a}^{x} (x - y)^{\alpha - 1} f(y) \, \mathrm{d}y \tag{23a}$$

$$(I^{\alpha}_{b-} f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_{x}^{b} (y - x)^{\alpha - 1} f(y) \, \mathrm{d}y \tag{23b}$$

for $f \in \mathsf{L}^1(\mathbb{J})$ [67, 92, 96, 171, 169, 27]. For $\alpha = 0$ the specification $(I^0_{a+} f)(x) = (I^0_{b-} f)(x) = f(x)$ completes the definition[5]. Some authors require piecewise continuity [145, p.45]. For $f \in \mathsf{L}^1(\mathbb{J})$ the fractional integral $(I^{\alpha}_{a+} f)(x)$ exists for almost every $x \in \mathbb{J}$. Then $I^{\alpha}_{a+} f \in \mathsf{L}^1((a, c))$ for every $a < c < b$ and its L^1-norm is finite. If $f \in \mathsf{L}^p(\mathbb{J})$ with $1 \leq p \leq \infty$ and $\alpha > 1/p$ then $(I^{\alpha}_{a+} f)(x)$ is finite for all $x \in \mathbb{J}$. On Lebesgue spaces $\mathsf{L}^p(\mathbb{J})$ for $1 \leq p \leq \infty$ the fractional integral I^{α}_{a+} is a bounded non-negative operator. It is unbounded (and non-negative) on $\mathsf{L}^p((a, \infty))$, i.e. for $b = \infty$. The definition of I^{α}_{a+} may be generalized to $\alpha \in \mathbb{C}$ with $\mathrm{Re}\, \alpha > 0$.

4.3 Weyl Interpretation. The Riemann-Liouville integral $I^{\alpha}_{a+} f$ of a periodic function $f(x) \sim \sum_k c_k e^{\mathrm{i}kx}$ will in general not be periodic. Integration of periodic functions amounts to division of the Fourier transform with the Fourier variable. This lead Weyl to interpret

$$I^{\alpha}_{\pm} f \coloneqq \mathscr{F}^{-1}\left\{(\pm \mathrm{i}k)^{-\alpha} \mathscr{F}\{f\}\right\} \tag{24}$$

as a Fourier multiplication operator [205], where $\mathscr{F}\{f\}(k)$ is the Fourier transform of $f(x)$, and $f(x) \sim \sum_k c_k e^{\mathrm{i}kx}$ with $c_0 = 0$ is periodic.

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5 The notation for fractional integrals has varied over time. Leibniz, Lagrange and Liouville used the symbol $\int^{\alpha}$ [123, 61, 130], Grünwald wrote $\int^{\alpha}[...\mathrm{d}x^{\alpha}]^{x=x}_{x=a}$, while Riemann used $\partial^{-\alpha}_{x}$ [163], Most $\mathrm{d}^{-\alpha}_{a}/\mathrm{d}x^{-\alpha}$ [151], Krug $\overset{x}{\underset{a}{D}}$ [116] and Weyl J^{α} [205]. The notation in (23a) is that of [171, 169, 79, 77]. Modern authors also use f_{α} [67], I^{α} [166], ${}_{a}I^{\alpha}_{x}$ [27], I^{α}_{x} [39], ${}_{a}D^{-\alpha}_{x}$ [145, 176, 156], or $\mathrm{d}^{-\alpha}/\mathrm{d}(x-a)^{-\alpha}$ [157] instead of I^{α}_{a+}.

4.4 Weyl Integrals. Let $\Omega = \mathbb{R}/2\pi\mathbb{Z}$ be the unit circle and let $f \in \mathsf{L}^p(\Omega)$ for $1 \le p < \infty$ be a 2π-periodic function such that its integral over a period vanishes. The *Weyl fractional integral of order* α is defined as

$$(I^\alpha_\pm f)(x) := \frac{1}{2\pi}\int_0^{2\pi} \sum_{\substack{k=-\infty\\k\neq 0}}^{\infty} \frac{e^{\mathrm{i}k(x-y)}}{(\pm\mathrm{i}k)^\alpha} f(y)\,\mathrm{d}y \tag{25}$$

for $0 < \alpha < 1$ [171, 27]. For such 2π-periodic f with vanishing integral over a period the relations $I^\alpha_+ f = I^\alpha_{a+} f$ with $a = -\infty$ and $I^\alpha_- f = I^\alpha_{b-} f$ with $b = \infty$ hold [212]. They motivate an extension of Weyl integrals as improper integrals

$$(I^\alpha_+ f)(x) = \lim_{a\to-\infty}(I^\alpha_{a+} f)(x), \qquad f \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}^-) \tag{26a}$$

$$(I^\alpha_- f)(x) = \lim_{b\to\infty}(I^\alpha_{b-} f)(x) \qquad f \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}^+) \tag{26b}$$

to locally integrable functions $f : \mathbb{R}^\pm \to \mathbb{R}$ [49, 171, 145, 27]. Then $I^\alpha_- f \in \mathsf{L}^1([a,b])$ for $0 < a < b$ and $I^\alpha_+ f \in \mathsf{L}^1([a,b])$ for $a < b < 0$. As operators on $\mathsf{L}^p(\mathbb{R}^\pm)$ with $1 \le p < \infty$ the domains are (see [137])

$$\mathsf{D}(I^\alpha_\pm) = \left\{ f \in \mathsf{L}^p(\mathbb{R}^\mp) : \exists\, x \lesseqgtr 0 \,\text{s.t.}\, (I^\alpha_\pm f)(x) \text{ exists and } I^\alpha_\pm f \in \mathsf{L}^p(\mathbb{R}^\mp) \right\}. \tag{27}$$

4.5 Convolution. For $f \in \mathsf{L}^1(\mathbb{R}^\pm)$ the Weyl fractional integral may be written as a convolution

$$(I^\alpha_\pm f)(x) = (K^\alpha_\pm * f)(x) \tag{28}$$

where the convolution kernels are defined as $K^\alpha_+ := K^\alpha$ and $K^\alpha_- := \mathrm{S}\,K^\alpha$ with K^α defined by extending K^n in eq. (17) from $n \in \mathbb{N}$ to $\alpha > 0$. For $\alpha = 0$ the definition is $K^0_+(x) = K^0_-(x) = \delta(x)$ with the Dirac distribution at 0. Convolution is defined pointwise as

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x-y)g(y)\,\mathrm{d}y \quad \text{resp.} \quad (f * g)(x) = \int_0^x f(x-y)g(y)\,\mathrm{d}y \tag{29}$$

for functions on $\mathbb{R}$ resp. $\mathbb{R}^+$ for $x \in \mathbb{R}^+$.

4.6 Riesz Integrals. Let $f \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R})$ be locally integrable. The *conjugate Riesz fractional integral* of order $\alpha > 0$ is defined as

$$(\widetilde{I^\alpha} f)(x) = \frac{1}{2\Gamma(\alpha)\sin(\alpha\pi/2)} \int_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-y)f(y)}{|x-y|^{1-\alpha}}\mathrm{d}y \tag{30}$$

where $\alpha \neq 2k, k \in \mathbb{Z}$. For $\alpha = 0$ one sets $(\widetilde{I^0 f)(x)} = f(x)$. Riesz fractional integration may be written as a convolution $(\widetilde{I^\alpha} f)(x) = (\widetilde{K}^\alpha * f)(x)$ with $\widetilde{K}^\alpha(x) = \frac{K^\alpha_+(x) - K^\alpha_-(x)}{2\sin(\alpha\pi/2)}$ for $\alpha \neq 2k, k \in \mathbb{Z}$, and $K^\alpha_\pm$ from eq. (28). For more information see [27],[82, Sec.2.2.5], [111, Sec.4.9] and Section 19 below.

5 Standard Interpretations for Derivatives

5.1 Riemann-Liouville Interpretation. Riemann [163, p.341] and Liouville [130] suggested to interpret fractional derivatives of order $n+\alpha-m$ with $\alpha>0, n\geq m$

$$D^{\alpha-m+n} \quad \bullet\!- \quad D^n\, I^{m-\alpha} \tag{31}$$

as derivatives of integer order $n\geq m$ of a fractional integral of order $m-\alpha$.

5.2 Fractional Derivatives. The Riemann-Liouville interpretation applies to all fractional integrals in Section 4. It can be generalized to all $\alpha\in\mathbb{C}$ with $\operatorname{Re}\alpha>0$ as

$$(\, D^{\alpha}_{a\pm}\, f)(x) := (\pm 1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}(I^{n-\alpha}_{a\pm}\, f)(x) \tag{32a}$$

$$(\, D^{\alpha}_{\pm}\, f)(x) := (\pm 1)^n \frac{\mathrm{d}^n}{\mathrm{d}x^n}(I^{n-\alpha}_{\pm}\, f)(x) \tag{32b}$$

$$(\, D^{\alpha}\, f)(x) := \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\widetilde{I^{n-\alpha}} f)(x) \tag{32c}$$

where $n=\lceil\operatorname{Re}\alpha\rceil := \min\{k\in\mathbb{Z} : k\geq\operatorname{Re}\alpha\}$ is the smallest integer larger than $\operatorname{Re}\alpha$. These fractional derivatives are named after Riemann-Liouville, Weyl and Riesz respectively[6]. Their domains $\mathsf{D}(D^{\alpha}) = \{f\in\mathsf{D}(I^{n-\alpha}) : \exists g\in\mathsf{D}(D^n) \text{ s.t. } g = I^{n-\alpha}\, f\}$, where D^n stands for $\mathrm{d}^n/\mathrm{d}x^n$, depend on those of D^n and $I^{n-\alpha}$.

5.3 Grünwald-Letnikov Interpretation. Already Liouville [129, p.107] suggested to interpret fractional derivatives

$$D^{\alpha} \quad \bullet\!- \quad \text{Equation (20)} \tag{33}$$

as a limit of n-th order finite difference quotients with $\alpha \quad \bullet\!- \quad n\notin\mathbb{N}$. The suggestion was later taken up in [62, 124, 125]. The *Grünwald-Letnikov fractional derivative of order* $\alpha>0$ is defined as the limit

$$(\,{}^{\mathsf{GL}}D^{\alpha}_{\pm}\, f)(x) := \lim_{h\to 0^+}\frac{1}{h^{\alpha}}(\Delta^{\alpha}_{\pm h} f)(x) \tag{34}$$

of fractional difference quotients whenever the limit exists. The Grünwald-Letnikov fractional derivative is called *pointwise* or *strong* depending on whether

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6 The notation is not standardized. Leibniz and Euler used d^{α} [123, 122, 50] Riemann wrote ∂^{α}_{x} [163], Liouville preferred $\mathrm{d}^{\alpha}/\mathrm{d}x^{\alpha}$ [130], Grünwald used $\{\mathrm{d}^{\alpha}f/\mathrm{d}x^{\alpha}\}^{x=x}_{x=a}$ or $D^{\alpha}[f]^{x=x}_{x=a}$ [62], Marchaud wrote $D^{(\alpha)}_{a}$, and Hardy-Littlewood used an index f^{α} [67]. The notation in (32a) follows [171, 169, 79, 77]. Modern authors also use $I^{-\alpha}$ [166], $I^{-\alpha}_{x}$ [39], ${}_{a}D^{\alpha}_{x}$ [27, 145, 176], $\mathrm{d}^{\alpha}/\mathrm{d}x^{\alpha}$ [210, 176], $\mathrm{d}^{\alpha}/\mathrm{d}(x-a)^{\alpha}$ [157] instead of D^{α}_{a+}.

the limit is taken pointwise or in the norm. For periodic functions in $\mathsf{C}(\mathbb{R}/2\pi\mathbb{Z})$ or $\mathsf{L}^p(\mathbb{R}/2\pi\mathbb{Z})$ with $1 \le p < \infty$ see [27], for non-periodic f see [171].

5.4 Marchaud-Hadamard Interpretation. Marchaud's idea is to interpret the fractional derivative of order $\alpha > 0$ directly

$$D^{\alpha} := I^{-\alpha} \tag{35}$$

as a fractional integral of negative order $-\alpha$ by subtracting divergent parts [136, 65]. For $f : \mathbb{R} \to \mathbb{R}$ this leads to

$$({}^{\mathsf{MH}}D^{\alpha}_{\pm} f)(x) = \frac{\alpha}{\Gamma(1-\alpha)} \int\limits_0^{\infty} \frac{f(x) - f(x \mp y)}{y^{\alpha+1}} \mathrm{d}y \tag{36}$$

and ${}^{\mathsf{MH}}D^{0}_{\pm} f = f$. Marchaud-Hadamard derivatives have a larger domain of definition than Riemann-Liouville derivatives. For the case of functions on bounded intervals and more details see [171, 76, 111].

6 Generalized Riemann-Liouville Interpretation

6.1 Interpretation. For $n = m = 1$ and $0 < \alpha < 1$ the interpretation (31) has been generalized to fractional derivatives of order α

$$D^{\alpha,\beta} := I^{\beta(1-\alpha)}\, D\, I^{(1-\beta)(1-\alpha)} \tag{37}$$

and type $0 \le \beta \le 1$ in [77, p.433]. Using eq. (32) yields *generalized Riemann-Liouville*, *generalized Weyl*, and *generalized Riesz derivatives*.

6.2 The type β of a fractional derivative allows to interpolate continuously from $D^{\alpha}_{a\pm} = D^{\alpha,0}_{a\pm}$ to $\widetilde{D}^{\alpha}_{a\pm} = D^{\alpha,1}_{a\pm}$. The fractional derivative $\widetilde{D}^{\alpha}_{a\pm}$ was introduced in [130, p.10]. A relation between fractional derivatives of the same order but different types is found in [77, p.434]. An operational calculus for generalized Riemann-Liouville derivatives was developed in [90].

6.3 Generalized Riemann-Liouville derivatives have been further generalized [207, 57, 102], reformulated [103] and found applications to telegraph-type equations [173], ultra-hyperbolic equations [38], nonlinear analysis in weighted spaces [55, 199], Ulam stability [198], functional differential inclusions [2], and implicit differential equations [1].

7 Localized Riemann-Liouville Interpretation

7.1 Interpretation. The Riemann-Liouville derivatives $D^{\alpha,\beta}_{a+}$ are nonlocal operators. Localization interprets

$$(\,D^{\alpha,\beta}\,f)(a) \;\;\bullet\!\!-\;\; (\,D^{\alpha,\beta}_{a+}\,f)(a) \text{ or } (\,D^{\alpha,\beta}\,f)(x) \;\;\bullet\!\!-\;\; (\,D^{\alpha,\beta}_{x-}\,f)(x) \tag{38}$$

as a limiting value at the boundary points of the interval (a, x).

7.2 For $-\infty < x < \infty$ the *localized Riemann-Liouville fractional derivative of order* $0 < \alpha < 1$ *and type* β is

$$\frac{\mathrm{d}^{\alpha,\beta} f}{\mathrm{d}x^{\alpha,\beta}}(x) := \lim_{a\to x}(\,D^{\alpha,\beta}_{a+}\,f)(x) = \lim_{b\to x}(\,D^{\alpha,\beta}_{b-}\,f)(x) \tag{39}$$

whenever the two limits exist and are equal. Localized fractional differentiability at x is related to regular variation at x [77]. Let $f : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ be monotonously increasing with $f(0) = 0$ and such that $(\,D^{\alpha,\beta}_{a+}\,f)(x)$ with $0 < \alpha < 1$ and $0 \le \beta \le 1$ is also monotonously increasing in $[a, a + \delta]$ for some $a \ge 0, \delta > 0$. Let Λ be slowly varying near a in the sense of [179], let $0 \le \lambda < \beta(1 - \alpha) + \alpha$ and $C \ge 0$. Then regular variation of f near a with index λ is equivalent to regular variation of $D^{\alpha,\beta}_{a+}\,f$ near a with index $\lambda - \alpha$ [77, p.438].

7.3 Localized fractional derivatives were introduced in [68, 69, 77] and immediately applied to the classification of phase transitions in [68, 69, 70, 72, 71, 77] (see also Section 20 below). Later the localized interpretation was appropriated by [112], who claimed inappropriately an absence of the lower limit.

8 Convolution Quotient Interpretation

8.1 The restriction of Heaviside's step function Θ with convention $\Theta(0) = 1$ as defined in eq. (18) to $\mathbb{R}^+_0$ is the constant function $\Theta(x) = 1$. Inserting $f = g = \Theta$ into eq. (29) shows $(\Theta * \Theta)(x) = x$. Iterating yields

$$(\underbrace{\Theta * \cdots * \Theta}_{n\text{-times}})(x) = \frac{x^{n-1}}{(n-1)!} = K^{n-1}(x), \qquad n \in \mathbb{N}, x \in \mathbb{R}^+_0 \tag{40}$$

with K^n from eq. (17) for the n-th convolution power of Θ. Convolution of Θ with $f \in \mathsf{C}(\mathbb{R}^+_0)$

$$(\Theta * f)(x) = \int\limits_0^x f(y)\,\mathrm{d}y \tag{41}$$

is the operator of integration. It is treated as if it were a multiplication.

8.2 Interpretation. The operational calculus of convolution quotients is based on interpreting fractional integration

$$I^\alpha f \quad \bullet\!- \quad (\Theta * \cdots * \Theta) f = \Theta^n f \tag{42}$$

as an n-fold convolution product with Θ where $\alpha \quad \bullet\!- \quad n \notin \mathbb{N}$.

8.3 Let $\big(\mathsf{C}(\mathbb{R}_0^+), +, *\big)$ denote the commutative ring (over $\mathbb{C}$) of complex-valued continuous functions $f : \mathbb{R}_0^+ \to \mathbb{C}$ with pointwise addition, pointwise multiplication with numbers and convolution as in (29). Then $\Theta \in \mathsf{C}(\mathbb{R}_0^+)$ and Eulers first integral implies the law of exponents $\Theta^\alpha * \Theta^\beta = \Theta^{\alpha+\beta}$ for $\operatorname{Re}\alpha > 0, \operatorname{Re}\beta > 0$.

8.4 The commutative convolution ring $\mathsf{C}(\mathbb{R}_0^+)$ does not contain any divisors of zero, because $f * g = 0$ implies that either $f = 0$ or $g = 0$. The ring $\mathsf{C}(\mathbb{R}_0^+)$ can therefore be extended to a field $\big(\mathsf{Q}(\mathbb{R}_0^+), +, \times\big)$ of convolution quotients in the same way as the ring of integers $\mathbb{Z}$ is extended to the field of rationals $\mathbb{Q}$. The elements of $\mathsf{Q}(\mathbb{R}_0^+)$ are ordered pairs $(f\!:\!g)$ of a convolution numerator f and a convolution denominator $g \neq 0$ defined such that

$$g * (f\!:\!g) = f, \qquad g \neq 0 \tag{43}$$

holds. Addition, multiplication and scalar multiplication in the field $\mathsf{Q}(\mathbb{R}_0^+)$ are defined as

$$(f\!:\!g) + (h\!:\!k) = (f * k + g * h\!:\!g * k) \tag{44a}$$

$$(f\!:\!g) \times (h\!:\!k) = (f * h\!:\!g * k) \tag{44b}$$

$$a\,(f\!:\!g) = (af\!:\!g) \tag{44c}$$

for $f, g, h, k \in \mathsf{C}(\mathbb{R}_0^+), g \neq 0, k \neq 0, a \in \mathbb{C}$. Note that $a\,(f\!:\!g) \neq a \times (f\!:\!g)$ where $a \times (f\!:\!g) = a\Theta \times (f\!:\!g) = a\,(\Theta * f\!:\!g)$, i.e. multiplication with a number is not the same as multiplication with a constant function in $\mathsf{Q}(\mathbb{R}_0^+)$.

8.5 The neutral element for multiplication $\delta = (\Theta\!:\!\Theta)$ acts like a Dirac δ-function. This suggests to interpret convolution quotients as generalized functions. The mappings

$$a \mapsto (a\Theta\!:\!\Theta)\,, \qquad a \in \mathbb{C} \tag{45a}$$

$$f \mapsto (\Theta * f\!:\!\Theta)\,, \qquad f \in \mathsf{C}(\mathbb{R}_0^+) \tag{45b}$$

$$f \mapsto (\Theta * f\!:\!\Theta)\,, \qquad f \in \mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_0^+) \tag{45c}$$

are embeddings of $\mathbb{C}$, $\mathsf{C}(\mathbb{R}_0^+)$ resp. $\mathsf{L}^1_{\mathrm{loc}}(\mathbb{R}_0^+)$ into the field $\mathsf{Q}(\mathbb{R}_0^+)$.

8.6 The definition of fractional integration as convolution with Θ^α for $\operatorname{Re}\alpha > 0$ can be extended also to all $\alpha \in \mathbb{C}$ with $\operatorname{Re}\alpha < 0$ as

$$\Theta^\alpha = \big(\Theta^{\alpha+n}\!:\!\Theta^n\big) \tag{46}$$

where $n = -(\lfloor \operatorname{Re}\alpha \rfloor - 1)$ is the smallest positive integer such that $\operatorname{Re}\alpha + n > 1$. For $\alpha = -1$ one finds

$$D = \Theta^{-1} = \big(\Theta : \Theta^2\big) \tag{47}$$

and this is interpreted as the differentiation operator D. The fractional derivative operators are $D^\alpha = \Theta^{-\alpha}$ with $\Theta^0 = \delta = D^0$ and obey $D^\alpha\, D^\beta = D^{\alpha+\beta}$.

8.7 The operational calculus in $\mathsf{Q}(\mathbb{R}_0^+)$ is called *Mikusinski calculus* [144, 208]. For applications see [144, 48, 208, 60] and [132] in this handbook. An operational calculus for generalized Riemann-Liouville derivatives was given in [90].

9 Distributional Interpretation

9.1 Let $f(x) \in \mathsf{L}^p(\mathbb{J}), g \in \mathsf{L}^q(\mathbb{J})$ with $1/p + 1/q \leq 1 + \alpha$, $p, q \geq 1$ and $p \neq 1$, $q \neq 1$ for $1/p + 1/q = 1 + \alpha$. Fractional integration by parts

$$\int_a^b f(x)(I^\alpha_{a+}\, g)(x)\mathrm{d}x = \int_a^b g(x)(I^\alpha_{b-}\, f)(x)\mathrm{d}x \tag{48}$$

can be used to extend fractional integrals to distributions, if g is viewed as a test function from a space mapped to itself by I^α_{b-}. Let $\mathsf{C}^\infty_\mathrm{c}(\mathbb{R})$ denote the space of test functions, i.e. smooth functions $f : \mathbb{R} \to \mathbb{R}$ with compact support. Its topological dual space consisting of continuous linear forms $u : \mathsf{C}^\infty_\mathrm{c}(\mathbb{R}) \to \mathbb{R}$ is the space of distributions and denoted as $\mathcal{D}'(\mathbb{R})$. Let $\mathcal{D}'_+(\mathbb{R})$ denote the set of distributions u such that there is an $a \in \mathbb{R}$ with $\operatorname{supp} u \subset [a, \infty)$.

9.2 **Interpretation.** The distributional interpretation views differentiation

$$D^\alpha\, u \quad \bullet\!\!-\quad \delta^{(n)} * u \tag{49}$$

as convolution with the n-th derivative $\delta^{(n)}$ of the Dirac-δ when $\alpha \quad \bullet\!\!- \quad n \notin \mathbb{N}$.

9.3 The distributional interpretation is based on the relations $D^n\, K^n = \delta$ and $D\, K^{n+1} = K^n$ for all $n \in \mathbb{N}$ in the sense of distributions, and the fact that

$$K^\alpha \in \mathcal{D}'(\mathbb{R}) \qquad \text{and } \operatorname{supp} K^\alpha \subset \mathbb{R}_0^+ \tag{50a}$$

$$K^{-n} = \delta^{(n)} \tag{50b}$$

$$K^\alpha * K^\beta = K^{\alpha+\beta} \tag{50c}$$

hold true for all $\alpha, \beta \in \mathbb{C}$ and $n \in \mathbb{N} \cup \{0\}$. As a result $I^\alpha_{0+} : \mathcal{D}'_+(\mathbb{R}) \to \mathcal{D}'_+(\mathbb{R})$ with $I^\alpha_{0+}\, u = K^\alpha * u$ is a bounded linear operator on $\mathcal{D}'_+(\mathbb{R})$ for all $\alpha \in \mathbb{C}$. It fulfills additivity $I^\alpha_{0+}\, I^\beta_{0+} = I^{\alpha+\beta}_{0+}$ and $I^{-\alpha}_{0+} = D^\alpha_{0+}$ [178, 58, 40].

10 Functional Calculus Interpretation

10.1 Interpretation. Let $A : \mathsf{X} \to \mathsf{X}$ be a closed operator (see A.4) on a Banach space X and $\mathscr{A}(\sigma(A))$ an algebra of functions $F : \sigma(A) \to \mathbb{C}$ on the spectrum of A. A *functional calculus for A* is interpreted as a mapping

$$\mathscr{A} \ni F \mapsto B \;\bullet\!\!-\; F(A) \in \mathcal{C}(\mathsf{X}) \tag{51}$$

that assigns to each $F \in \mathscr{A}$ a closed operator $B : \mathsf{X} \to \mathsf{X}$ interpreted as $F(A)$ in such a way that for $B \;\bullet\!\!-\; F(A)$ and $C \;\bullet\!\!-\; G(A)$ also $B \circ C \;\bullet\!\!-\; (F \circ G)(A)$.

10.2 The natural powers A^n from paragraph 2.5 suffice to define polynomial or rational functions of A as examples. Let $F(z) = \sum_{k=0}^{\deg(F)} a_k z^k$ be a polynomial of degree $\deg(F)$ with complex coefficients $a_k \in \mathbb{C}$. Then the operator

$$F(A) := \sum_{k=0}^{\deg(F)} a_k\, A^k, \qquad \mathsf{D}(F(A)) = \mathsf{D}(A^{\deg(F)}) \tag{52}$$

is well defined. If $\rho(A) \neq \emptyset$, then $F(A)$ is a closed operator for each polynomial F in the polynomial ring $\mathbb{C}[z]$. The spectral mapping theorem $\sigma((F(A)) = F(\sigma(A))$ holds. If a bounded operator commutes with A, then it commutes also with $F(A)$. The mapping $\mathbb{C}[z] \ni F \mapsto F(A)$ is a functional calculus for polynomials.

10.3 Let $F, G \in \mathbb{C}[z]$ be two polynomials and let G be such that its set of zeros $\{\lambda \in \mathbb{C} : G(\lambda) = 0\} \subset \rho(A)$ is contained in the resolvent set of A (see A.5). The rational function

$$h(A) := F(A)G(A)^{-1} \tag{53}$$

of A is well defined with domain

$$\mathsf{D}\left(h(A)\right) = \begin{cases} \mathsf{D}\left(A^{\deg(F)-\deg(G)}\right), & \text{if } \deg(F) \geq \deg(G) \\ \mathsf{X} & \text{otherwise.} \end{cases} \tag{54}$$

Again $h(A)$ is a closed operator. Its spectrum obeys $h(\overline{\sigma}(A)) \subset \overline{\sigma}(h(A))$ where

$$\overline{\sigma}(A) = \begin{cases} \sigma(A), & \text{if } A \text{ is bounded} \\ \sigma(A) \cup \{\infty\} & \text{otherwise} \end{cases} \tag{55}$$

is the *extended spectrum of A*. The mapping $h \mapsto h(A)$ is a rational calculus for linear operators on X.

11 Spectral Projection Interpretation

11.1 In finite dimensional spaces X an operator $A : \mathsf{X} \to \mathsf{X}$ is a matrix. If it can be transformed into diagonal form $A = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, the eigenvalues λ_i appear on the diagonal. The fractional power A^α is then defined as $A^\alpha = \mathrm{diag}(\lambda_1^\alpha, \dots, \lambda_n^\alpha)$. This A^α has the same eigenvectors as A, and, if λ is an eigenvalue of A, then λ^α is an eigenvalue of A^α. The finite dimensional calculus is extended to Hilbert spaces using the spectral theorem.
11.2 Let $A : \mathsf{H} \to \mathsf{H}$ denote a selfadjoint operator on a Hilbert space H with scalar product $(\cdot, \cdot)$. Its domain is denoted as $\mathsf{D}(A)$, its spectrum as $\sigma(A)$ and its spectral family as E_λ. Then

$$(A\,u, v) = \int\limits_{\sigma(A)} \lambda \,\mathrm{d}(E_\lambda u, v) \tag{56}$$

holds for all $u, v \in \mathsf{D}(A)$. The fractional power A^α is defined by

$$(A^\alpha\,u, u) := \int\limits_{\sigma(A)} \lambda^\alpha \,\mathrm{d}(E_\lambda u, v) \tag{57}$$

on the domain $\mathsf{D}(A^\alpha) = \{u \in \mathsf{H} : (A^\alpha\,u, u) < \infty\}$. Generally, for any Borel measurable function $g : \sigma(A) \to \mathbb{C}$ the operator $g(A)$ is defined by replacing the integrand in eq. (57) with $g(\lambda)$. The mapping $g \mapsto g(A)$ is sometimes referred to as *spectral calculus*.
11.3 Fractional spectral calculus was studied in [127]. It was shown that the domains $\mathsf{D}(A^\alpha)$ for m-accretive operators coincide with certain real interpolation spaces constructed by the trace method.

12 Cauchy Integral Interpretation

12.1 Interpretation. Let $F : \Omega \to \mathbb{C}$ be a holomorphic function in an open domain $\Omega \subset \mathbb{C}$ and let $\mathscr{C} : [0,1] \to \Omega$ be a closed path inside Ω so that $\mathscr{C}(0) = \mathscr{C}(1)$. Let $a \in \Omega$ be a point in the interior of the region encircled by $\mathscr{C}$. Then Cauchy's integral formula

$$F(a) = \frac{1}{2\pi\mathrm{i}} \int\limits_{\mathscr{C}} \frac{F(z)}{z-a}\,\mathrm{d}z \tag{58}$$

holds, where the path is traversed counter clockwise. The interpretation $F(a) \;\bullet\!\!-\; F(A)$ of the number $F(a)$ as the operator $F(A)$ is based on the interpretation

$$(z-a)^{-1} \quad \bullet\!\!- \quad R(z,A) \tag{59}$$

of the function $(z-a)^{-1}$ as the resolvent operator of A at z defined in A.5.

12.2 Let $A \in \mathcal{B}(\mathsf{X})$ be a bounded operator on a Banach space X with spectrum $\sigma(A) \subset \Omega \subset \mathbb{C}$ contained within an open set Ω. Let $\mathsf{Hol}(\Omega)$ denote the algebra of holomorphic functions $F : \Omega \to \mathbb{C}$ and define the mapping $\Phi : \mathsf{Hol}(\Omega) \to \mathcal{B}(\mathsf{X})$

$$\Phi(F) = F(A) := \frac{1}{2\pi\mathrm{i}} \int\limits_{\mathscr{C}} F(\lambda) R(\lambda, A)\, \mathrm{d}\lambda \tag{60}$$

by Cauchy's integral. Here $\mathscr{C}$ is a path encircling the spectrum $\sigma(A)$ of A in a positive sense. Then Φ is characterized as the unique map satisfying:

(a) Φ is an algebraic homomorphism.

(b) If P_α are the functions $P_\alpha(z) = z^\alpha$ for $\alpha \in \mathbb{C}$, then $\Phi(P_0) = 1_{\mathsf{X}}$ and $\Phi(P_1) = A$.

(c) If a sequence $F_n \in \mathsf{Hol}(\Omega)$ converges unifomly on compact sets to $F \in \mathsf{Hol}(\Omega)$, then $\Phi(F_n) \to \Phi(F)$ in $\mathcal{B}(\mathsf{X})$.

The mapping Φ is called *Riesz-Dunford calculus* for bounded operators.

12.3 If the domain Ω of F is such that $\mathbb{C} \setminus \Omega \cap \sigma(A) \neq \emptyset$, then the integral is singular. The problem can be overcome by finding a function $g : \Omega_g \to \mathbb{C}$ such that $\mathbb{C} \setminus \Omega_g \cap \sigma(A) = \emptyset$ and $\mathbb{C} \setminus \Omega_h \cap \sigma(A) = \emptyset$ where $h = Fg^{-1}$, and such that eq.(60) can be used to define $g(A)$ and $h(A)$. Then $F(A)$ is defined as $F(A) := g(A)h(A)$. Examples where $g(z) = (1+z)^n$ is a polynomial can be found in [9, 138].

12.4 This Riesz-Dunford calculus for $\mathcal{B}(\mathsf{X})$ can be generalized to sectorial operators [138]. Because sectorial operators (defined below in A.6) can have unbounded spectra, the path $\mathscr{C}$, as a curve on the Riemann sphere, passes through the point at ∞. The function F therefore must decrease sufficiently rapidly at ∞ for the Cauchy integral to make sense. Suitable functions $F : \mathbb{S}_\phi \to \mathbb{C}$ on a sector $\mathbb{S}_\phi$ (see eq. (103)) are those for which there exists $C \geq 0$ and $\gamma > 0$ such that $|F(z)| \leq C \min\{|z|^\gamma, |z|^{-\gamma}\}$ for all $z \in \mathbb{S}_\phi$. For such functions an operator $F(A) \in \mathcal{C}(\mathsf{X})$ can be defined by the Cauchy algorithm for a suitably restricted class of functions [5]. The complex powers defined in this way coincide with the complex powers studied in [113].

12.5 The benefits of Cauchy integrals for fractional calculus were highlighted in [116] and an incipient functional calculus for compact operators emerged from [54, 164, 206]. Later developments [192, 131, 41] were summarized in [42] (see also [170, Chapter 10]. The application of Cauchy integrals to fractional powers of sectorial operators was studied in [137].

13 Laplace Transform Interpretation

13.1 Interpretation. Let $\mathcal{M}(\mathbb{R}_0^+)$ be the set of all complex Borel measures on $\mathbb{R}_0^+$. The Laplace transform $F := \mathscr{L}\{\mu\}$ of a measure $\mu \in \mathcal{M}(\mathbb{R}_0^+)$ is

$$F(u) = \int_0^\infty e^{-ut} \mathrm{d}\mu(t) \tag{61}$$

for all $u \in \mathbb{C}$ with $\operatorname{Re} u > 0$. The interpretation $F(a) \;\bullet\!\!-\; F(-A)$ of the number $F(a) \in \mathbb{C}$ as an operator $F(-A)$ is now based on the interpretation

$$e^{-at} \;\bullet\!\!-\; e^{At} \;\bullet\!\!-\; T(t) \tag{62}$$

of the function e^{-at} as a semigroup $T(t) = e^{At}$ with infinitesimal generator A.

13.2 Let X be a Banach space with norm $\|\cdot\|$. A one-parameter family of operators $\{T(t)\}_{t\geq 0} \subset \mathcal{B}(\mathsf{X})$ is called a *bounded strongly continuous semigroup* if
(a) $T(t) \in \mathcal{B}(\mathsf{X})$ for all $t \geq 0$ and $\sup\{\|T(t)\| : t \geq 0\} < \infty$
(b) $T(0) = 1_{\mathsf{X}}$ and $T(t)T(s) = T(t+s)$ for all $t, s \geq 0$
(c) $\lim_{t\to 0^+} \|T(t)f - f\| = 0$ for all $f \in \mathsf{X}$.

The operator

$$A := \lim_{t\to 0^+} \frac{T(t)f - f}{t} \tag{63}$$

with $\mathsf{D}(A) = \{f \in \mathsf{X} : \text{the limit (63) exist}\}$, is called *infinitesimal generator of the semigroup*. Generators of strongly continuous semigroups are closed and densely defined operators that uniquely determine $T(t)$. Their spectrum lies in the left half plane $\mathbb{C} \setminus \mathbb{S}_{\pi/2}$ [92, 28, 46].

13.3 Denote the set of Laplace transformations of measures from $\mathcal{M}(\mathbb{R}_0^+)$ as

$$\mathscr{L}\left\{\mathcal{M}(\mathbb{R}_0^+)\right\} := \{F : \overline{\mathbb{S}}_{\pi/2} \to \mathbb{C} \mid F = \mathscr{L}\{\mu\} \text{ for some } \mu \in \mathcal{M}(\mathbb{R}_0^+)\} \tag{64}$$

and let A be the infinitesimal generator of a bounded strongly continuous semigroup on X. If the function $F \in \mathsf{Hol}(\overline{\sigma}(-A)) \cap \mathscr{L}\left\{\mathcal{M}(\mathbb{R}_0^+)\right\}$ corresponds to the measure $\mu \in \mathcal{M}(\mathbb{R}_0^+)$, then the bounded linear operator defined as

$$f \mapsto \int_0^\infty T(t)f \,\mathrm{d}\mu(t), \qquad f \in \mathsf{X} \tag{65}$$

defined for all $f \in \mathsf{X}$ equals the operator $F(-A)$ obtained from the extended Riesz-Dunford calculus in paragraph 12.4 above. It is written as $F(-A)$ or $\Phi(\mu) \in \mathcal{B}(\mathsf{X})$.

13.4 The triple $(\mathcal{M}(\mathbb{R}_0^+), +, *)$ is an algebra with convolution of measures as multiplication. The triple $(\mathscr{L}\left\{\mathcal{M}(\mathbb{R}_0^+)\right\}, +, \cdot\,)$ is the corresponing function algebra by virtue of the convolution theorem. The mapping

$$\begin{aligned} \Phi : \mathscr{L}\left\{\mathcal{M}(\mathbb{R}_0^+)\right\} &\to \mathcal{B}(\mathsf{X}) \\ F &\mapsto \Phi(\mu) = F(-A) \end{aligned} \tag{66}$$

with $F(-A)$ from eq.(65) above is the unique map satisfying:

(a) Φ is an algebraic homomorphism.

(b) if $G_z(\lambda) = (z-\lambda)^{-1}$ with $\operatorname{Re} z < 0$, then $\Phi(G_z) = (z-A)^{-1} = R(z, A)$

(c) if the sequence F_n corresponding to the sequence of measure μ_n converges weakly to the limit F in $\mathscr{L}\left\{\mathcal{M}(\mathbb{R}_0^+)\right\}$, then $\lim_{n\to\infty} \|\Phi(F_n)f - \Phi(F)f\| = 0$ for all $f \in \mathsf{X}$.

The mapping Φ is called *generalized Hille-Phillips calculus*. The Laplace transform interpretation was developed originally for functions in [159, 92] and then extended to distributions in [154, 203].

13.5 Rescaling the measure μ as $\mu(\cdot/s)$ or the semigroup T as $T(\,\cdot\, s)$ with a parameter $s > 0$ gives a one-parameter family of bounded linear operators

$$\Phi_s(\mu) = \int_0^\infty T(ts)\,\mathrm{d}\mu(t) = \int_0^\infty T(t)\mathrm{d}\mu(t/s) \tag{67a}$$

on X that can be used to obtain several known interpretations of fractional powers via integral representations. Let δ_x denote the Dirac measure at $x \geq 0$ and $D\,\delta_x$ its distributional derivative. Then

$$\mu = \delta_0 \qquad \longrightarrow \qquad \Phi_s(\mu) = 1_{\mathsf{X}} \tag{67b}$$

$$\mu = D^n\,\delta_0 \qquad \longrightarrow \qquad \Phi_s(\mu) = (-A)^n \tag{67c}$$

for all $s > 0$ and $n \in \mathbb{N}$ [204, Thm 2.5], while

$$\mu = \delta_1 \qquad \longrightarrow \qquad \Phi_s(\mu) = T(s) \tag{67d}$$

reproduces the semigroup for the rescaling parameter, and

$$\mu = \Theta(t)e^{-t\arg s}\mathrm{d}t \qquad \longrightarrow \qquad \Phi_{|s|}(\mu) = (s-A)^{-1} = R(s, A) \tag{67e}$$

yields its resolvent family for $s \in \mathbb{C}$ and $\operatorname{Re} s > 0$. The examples

$$\mu = D^n(\delta_0 - \delta_1) \qquad \longrightarrow \qquad \Phi_s(\mu) = [1_{\mathsf{X}} - T(s)]^n \tag{67f}$$

$$\mu = D^n(\delta_0 - e^{-t}\mathrm{d}t) \qquad \longrightarrow \qquad \Phi_s(\mu) = \left[1_{\mathsf{X}} - (1_{\mathsf{X}} - sA)^{-1}\right]^n \tag{67g}$$

with $n \in \mathbb{N}$ were studied in [188]

13.6 For a large set of so-called n-measures the fractional powers, when interpreted as the limit

$$(-A)^\alpha := \lim_{\varepsilon\to 0} C_{\alpha,n} \int_\varepsilon^\infty s^{-\alpha-1}\Phi_s(\mu)\,\mathrm{d}s, \qquad 0<\alpha<n, n\in\mathbb{N}, \tag{68}$$

all define the same operator for a suitable constant $C_{\alpha,n}$ and a domain consisting of all $f \in \mathsf{X}$ such that the limit exists [188]. Indeed eq. (67f) for $n = 1$ and $C_{\alpha,1} = \Gamma(\alpha)/\Gamma(1-\alpha)$ was used in [159] and for general n in [128] to define fractional powers. Example (67g) was used in [114] for that purpose. Balakrishnan's interpretation [10]

$$-(-A)^\alpha = \frac{\sin(\alpha\pi)}{\pi}\int_0^\infty AR(s,A)\; s^{\alpha-1}\mathrm{d}s \tag{69a}$$

is obtained from eq. (67f) using the integral representation

$$x^\alpha = \int_0^\infty (1-e^{-xy})\mu(y)\mathrm{d}y = \int_0^\infty \frac{x}{x+y}\mathrm{d}\nu(y) \tag{69b}$$

with $\mu(x) = \alpha\, x^{-1-\alpha}/(\Gamma(1-\alpha))$, $\nu(x) = \sin(\alpha\pi)\; x^{\alpha-1}/\pi$ and $0<\alpha<1$.

14 Integral Transform Interpretations

14.1 An interpretation based on the double Laplace (or Stieltjes) transforms applies to functions f representable as $f(\lambda) = \int_0^\infty\int_0^\infty e^{\lambda z}e^{-zt}\mathrm{d}\mu(t)\mathrm{d}z$ for $\mu \in \mathcal{M}(\mathbb{R}_0^+)$. It has been studied in [93, 94] and is reviewed in [137, Ch. 4].
14.2 An interpretation using Fourier transforms instead of Laplace transforms has been given in [118]. It goes beyond generators of strongly continuous semigroups and considers also weak topologies. In this way it extends and unifies earlier approaches.
14.3 An interpretation in terms of Mellin transforms was developed in [161, 197]. It is well suited for studying purely imaginary powers.

15 Stochastic Interpretation of α

15.1 A stochastic process $(X_t)_t \geq 0$ with state space $\mathbb{R}^d$ is called a Levy process if it has stationary and independent increments, and if its sample paths are right continuous and have left limits [18]. The process is completely determined by its characteristic exponent $\zeta : \mathbb{R}^d \to \mathbb{C}$ defined via the relation

$$E\left(e^{\mathrm{i}X_t \cdot k}\right) = e^{-t\zeta(k)}. \tag{70}$$

where E denotes the operation of taking expectation values and $X_t \cdot k$ the scalar product. The characteristic exponent ζ is continuous and negative definite (see eq. (104)). The characteristic exponent ζ contains all information about the process. For example, the stochastic process is conservative if and only if $\zeta(0) = 0$.

15.2 Interpretation. The stochastic interpretation identifies α from the behaviour of ζ near the origin as

$$\alpha := \frac{\log g_\lambda}{\log \lambda}, \tag{71}$$

i.e. as the index of regular variation [104, 179]. Here

$$g_\lambda := \lim_{|k|\to\infty} \frac{\left|\zeta\left(\lambda |k|^{-1}\right)\right|}{\left|\zeta\left(|k|^{-1}\right)\right|}, \tag{72}$$

provided the limit exists on a set of $\lambda > 0$ with positive measure.

15.3 The stochastic interpretation is the basis for many physical interpretations at some mesoscopic scale. It is also the basis of stochastic differential (Langevin) equations and the continuous time random walk interpretation in Section 22. Its mathematical origin is potential theory (see also Section 19). The positive hyperharmonic functions for the Laplace equation are the excessive functions of the Brownian semigroup, or equivalently, the harmonic measures in potential theory are the hitting distributions for Brownian motion.

15.4 For diffusion with fractional Laplaceans the stochastic interpretation was pioneered in [126, 23, 24, 51], for time fractional diffusion in [89]. The interpretation has been generalized to Feller processes and pseudodifferential equations [100].

16 Geometric Interpretation

16.1 Leibniz' question „... SED QUID EST IN GEOMETRIA?“[7] from September 30th, 1695 seems unanswered to this day. Leibniz' rule (8) has over centuries evolved into the basis for the modern geometric interpretation of derivatives as

7 *„... but what is it in geometry?“*

vectors and tensors [33]. The modern interpretation of D applies also in cases where a classical interpretation of $(D\,f)(x)$ as the slope of the tangent to the graph of f at x, fails [26, 119].

16.2 Iteration of Leibniz' rule (8) yields Leibniz' formula (3) for D^n. Extending eq. (3) from $n \in \mathbb{N}$ to $\alpha \in \mathbb{C}$ leads to a binomial series [92]

$$D^\alpha(fg) = \sum_{k=0}^{\infty} \binom{\alpha}{k} (D^k\, f)(D^{\alpha-k}\, g) \tag{73}$$

where the generalized binomial coefficient is given in eq. (5a). This formula appears already in [129, p. 117] and has been discussed in [158]. Because eq. (73) differs from eq. (3) for $\alpha \neq n, n \in \mathbb{N}$, a geometrical interpretation of fractional derivatives appears remote and difficult.

16.3 Notwithstanding the failure of Leibniz' rule some authors proposed a "fractional curl operator" [47] or "fractional vector calculus" [139]. Inspecting the definition of curl^α in [47, eq.(9)] reveals however that $\mathrm{curl}^0 = 1 \neq \lim_{\alpha\to 0^+} \mathrm{curl}^\alpha$ rendering the definition discontinuous at $\alpha = 0$. More disturbingly there is no indication that the curl of a vector field is a 2-form, i.e. a tensor of rank 2. Inspecting in turn the definitions of fractional curl, divergence and gradient operators in [139, Sec. 3] reveals that the "mixing measure" survives the limit $\alpha \to 1$, so that the definitions do not reduce to their vector calculus counterpart in that limit.

16.4 There exist also publications [12, 37] that speak of "fractional differential forms". The proposed interpretation based on eq.(11) in [37] fails, because eq. (11) in [37] is not defined for functions on $\mathbb{R}^n$ if $n \neq 1$. As a consequence eq. (19) in [37, p.2006] lacks meaning.

16.5 Note, that the localized Riemann-Liouville derivative $\mathrm{d}^{\alpha,\beta} f(x)/\mathrm{d}x^{\alpha,\beta}$ has a certain geometrical interpretation. For $0 < \alpha < 1$ it approximates f at x by a cusp instead of a straight line. The slope of the tangent at x correspond to the "opening coefficient" of the cusp at x. The geometric interpretation was exploited for the generalized Ehrenfest classification [68, 69, 70, 72, 71, 77] (see Section 20 below).

17 Type Changing Interpretation

17.1 Interpretation. An unusual mathematical interpretation of the fractional derivative $D^{\alpha,1}_{0+}$ of order α and type $\beta = 1$ [130, p.10] [29, 30] has become popular in the physics literature [184, eq.(22)] [34, eq.(34)] [32, eq.(2)]. It interprets fractional differential equations of type $\beta = 1$

$$D_{0+}^{\alpha,1} f = g \qquad \bullet\!\!- \qquad D\,f = D_{0+}^{1-\alpha,0}\, g \tag{74}$$

as equations of type $\beta = 0$ with a second differential operator D entering the type-zero-equation.

17.2 To exhibit the unusual character consider $D_{0+}^{\alpha,1} : \mathsf{X} \to \mathsf{Y}$ as a linear operator between two Banach spaces X, Y. Usually its domain $\mathsf{D}(D_{0+}^{\alpha,1})$ and its range $\mathsf{R}(D_{0+}^{\alpha,1})$ are the spaces given in eq.(95), namely

$$\mathsf{D}\left(D_{0+}^{\alpha,1}\right) := \left\{ f \in \mathsf{X} : \exists\, g \in \mathsf{Y} \text{ s.t. } (f,g) \in D_{0+}^{\alpha,1} \right\} \tag{75a}$$

$$\mathsf{R}\left(D_{0+}^{\alpha,1}\right) := \left\{ g \in \mathsf{Y} : \exists\, f \in \mathsf{X} \text{ s.t. } (f,g) \in D_{0+}^{\alpha,1} \right\}. \tag{75b}$$

17.3 In [184, eq.(22)] or [34, eq.(34)] however, the authors interpret the operator $D_{0+}^{\alpha,1} : \mathsf{X} \to \mathsf{Y}$ indirectly by the first order derivative $D : \mathsf{X} \to \mathsf{Y}$ and the fractional Riemann-Liouville derivative $D_{0+}^{1-\alpha,0} : \mathsf{X} \to \mathsf{Y}$ of order $1 - \alpha$ and type $\beta = 0$. Indeed, eq. (74) suggests that the authors interpret $D_{0+}^{\alpha,1}$ as having domain and range given by

$$\mathsf{D}\left(D_{0+}^{\alpha,1}\right) := \left\{ f \in \mathsf{D}(D) : \exists\, g \in \mathsf{Y} \text{ s.t. } (D\,f, g) \in D_{0+}^{\alpha,1} \right\}, \tag{76a}$$

$$\mathsf{R}\left(D_{0+}^{\alpha,1}\right) := \left\{ g \in \mathsf{D}(D_{0+}^{1-\alpha,0}) : \exists\, f \in \mathsf{X} \text{ s.t. } (f, D_{0+}^{1-\alpha,0}\, g) \in D_{0+}^{\alpha,1} \right\}, \tag{76b}$$

which is puzzling. This unusual interpretation seems to restrict the domain without any physical or mathematical motivation or justification.

17.4 In addition, as pointed out already in [82, p.46], this interpretation gives rise to an unusual form of the eigenvalue equation. Nevertheless this interpretation is adopted by numerous authors in physics [14, 185, 174, 184, 143, 209, 43].

18 Physical Interpretations

18.1 While mathematical interpretations of fractional derivatives and integrals abound, physical interpretations are often questionable. Fundamental theories of physics generally contain only integer order derivatives. This raises at least two fundamental questions discussed in [82]:

(a) Are mathematical models with fractional derivatives consistent with the fundamental laws and fundamental symmetries of nature ?

(b) Can the fractional order α of differentiation be related to, or derived from, established theories of physics?

A partially positive answer to (a) is given below in Sections 20 and 21. It suggests

Definition 2. A *physical interpretation* of a fractional derivative D^α or integral I^α is an identification

$$\alpha \quad \bullet\!- \quad (x \longleftrightarrow H) \tag{77}$$

of the fractional order $\alpha \notin \mathbb{N}$ with a quantity x that can be related to (or computed from) the energy H (Hamiltonian) of a physical system. A physical interpretation is called *tentative*, if an established phenomenological but nonrigorous relation $x \longleftrightarrow H$ exists that is supported by experimental or numerical evidence. Other interpretations are called *questionable*.

18.2 Numerous physical interpretations have been attempted in the literature. Readers might consult [76, 202, 108, 107, 196] for reviews and references reflecting the time of their publication. Physical interpretations of fractional derivatives are significantly more difficult than mathematical interpretations. Besides being well-defined a physical interpretation must not contradict established theory or experiment. Given such fundamental constraints surprisingly few proposed interpretations mention, discuss or contemplate the basic questions above.

18.3 Fractional models that arise from reformulating models without fractional operators, will not be considered here. For examples see [157, Ch.10]. Also, questionable physical interpretations will mostly be left out.

18.4 An example for a questionable interpretation is ”fractional duality in electromagnetism” [47]. Equations (18) and (19) in [47] would seem to be in direct conflict with electrodynamics and relativity theory, because the vectors appearing in them cannot be vectors in the usual sense (see also 16.3 above).

18.5 Consider next the “fractional time Schrödinger equation” [99, eq.(8)]. While the Hamiltonian $\widehat{H}$ in [99, eq.(8)] is a (non-dimensional) energy, the operator $(ih)^\alpha \partial^\alpha/\partial t^\alpha$ on the left hand side is a non-dimensional (energy)$^\alpha$ instead. In other words, the left hand side operator cannot be a physical interpretation of the right hand side operator for $\alpha \neq 1$. The error can be traced to [153] where, curiously, also the speed of light and the gravitational constant appear in a non-relativistic equation.

18.6 Similar problems appear in eq. (1.118) in [191, p.39]. Using the notation of [191] and inserting $\rho = R_0^3\rho'$ and $x = x'/R_0, y = y'/R_0, z = z'/R_0$ into eq. (1.118) in [191, p.39], the left hand side has units of charge [C], while the right hand side has units of [Cm^{3-D}] where $D \neq 3$. The same error appears in [191] (e.g. eq. (1.88)) and numerous other publications.

19 Nonlocal Interpretations

19.1 Fractional derivatives with respect to position are nonlocal operators. Locality is a deep and well established fundamental principle of physics [190, 64, 82]. An increasing number of publications “generalize” partial differential equations for physical phenomena by replacing the local Laplace operator Δ with a nonlocal fractional power $-(-\Delta)^{\alpha/2}$. The generalization is usually accompanied by postulating a “mesoscopic” stochastic process along the lines of Section 15. Independent experimental evidence for this process is often absent.

19.2 The mathematical background for such generalizations is fractional potential theory [166]. Let ϱ be a positive measure on $\mathbb{R}^d$ that is absolutely continuous with respect to the d-dimensional Lebesgue measure $\mathrm{d}x$, and let $\rho = \mathrm{d}\varrho/\mathrm{d}x$ be its density function $\rho : \mathbb{R}^d \to \mathbb{R}$. Then for $d \in \mathbb{N}, d > 2$ the integral

$$\Phi_{\varrho,d}(x) = \int\limits_{\mathbb{R}^d} \frac{\mathrm{d}\varrho(y)}{|x-y|^{d-2}}, \qquad x \in \mathbb{R}^d \tag{78}$$

is called the *potential of ϱ (or ρ) at x*, because $-\Phi_{G\varrho,3}$ is the specific potential energy of gravitation (with units [J/kg]), if $G \approx 6.67 \times 10^{-11}\mathrm{kg}^{-1}\mathrm{m}^3\mathrm{s}^{-2}$ is the gravitational constant and ρ is mass density.

19.3 **Interpretation.** Let Δ_d denote the Laplace operator in $\mathbb{R}^d$. The d-dimensional Riesz integral $I^\alpha\,\rho$ (or Riesz derivative $D^\alpha\,\rho$) of a density function $\rho = \mathrm{d}\varrho/\mathrm{d}x$

$$I^\alpha\,\rho \;:=\; \frac{\Gamma\left((d-\alpha)/2\right)}{2^\alpha \pi^{d/2}\Gamma\left(\alpha/2\right)}\,\Phi_{\varrho,d+2-\alpha} \;:=\; (-\Delta_d)^{-\alpha/2}\,\rho \;=:\; D^{-\alpha}\,\rho \tag{79}$$

is interpreted as a Newtonian potential in $d+2-\alpha$ “fractional dimensions”, or as a “fractional potential” in d dimensions [166]. The parameter range $0 < \alpha < d/2$ can be analytically continued to all $\alpha \in \mathbb{C}$ with $\alpha \neq \pm(d+2k), k \in \mathbb{N} \cup \{0\}$.

19.4 Many “generalized” partial differential equations are based on this nonlocal interpretation and thus contain the fractional Dirichlet problem as a special case. The fractional Dirichlet problem for a domain $\mathbb{B}(z,R)$ and fractional order $0 < \alpha \leq 2$ is to find a suitably regular function $f : \mathbb{R}^d \to \mathbb{R}$ obeying [84]

$$(-\Delta)^{\alpha/2} f(x) = 0 \qquad x \in \mathbb{B}(z,R) \tag{80a}$$

$$f(x) = g(x) \qquad x \in \mathbb{R}^d \setminus \mathbb{B}(z,R) \tag{80b}$$

for suitably regular data g with

$$\int\limits_{\mathbb{R}^d\setminus\mathbb{B}(z,R)} \frac{|g(x)|}{1+|x|^{d+\alpha}} < \infty. \tag{81}$$

Here $\mathbb{B}(z,R)=\{x\in\mathbb{R}^d : |x-z|<r\}$ denotes a ball of radius $R>0$ centered at $z\in\mathbb{R}^d$. The solution of the fractional Riesz-Dirichlet problem is the fractional Poisson integral [117]

$$f(x)=\frac{\Gamma\left(\frac{d}{2}\right)\sin\left(\frac{\pi\alpha}{2}\right)}{\pi^{\frac{d}{2}+1}}\int\limits_{\mathbb{R}^d\setminus\mathbb{B}(z,R)}\frac{\left|R^2-|x-z|^2\right|^{\frac{\alpha}{2}}}{|R^2-|y-z|^2|^{\frac{\alpha}{2}}\;|x-y|^d}g(y)\,\mathrm{d}^d y \tag{82}$$

for $x\in\mathbb{B}(z,R)$. The solution reduces to the standard Poisson integral for $\alpha\to 2$. The crucial difference between the cases $\alpha=2$ and $\alpha<2$ is the dimensionality of the domain of integration. Although it has been known for 80 years [165], the fractional Poisson formula (82) seems to have escaped the attention of many contemporary authors.

19.5 The problem with eq. (82) was pointed out in [82, 84]. While for $\alpha=2$ the solution depends only on values of g on the boundary, for $\alpha<2$ it depends on all values of g in an unbounded infinite exterior domain. For this reason the pioneers never ventured into proposing physical interpretations of their generalized diffusion equations. Indeed, Bochner cautions his readers explicitly writing "*Whether this might have a physical interpretation, is not known to us.*" [23, eq. (7)]. The nonlocality of $-(-\Delta)^{\alpha/2}$ for $\alpha<2$ implies action at a distance and makes it impossible to isolate the physical system from influences of the environment [82, 84]. The skeptical attitude of Bochner remains adequate as long as action at a distance remains unproven in experiment.

19.6 Contrary to Bochner's skepticism towards physical interpretations, it is claimed in "fractional quantum mechanics" [120, eq.(3),eq.(7)] that α is a "*fundamental* [sic!] *parameter in standard quantum and classical mechanics*". In other words, page 395 in [120] claims that physical interpretation is no problem at all. The exponent α is interpreted in [120] along the lines of Section 15 as $\alpha \;\bullet\!-\; \mathrm{d}^{\mathrm{Levy}}_{\mathrm{fractal}}$ the fractal dimension of random paths. But Feynman paths are neither basic nor needed in "standard" quantum and classical mechanics. They have yet to be observed in experiment. Similarly, in mathematics a joint spectral measure for two non-commuting physical observables is yet to be derived.

19.7 On top of postulating a new parameter α for quantum physics, a second fundamental constant D_α with strange dimensions is also postulated in [120]. The *ad hoc* postulate of α necessitates the introduction of D_α as a fundamental constant of nature, above and beyond Planck's constant. But it is unclear how to observe, measure or interpret the fractional constant D_α in experiment.

19.8 Finally, there has been some debate whether the eigenvalues and eigenfunctions for infinite potential wells published by numerous authors are valid or not [101, 133]. Besides such problems with technical aspects, the physical interpretation suggested in [120] currently has no derivation from a Hamiltonian, no phenomenological basis in theoretical physics, and no experimental support.

20 Thermodynamic Interpretation

20.1 A genuine physical interpretation of fractional derivatives in the sense of Definition 2 was established in thermodynamics [68, 69]. Fractional derivatives of thermodynamic potentials permit a generalization of Ehrenfest's classification scheme [44] for thermodynamic phase transitions.

20.2 **Interpretation.** The fractional order α is interpreted as the generalized Ehrenfest order of a thermodynamic phase transition

$$\alpha \;\cdot\!-\; 2-\varphi_1 \tag{83}$$

where $\varphi_1 = \max_i\{\varphi_i\}$ is the largest thermodynamic fluctuation exponent.

20.3 The thermodynamic fluctuation exponents characterize a thermodynamic system in the vicinity of a phase transition [189, 66, 52]. The exponents form a partially ordered set. Its maximum, denoted as φ_1, characterizes fluctuations of the order parameter [200, 160]. For a liquid-gas system the maximal fluctuation exponent is related as

$$\delta = \frac{1}{1-\varphi_1} \tag{84}$$

to the equation of state exponent δ. In this way α is directly given by the equation of state of the physical system.

20.4 Thermodynamic equations of state are measurable experimentally [8]. Theoretically they follow from thermodynamic potentials [59]. Thermodynamic potentials are in turn obtained from the Hamiltonian H of the physical system via the basic formula $U = \langle H \rangle$ for the thermodynamic internal energy U. Here the expectation value map $\langle\,\cdot\,\rangle : \mathscr{A} \to \mathbb{R}^+$ is a continuous, positive and normalized linear functional on the C*-algebra of observables of the physical system [64]. For given (inverse) temperature $\beta \in \mathbb{R}^+$ it can be defined for all $A, B \in \mathscr{A}$, $t \in \mathbb{R}$ by the KMS-condition [26] $\big\langle (T^t A)B \big\rangle = \big\langle B(T^{t+i\beta}A) \big\rangle$ where the map $T^z : \mathscr{A} \to \mathscr{A}, z \in \mathbb{C}$ is defined as $T^z A = \exp(iHz)A\exp(-iHz)$ for all $A \in \mathscr{A}$. The KMS-characterization links Hamiltonian mechanics, equilibrium statistical mechanics and thermodynamics with each other.

20.5 Localized fractional derivatives with respect to the field h conjugate to the order parameter of the phase transition appear in the fractional Clausius-Clapeyron equation in [77, p. 458f]. Examples are derivatives of order $\alpha = 4/3$

for mean-field theory, of order $\alpha = 16/15$ for the two-dimensional Ising model, or $\alpha = 2d/(d+2)$ for the spherical model in $2 < d < 4$ dimensions. Numerical examples are derivatives of order $\alpha \approx 1.208$ for the three-dimensional Ising model or $\alpha \approx 1.216$ for the three-dimensional Heisenberg model.

20.6 The classification of phase transitions and the thermodynamic interpretation of α as $\alpha \;\cdot\!-\; 1 + (1/\delta)$ was introduced in [68] and further developed in [69, 70, 72, 71, 77]. It has subsequently been extended to topological pressure functionals for dynamical systems in [172].

21 Classification of Long Time Limits

21.1 This section gives a partial answer to question (a) in Section 18. The classification of limits is a physical interpretation in the sense that it gives a general bound $0 < \alpha < 1$ on α, and links α to a subset of measure zero of the microscopic state or phase space of a physical system. It does not establish a direct link with the Hamiltonian. Instead, the classification provides a mathematical framework for the concept of local equilibrium in nonequilibrium statistical physics. This answers a fundamental question formulated in [187, Sec.2.4,p.25].

21.2 Mathematically, the problem studied is that of induced automorphisms on subsets of measure zero in ergodic theory [36]. Given the transition map $T : \mathsf{X} \to \mathsf{X}$ of a dynamical system between any two time instants $t_0 < t_1 \in \mathbb{R}$ its iterates $T^k f$, $k \in \mathbb{N}, f \in \mathsf{X}$ represent the state of the system on the arithmetic progression of time instants

$$\mathbb{A} = \{t_0 + k\tau : k \in \mathbb{N}\} \subset \mathbb{R} \tag{85}$$

where $\tau = t_1 - t_0 > 0$ and t_0 is the initial instant. The classification arises from investigating the induced automorphism $T_{\mathsf{Y}} : \mathsf{Y} \to \mathsf{Y}$ induced by T for $k \to \infty$ on a subset $\mathsf{Y} \subset \mathsf{X}$ of small or zero measure. Its iterates T_{Y}^N represent the state of the system on the arithmetic progression of time instants

$$\mathbb{A} = \{Nt_0 + k\tau : k \in \mathbb{N}, N \in \mathbb{N}\} \tag{86}$$

for large $k \to \infty$. The classification emerges from the limit $N \to \infty$ while rescaling the time axis.

21.3 The result of taking the limit and rescaling the time axis yields a one-parameter family of semigroups

$$T_\alpha(t) = \Phi_t(\mu_\alpha), \qquad t \geq 0 \tag{87a}$$

of the form of eq. (67) with parameter $0 < \alpha \leq 1$. The probabiliy measure $\mathrm{d}\mu_\alpha = h_\alpha(t)\,\mathrm{d}t$ has the density function

$$h_\alpha(x) = \begin{cases} 0 & \text{for } x \le 0 \\ \dfrac{1}{x}\displaystyle\sum_{j=0}^{\infty} \dfrac{(-1)^j x^{-\alpha j}}{j!\,\Gamma(-\alpha j)} & \text{for } x > 0. \end{cases} \tag{87b}$$

with respect to Lebesgue measure. The semigroups T_α arise from subordination [167, 24] with a stable subordinator [208, Sec.IX.11]. Concrete examples are given in Sections 22 and 23 below.

21.4 Originally, the classification of long time limits emerged from the classification of phase transitions of Section 20. The main result (87) was described in [71, Sec.V]. Equation (87) was initially obtained for classical systems in [75, 74]. It was then rederived by interpreting convolution as time averaging in [79].

21.5 The inequality $t \ge 0$ in (87a) reflects irreversibility of time. The result $t \ge 0$ provides *"a general and model-independent mechanism for the origin of macroscopic time irreversibility"* [75] as well as additional *"insight into the longstanding irreversibility paradox"* [74, p.544]. This insight was enunciated as the "reversed irreversibility problem" introduced and solved in [81, 83]. The "reversed irreversibility problem" is the problem to explain the abundance of time reversal invariant dynamical systems in physics given that time evolution is irreversible at the fundamental level [81, p. 235],[83, 85].

21.6 Ergodicity breaking understood as *"invariance breaking"* or *"stationarity breaking"* was introduced in [75, 74]. The phenomenon was called *"fractional ergodicity"* in [75] or *"fractional stationarity"* in [74], and it emerges spontaneously from the dynamics. Its relevance for aging phenomena in glasses and other systems has long been appreciated [135, 45, 4, 6, 186, 25, 80, 141, 86].

21.7 More recently the basic result (87) has been generalized to quantum systems [85, 86, 88]. The invariance breaking expressed by eq. (87) resolves the fundamental puzzle [187, Sec.2.4,p.25] of local equilibrium in nonequilibrium statistical physics.

22 CTRW Interpretation

22.1 The continuous time random walk (CTRW) interpretation, discovered in [71, 89, 73], emerged from the classification of long time limits (Section 21) and the stochastic process interpretation (Section 11). It illustrates and exemplifies eq. (87) for the case of master equations and Fokker-Planck equations [56].

22.2 Continuous time random walks are parametrized by a waiting time density $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ and a transition probability $\lambda : \Omega \times \Omega \to [0, 1]$ for transitions between two states in Ω. The integral equation for CTRW's reads

$$p(z,t) = \delta_{zz_0}\Phi(t) + \int\limits_0^t \psi(t-s) \sum_{z'\in\Omega} \lambda(z,z')p(z',s)\,\mathrm{d}s, \tag{88}$$

where $z, z_0 \in \Omega$, $t \geq 0$, $\Phi(t) = 1 - \int_0^t \psi(s)\mathrm{d}s$, and $p(z,t)$ is the probability to be in state z at time t if the walker started from state z_0 at time 0.

22.3 It was shown in [89] that eq. (88) is exactly equivalent to the fractional master equation

$$D^{\alpha,1}_{0+}\, p(z,t) = \sum_{z'} w(z,z')p(z',t), \qquad p(z,0) = \delta_{zz_0}, \tag{89}$$

for all $t \geq 0$, if and only if $\lambda = 1 + \tau^\alpha w$ and $\psi = \psi_\alpha/\tau$ holds true. Here

$$\psi_\alpha\left(\frac{t}{\tau}\right) = \left(\frac{t}{\tau}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-\frac{t^\alpha}{\tau^\alpha}\right) = \frac{1}{t}\sum_{k=0}^{\infty} \frac{(-t/\tau)^{\alpha(k+1)}}{\Gamma(\alpha(k+1))}, \tag{90}$$

$\tau > 0$ is a time constant, $w : \Omega\times\Omega \to \mathbb{R}$ are the transition rates between two states [56], and $E_{\alpha,\alpha}$ denotes the generalized Mittag-Leffler function [97, 3, 98, 181]. For $\alpha \to 1$ one has $\psi_1(x) = e^x$ and the result was known [21]. Recently the result was extended to composite CTRW's [87], where a binomial Mittag-Leffler function (from [90]) appears for the waiting time density ψ.

22.4 **Interpretation.** The CTRW interpretation

$$\alpha \text{ in eq. (89)} \;\bullet\!\!-\; \text{parameter } \alpha \text{ in eq. (90)} \tag{91}$$

holds that the fractional order is the parameter of the waiting time density.

22.5 Continuous time random walks were introduced by Montroll and coworkers into solid-state, chemical and statistical physics as an idealization on a mesoscopic level of description [149, 148, 150, 211, 201, 95]. They were later studied also in mathematics [140, 20, 19] and have found applications to exciton trapping [146, 152], β''-alumina superionic conductors [91], organic photoconductors [17], dielectric relaxation [147, 22], turbulent plasmas [13], semiconductors [121], electron transport in noncrystalline electrodes [155], or transient photocurrents in amorphous solids [175], to name but a few.

22.6 The fractional master equation for continuous time random walks was introduced only much later in [89]. Contrary to the presentation in [184, p.50f] there is no mention of fractional derivatives in [11] and no mention of CTRW's in [177]. Although the relation between continuous time random walks and generalized master equations [105, 21], as well as the asymptotic Fourier-Laplace solution [195, eq.(21),p.402] [182, eq.(23),p.505] [106, eq.(29),p.3083] were known, the connection to fractional master equations and fractional diffusion had been overlooked.

23 Anomalous Diffusion Interpretation

23.1 The CTRW interpretation „*plays a particularly important role in the theory of fractal time processes by virtue of its universality [71, 75]*“ as emphasized in [73]. For lattice walks with lattice constant $\sigma > 0$, where $\Omega = \sigma\mathbb{Z}^d$, the CTRW interpretation leads asymptotically to the fractional diffusion equation

$$D^{\alpha;1}_{0+}\, p(\mathbf{r}, t) = C_\alpha\, \Delta p(\mathbf{r}, t), \qquad p(\mathbf{r}, 0) = \delta_{\mathbf{r}\mathbf{r}_0}, \tag{92}$$

for any waiting time density ψ from the class of regularly varying functions with index $0 < \alpha \leq 1$ (see A.8 for the definition). Asymptotically means that position $\mathbf{r}$ and time t are rescaled in a suitable way. Equivalently, the limits $\sigma \to 0, \tau \to 0$ are taken such that

$$\frac{\sigma^2}{2\tau^\alpha} \xrightarrow[\tau\to 0]{\sigma\to 0} C_\alpha \tag{93}$$

is the fractional diffusion constant C_α in (92) (see [73, 80],[79, Sec. 3.4]).

23.2 **Interpretation.** The anomalous diffusion interpretation holds that

$$\alpha \text{ in eq. (92)} \quad \mathrel{\cdot\!-} \quad \text{index of regular variation of } \psi \text{ in eq. (88)} \tag{94a}$$

or, equivalently,

$$\alpha \quad \mathrel{\cdot\!-} \quad 1 + \frac{1}{\log b} \log\left(\lim_{t\to\infty} \frac{\psi(bt)}{\psi(t)} \right) \tag{94b}$$

where $b > 0$ is arbitrary as long as the limit exists for more than countably many values of b.

23.3 Note, that Proposition A „$p(\mathbf{r}, t)$ *satisfies a fractional diffusion equation*“ and Proposition B „$p(\mathbf{r}, t)$ *is the solution of a CTRW with long time tail*“ are not equivalent [80]. Some claims in this direction are too general [35, 184, 142, 15] and some comments in [16] are invalid. The simultaneous limits $\sigma \to 0, \tau \to 0$ can and must be taken in various ways to explore the different asymptotic regions in the parameter space of a given lattice CTRW-model [80]. Note that [80, eq.(22)] does not imply $p(\mathbf{r}, t) = 0$. As referenced in [80, p.38f] it is well known [18, p.202], that whenever a sequence $\phi_n(k)$ of characteristic functions converges pointwise to some function $\phi(k)$ for all k, then the propositions (a) $\phi(k)$ is the characteristic function of some random variable, (b) $\phi(k)$ is a continuous function of k, and (c) $\phi(k)$ is continuous at $k = 0$, are equivalent.

23.4 The fundamental solutions for the integral formulation of eq. (92) were studied in [177, 109, 110]. In [73, Table 1] their stretched Gaussian asymptotic behaviour, their cusp at the origin and their relation with CTRW's were found. Fundamental solutions for eq. (92) with arbitrary types $0 \leq \beta \leq 1$ are given in [79, Sec. 3.3]. Fractional diffusions of type $\beta \neq 1$ are not expected to arise asymptotically from a CTRW-model, because they do not have a probabilistic interpretation (see [78] and [79, Sec. p.116ff]).

23.5 The continuous time random walk interpretation and the anomalous diffusion interpretation of α, in eq. (94) are tentative in the sense of Definition 2. Theoretical arguments for continuous time random walks [211] fall short of a rigorous derivation from the Hamiltonian of a physical system. While there is some experimental support (see point 22.5 above), the waiting time density ψ remains a hypothetical mesoscopic quantity. It is difficult to measure ψ directly in an experiment.

A Appendix

A.1 Let $\mathsf{X}, \mathsf{Y}, \mathsf{Z}$ be Banach spaces. A *linear operator* $A : \mathsf{X} \to \mathsf{Y}$ is a linear subspace of the direct sum $\mathsf{X} \oplus \mathsf{Y}$. The *domain* $\mathsf{D}(A)$, *range* $\mathsf{R}(A)$ and *kernel* (or *null space*) $\mathsf{N}(A)$ of a linear operator A are

$$\mathsf{D}\,(A) := \{f \in \mathsf{X} : \exists\, g \in \mathsf{Y} \text{ s.t. } (f, g) \in A\} \tag{95a}$$

$$\mathsf{R}\,(A) := \{g \in \mathsf{Y} : \exists\, f \in \mathsf{X} \text{ s.t. } (f, g) \in A\} \tag{95b}$$

$$\mathsf{N}\,(A) := \{f \in \mathsf{X} : (f, 0) \in A\} \tag{95c}$$

and A is called *injective* if $\mathsf{N}(A) = 0$ and *surjective* if $\mathsf{R}(A) = \mathsf{Y}$. The set of all linear operators from X to Y is denoted $\mathcal{A}(\mathsf{X}, \mathsf{Y})$ and $\mathcal{A}(\mathsf{X}, \mathsf{X}) = \mathcal{A}(\mathsf{X})$ for short.

A.2 For $\lambda \in \mathbb{C}$ the *scalar multiple* $\lambda\, A$ and the *inverse* A^{-1} of A are defined as

$$\lambda\, A := \{(f, \lambda g) \in \mathsf{X} \oplus \mathsf{Y} : (f, g) \in A\}\,, \qquad \mathsf{D}(\lambda\, A) = \mathsf{D}(A) \tag{96}$$

$$A^{-1} := \{(g, f) \in \mathsf{Y} \oplus \mathsf{X} : (f, g) \in A\}\,, \qquad \mathsf{D}(A^{-1}) = \mathsf{R}(A). \tag{97}$$

For $A, B \in \mathsf{X} \oplus \mathsf{Y}$ their *sum* is defined as

$$A + B := \{(f, g + h) \in \mathsf{X} \oplus \mathsf{Y} : (f, g) \in A, (g, h) \in B\} \tag{98}$$

with $\mathsf{D}(A + B) = \mathsf{D}(A) \cap \mathsf{D}(B)$. For $A \in \mathsf{X} \oplus \mathsf{Y}, B \in \mathsf{Y} \oplus \mathsf{Z}$ the *composition* $B \circ A : \mathsf{X} \to \mathsf{Z}$ is the linear operator defined as

$$B \circ A := \{(f, h) \in \mathsf{X} \oplus \mathsf{Z} : \exists g \in \mathsf{Y} \text{ s.t. } (f, g) \in A \text{ and } (g, h) \in B\} \tag{99}$$

with $\mathsf{D}(B \circ A) = \{f \in \mathsf{D}(A) : \exists g \in \mathsf{D}(B) \text{ s.t. } (f, g) \in A\}$. The *identity operator* $\mathsf{1} : \mathsf{X} \to \mathsf{X}$ is defined as

$$\mathsf{1} := \{(f, f) : f \in \mathsf{X}\} \tag{100}$$

and its scalar multiples will be abbreviated as $\lambda 1 = \lambda$.

A.3 An operator A is called *continuous* if there exists a constant $c \geq 0$ such that $\| A f\| \leq c\|f\|$ for all $f \in \mathsf{D}(A)$. An operator $A : \mathsf{X} \to \mathsf{Y}$ is called *bounded*, if it is continuous and $\mathsf{D}(A) = \mathsf{X}$. The set of all bounded linear operators from X to Y is denoted $\mathcal{B}(\mathsf{X}, \mathsf{Y})$ and $\mathcal{B}(\mathsf{X}) := \mathcal{B}(\mathsf{X}, \mathsf{X})$ for short.

A.4 An operator $A : \mathsf{X} \to \mathsf{X}$ is called *closed* if its graph $\{(f, A f) : f \in \mathsf{D}(A)\}$ is a closed subspace in $\mathsf{X} \oplus \mathsf{X}$. Equivalently, if its domain $\mathsf{D}(A)$ endowed with the graph norm $\|f\|_A := \|f\| + \| A f\|$ is a Banach space. The set of closed operators is denoted $\mathcal{C}(\mathsf{X})$.

A.5 For a linear operator $A : \mathsf{X} \to \mathsf{X}$ the set

$$\rho(A) := \left\{\lambda \in \mathbb{C} : (\lambda - A)^{-1} \in \mathcal{B}(\mathsf{X})\right\} \tag{101}$$

is called *resolvent set* and $\sigma(A) := \mathbb{C} \setminus \rho(A)$ *spectrum* of A. The operator

$$R(\lambda, A) := (\lambda - A)^{-1} \tag{102}$$

is called *resolvent operator at* λ.

A.6 An operator A is called *non-negative* if $\mathbb{R}^-$ is contained in its resolvent set and $\|\lambda(\lambda+A)^{-1}\| \leq M$ for $0 < \lambda < \infty$ [115]. An operator A is called *positive* if it is non-negative and $0 \in \rho(A)$. Let

$$\mathbb{S}_\theta := \{z \in \mathbb{C} : z \neq 0, |\arg z| < \theta\} \tag{103}$$

denote a *sector of angle* θ in the complex plane and $\overline{\mathbb{S}}_\theta$ its closure. An operator A is called *sectorial* of angle $\theta < \pi$ if $\sigma(A) \subset \overline{\mathbb{S}}_\theta$ and $\sup\{\|\lambda R(\lambda, A)\| : \lambda \notin \overline{\mathbb{S}}_{\theta'}\} < \infty$ for all $\theta < \theta' < \pi$.

A.7 A function $f : \mathbb{R}^n \to \mathbb{C}$ is called *negative definite*, if

$$\sum_{i,j=1}^{n} (f(x_i) + \overline{f(x_j)} - f(x_i + x_j))c_i\overline{c}_j \geq 0 \tag{104}$$

for all $x_1, \ldots, x_n \in \mathbb{R}^n$ and $c_1, \ldots, c_n \in \mathbb{C}$ [24].

A.8 A measurable function $f : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to *vary regularly at infinity with index* α, if

$$\lim_{x\to\infty} \frac{f(bx)}{f(x)} = b^\alpha \tag{105}$$

for all $b > 0$ [179]. For this to hold true it suffices that the limit exists on a set of b with positive measure.

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