Tutorial

The Finite Difference and Finite element methods

Joan J. Cerdà *

December 9, 2009
ICP, Stuttgart

Contents

1 Introduction

This tutorial is intended to strengthen your understanding on the finite difference method (FDM) and the finite element method (FEM). These two techniques will allow you to solve numerically many ordinary and partial differential equations. Such techniques have been developed and improved through many years, here, we just play around with the basics of those methods in order that you become more aware of the potential of such techniques to solve partial differential equations (PDE). If you want to learn more about FDM and FEM methods, we refer you to the "To learn more" section at the end of this tutorial.

To do the tutorial you must decompress the file *"tutorial-finite-diff-and-elements.tar"*(use tar -xzf file_name). You will find a set of folders containing the programs needed for each section. **BE CAREFUL**, errors have been introduced in the C-Codes on purpose in order to make sure you will read and understand the codes before using them.

*jcerda put here an at symbol icp.uni-stuttgart.de

2 Finite Difference Methods

Let's assume for simplicity we have just the time t and one spatial coordinate x . We further assume that space and time are discretized into steps Δx and Δt respectively. Thus

$$\Phi(i,j) \equiv \Phi(\ x\ =\ i\,\Delta x\ ,\ t\ =\ j\,\Delta t\ ). \tag{1}$$

Given this notation, it is possible to write down the first and second central partial differences as:

$$\left(\frac{\partial\Phi}{\partial x}\right)_{i,j} \approx \frac{\Phi(i+1,j)-\Phi(i-1,j)}{2\,\Delta x} \tag{2}$$

$$\left(\frac{\partial\Phi}{\partial t}\right)_{i,j} \approx \frac{\Phi(i,j+1)-\Phi(i,j-1)}{2\,\Delta t} \tag{3}$$

$$\left(\frac{\partial^2\Phi}{\partial x^2}\right)_{i,j} \approx \frac{\Phi(i+1,j)-2\,\Phi(i,j)+\Phi(i-1,j)}{(\Delta x)^2} \tag{4}$$

$$\left(\frac{\partial^2\Phi}{\partial t^2}\right)_{i,j} \approx \frac{\Phi(i,j+1)-2\,\Phi(i,j)+\Phi(i,j-1)}{(\Delta t)^2}. \tag{5}$$

In 1974, Crank and Nicholson proposed an implicit formula to compute the $\frac{\partial^2\Phi}{\partial x^2}$ that consist on taking the average of the central difference formulas for the time steps j and $j+1$, namely:

$$\begin{aligned}\left(\frac{\partial^2\Phi}{\partial x^2}\right)_{i,j} \approx \frac{1}{2}\Bigg[&\frac{\Phi(i+1,j)-2\,\Phi(i,j)+\Phi(i-1,j)}{(\Delta x)^2}\\ &+\frac{\Phi(i+1,j+1)-2\,\Phi(i,j+1)+\Phi(i-1,j+1)}{(\Delta x)^2}\Bigg],\end{aligned} \tag{6}$$

this last scheme is said to be *implicit* because if you have discretized your total space L into $n = L/\Delta x$ segments, then to solve the differential equation for at time step $j = 0$ you will need to solve n simultaneous algebraic equations with n unknown values of Φ, for the time step $j = 1$ another n algebraic equations, and so on, so forth ....

2.1 The diffusion equation (an example of parabolic PDE)

Let's suppose we have a rod of length $L = 2m$ which is initially at a temperature of $T = 473K$. Let's assume the rod is isolated except at the two ends ($x = 0$ and $x = L$). At time $t = 0$ we put the two ends in contact with an infinite reservoir of ice that keep them constantly at a temperature of $273K$. We want to know how the temperature changes as a function of position and time.

The previous problem is related to solving the diffusion equation for the internal temperature $T(x,t)$

$$\alpha\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}, \tag{7}$$

subject to boundary conditions

$$T(0,t) = 273K \quad (8)$$
$$T(2,t) = 273K \quad (9)$$

and to the initial condition

$$T(x,0) = 473K. \quad (10)$$

In the previous equation α is the so-called *thermal diffusivity* (to know more, see for instance the Wikipedia: *http://en.wikipedia.org/wiki/Heat_equation* and *http://en.wikipedia.org/wiki/ Thermal_diffusivity*. There, you can find that for Carbon steel at 1% the thermal diffusivity is $\alpha = 1.172 \times 10^{-5} m^2/s$.

Tasks

1. Derive an expression to compute $T(i, j+1)$ using $T(i+1, j)$, $T(i, j)$ and $T(i+1, j)$. Can you use the centered difference for time? Justify your answer. **Hint: Use as notation** $r \equiv \alpha \Delta t/(\Delta x)^2$.

2. Modify the c code "./sources/example-finite-differences-1.c" to solve the previous problem. Plot the temperature profiles (T vs x) at $t = 1.0s,\ 10s,\ 100s,\ 1000s$ and $t = 10^5 s$. How does it look the stationary solution this system? How it will look if the central point of the rod is always maintained at $T = 473K$?

3. Is it possible to use whatever value for r (see task 1)? In case it is not possible , show an example illustrating what occurs. and try to explain why this happens? If you think it is possible to use whatever value of r, plot the temperature profiles using $\Delta x = 0.01$ and $\Delta t \geq 5$. Justify your answer.

4. The analytic solution to the previous problem is

$$T(x,t) = 273 + \frac{4(473-273)}{\pi} \sum_{k=0}^{k=\infty} \frac{1}{2k+1} sin((2k+1)\pi x/L) exp(-\alpha t (2k+1)^2 \pi^2/L^2). \quad (11)$$

   Check your numerical solutions against the exact solution. For a derivation of the previous analytical formula, see for instance the web-page: *http://www.tmt.ugal.ro/crios/ Support/ANPT/Curs/math/s10/ex10_2/ex10_2.html* . Notice that previous formula is not valid in case the two tips of the rod have a different temperature.

5. Write down the scheme of the matrix to be solved if one wants to use the Crank-Nicholson algorithm to solve the previous problem. **Hint: Use again as notation** $r \equiv \alpha \Delta t/(\Delta x)^2$.

6. Use now the c code "./sources/example-finite-differences-2.c" which is based on the Crank-Nicholson scheme and check if one is able to get right solutions for whatever value of r. Check for instance the case $\Delta x = 0.01$ and $\Delta t \geq 5$. Do the two numerical methods agree?

7. Suppose now that our rod has a length $L = 5cm$ and one end of the rod is kept at $300K$ and the other at $373K$. Initially the rod is at $T = 300K$. How is now the temperature profile at $t = 0.5s,\ 1.0s,\ 10s,\ 50s$ and $t = 500s$. How will look the stationary profile? Compare those profiles with those of a rod made of wood ($\alpha = 8.2 \times 10^{-8} m^2/s$). If you want to make a handle for a pot, which kind of material would you prefer, why? Is it enough to look at the temporal evolution of the temperature profiles or are there some other factors one should also take into account? Justify your answer.

2.2 The wave equation (an example of Hyperbolic PDE)

Let's suppose we want to solve the wave equation

$$\frac{\partial^2 \Phi}{\partial x^2} = \frac{\partial^2 \Phi}{\partial t^2} \tag{12}$$

in the spatial region $0 < x < 1$ and for $t \geq 0$ subject to the initial conditions $\Phi(x, 0) = sin(\pi\, x)$ and $\frac{\partial \Phi}{\partial t} = 0$ for $0 < x < 1$. The boundary conditions are $\Phi(0, t) = \Phi(1, t) = 0$.

Tasks

1. Derive an expression relating $\Phi(i, j+1)$ to $\Phi(i, j)$, $\Phi(i+1, j)$, $\Phi(i-1, j)$ and $\Phi(i, j-1)$.

2. Is there some trouble to modify the program "./sources/example-finite-differences-1.c" and get a solution for the PDE?

3. Try to solve the PDE using the starting formula

$$\Phi(i, 1) = (1 - r)\Phi(i, 0) + \frac{r}{2}\left[\Phi(i - 1, 0) + \Phi(i + 1, 0)\right] \tag{13}$$

   where

$$r \equiv \left(\frac{\Delta t}{\Delta x}\right)^2. \tag{14}$$

   Can you explain how the previous starting formula is obtained? **Hint: we recommend you to use r=1 to solve the PDE, as you can see many terms will then simplify.**

4. Compare your numerical solution to the analytical solution

$$\Phi(x, t) = sin(\pi x)\, cos(\pi t) \tag{15}$$

   Which is the range of values of r suitable to get an accurate solution of the PDE? Can you proof why should be that the range of validity of r?

2.3 The Poisson's equation (an example of Elliptic PDE)

Let's suppose we want to solve the following two-dimensional Poisson's equation:

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = g(x, y). \tag{16}$$

If we use central differences, and for simplicity we set $\Delta x = \Delta y = h$ then we obtain

$$\Phi(i,j) = \frac{1}{4}\left[\Phi(i+1,j) + \Phi(i-1,j) + \Phi(i,j+1) + \Phi(i,j-1) - h^2 g(i,j)\right] \tag{17}$$

which leads to solving a system of algebraic equations of the style $[A][X] = [B]$ where $[X]$ are the unknown values, and $[B]$ is a colum matrix containing the know values of Φ at the fixed nodes. There are several methods to find $[X]$ which becomes very tricky when we have many grid nodes because the matrixes are in that case very large and it takes a lot of time to find numerically the solution. Here we will use a very simple method called successive over-relaxation (SOR) which basically consist on doing the following: we define the residual at node (i,j) as

$$R(i,j) \equiv \Phi(i+1,j) + \Phi(i-1,j) + \Phi(i,j+1) + \Phi(i,j-1) - 4\Phi(i,j) - h^2 g(i,j), \tag{18}$$

i.e., the amount by which Φ at point (i,j) does not satisfy the Poisson's equation. We iterate the process as follows, the $k+1$ iteration is obtained from the k iteration as

$$\Phi^{(k+1)}(i,j) \equiv \Phi^{(k)}(i,j) + \frac{\omega}{4} R^{(k)}(i,j) \tag{19}$$

the optimal value of the parameter ω, i.e. the one that speeds at maximum the convergence must be found by trial and error. When $\omega = 1$ the method is known as successive relaxation. Alternatively, in order to speed calculations, one can use for the iterations

$$\begin{aligned} \Phi^{(k+1)}(i,j) \quad \equiv \quad & \Phi^{(k)}(i,j) + \frac{\omega}{4}\Big[\Phi^{(k)}(i+1,j) + \Phi^{(k+1)}(i-1,j) + \Phi^{(k)}(i,j+1) \\ & + \Phi^{(k+1)}(i,j-1) - 4\Phi^{(k)}(i,j) - h^2 g(i,j)\Big] \end{aligned} \tag{20}$$

When this last expression is used, it is known that the optimal value of $\omega \in [1,2]$.

Let's suppose we want to solve the Poison equation in square of size $L = 1$ using a grid of 5×5 nodes ($i \in [0,4]$, and $j \in [0,4]$], where the following boundary conditions exist:

$$\begin{aligned} \Phi(i,0) &= 0 \quad for\ i = 1,2,3 \\ \Phi(i,4) &= 20 \quad for\ i = 1,2,3 \\ \Phi(0,j) &= -10\ for\ j = 1,2,3 \\ \Phi(4,j) &= +10\ for\ j = 1,2,3 \\ \Phi(0,0) &= -5 \\ \Phi(4,0) &= +5 \\ \Phi(0,4) &= +5 \\ \Phi(4,4) &= +15 \end{aligned}$$

and $g(x,y) = x\,(y-1)$.

Tasks

1. Proof equation 17.

2. Use the c code "./sources/example-finite-differences-3.c" to solve the previous PDE. Is there some difference in the values of ω one can use when using eq. 20 or eq. 19.

3. Refine the mesh to $N \times N$ grid points , with $N = 5, 10, 20, 50, 100, 500, 1000$. Use the eq. 20 to do the calculations. When N increases, you get convergence faster or slower (try to tune w to an optimal value by trial and error) ? Compare ω values you have found with the theoretical prediction that states that when using eq. 20 in a rectangular region, the optimum over-relaxation factor is given by the smaller root of the equation: $t^2\,\omega^2 - 16\,\omega + 16 = 0$ where $t = cos(\pi/N_x) + cos(\pi/N_y)$ and $N_x = N_y = N$ in the case of a square region. How does the time needed to solve the equation evolve with N. **Hint: use the command *"time"*** .

4. Check your numerical solution against the know exact solution given in the following table:

Table 1: Values for the exact solution

x	y	Exact Value
1/4	1/4	-3.429
1/4	2/4	-2.029
1/4	3/4	4.277
2/4	1/4	-0.1182
2/4	2/4	2.913
2/4	3/4	9.593
3/4	1/4	2.902
3/4	2/4	6.065
3/4	3/4	11.13

5. In the section 2.1, when dealing with the Crank-Nicholson algorithm: can we use the (SOR) method to solve the system of equations that appear there? Justify your answer.

3 Finite Element Methods

In this section we plan to build up a very simple one-dimensional FEM method which will not allow you to caress the whole power of the FEM method but it will help you to understand the fundamentals of the method. In fact, the example we will work out is just selected for pedagogical reasons, the resolution of that particular problem could be done more easily using FDM. The powerfulness of FEM arises when more complex geometries are needed, and FDM becomes harder and harder to implement and its demands of computational power increase excessively.

In our tutorial we will study the case of a cylindrical rod of radius R and length L with one end insulated an the other held to constant temperature T_{tip}, while the surrounding environment has a temperature T_{env} (see figure 1). We assume the cylinder lies along the x-axis, and the insulated tip is located at $x = 0$. The lost of heat occurs via the lateral surface of the cylinder at a rate characterized by the heat transfer coefficient h which units are $W/(m^2\ K)$, we assume that the thermal conductivity for the bar is k. For such system, we can use the following coordinates

$$\hat{\theta} = (T - T_{env}) \tag{21}$$

$$\eta = \frac{x}{L} \tag{22}$$

$$\mu^2 = L^2\,\frac{2h}{kR} \tag{23}$$

and it can be proved that the governing differential equation of the system is:

$$\frac{d^2\hat{\theta}}{d\eta^2} - \mu^2\hat{\theta} = 0 \tag{24}$$

with the following boundary conditions

$$\frac{d\hat{\theta}}{d\eta} = 0 \qquad at\ \ \eta = 0 \tag{25}$$

$$\hat{\theta} = \hat{\theta}_{tip} = (T_{tip} - T_{env}) \qquad at\ \ \eta = 1. \tag{26}$$

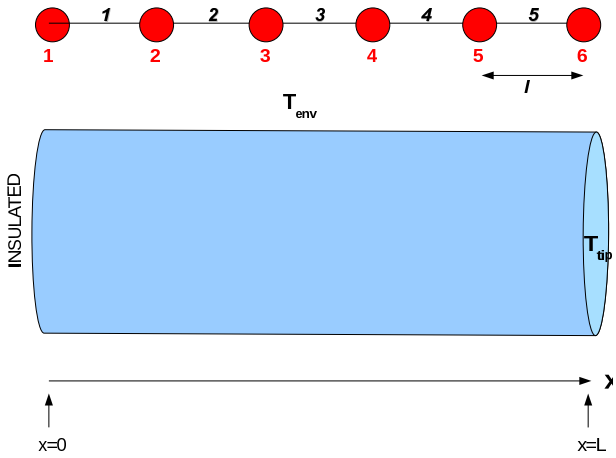

Figure 1: The system we want to study using Galerkin FEM. The plot corresponds to using 5 one-dimensional linear elements, i.e. 6 nodes.

We will use a FEM method known as *the Galerkin finite element method*. As we will see, the name comes from the request we do: we want the approximate solution to obey for each element the *Galerkin constraint*. For our very simple one dimensional problem we choose the *elements* as a straight segments with two *nodes* located at the two ends of each segment, see figure 2. Therefore, we will split the rod bar in N elements, where two adjacent elements have always

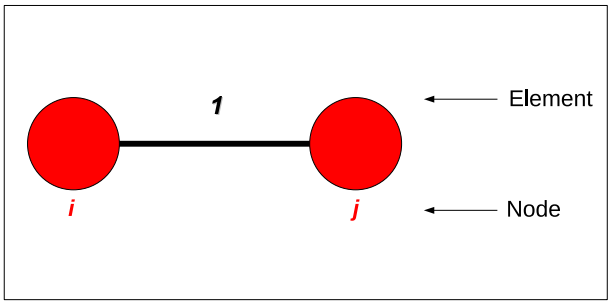

Figure 2: A single one dimensional element

one node in common. Thus, for instance, in the case $N = 5$ elements, we will have a total number of 6 nodes (see figure 1). Let's call l to the length of each one of the elements. What we have done until now is the first step in any FEM problem, the *discretization of the continuum*: we divide the solution region into non-overlapping sub-regions that we call *elements*.

The second step in any FEM method is to choose which type of function will represent the variation of the field (in our case θ) along each element. This second stage is know as *Selection of the shape functions*. The Shape functions are also known as *interpolation functions* or *basis functions*. Here, we will do the most basic assumption, i.e., for a segment with two nodes i and j which have values for θ function that are θ_i and θ_j, respectively, the change from one value to the other will be given by a linear function. Thus,

$$\theta(x) = a_1 + a_2\, x \tag{27}$$

where the constants a_1 and a_2 can be obtained via the two conditions

$$\theta_i = a_1 + a_2\, x_i \tag{28}$$
$$\theta_j = a_1 + a_2\, x_j \tag{29}$$

it is simple to show that one can rewrite equation 27 as

$$\theta(x) = N_i(x)\, \theta_i + N_j(x)\, \theta_j \tag{30}$$

where

$$N_i(x) = \frac{x_j - x}{l}, \tag{31}$$
$$N_j(x) = \frac{x - x_i}{l}, \tag{32}$$

are the so-called *Shape functions* for the 1D linear finite element. You should take into account that this shape functions, are defined to be strictly zero when we are out of the space assigned to the element associated to them. One should remark that using a first order polynomial like eq. 27 (*lineal element*) to account for the variation of the temperature θ inside an element is a quite coarse and rough approach. Thus for instance, one can assume a second degree polynomial like,

$$\theta(x) = a_1 + a_2\, x + a_3\, x^2 \tag{33}$$

but then, in order to calculate the three constants we need a third node to exist in the element (usually placed at the mid-point of the segment). Such type of element is know as a *quadratic element*. The quadratic element (or higher order elements), allow for a better description of what happens inside the element, but of course, the FEM method becomes more complex. At this moment we will keep things a simple as possible and we will assume we use a linear element such that equations 27 and 30 hold.

The third stage in any FEM method is know as the *Formulation* or the *obtaining of the element characteristics*. Basically in this step we determine the matrix equations that will govern the behavior of one single element. To do that, of course, we need to know which governing differential equation does the system (and therefore the element) obey, as well as what are the boundary conditions. In our simple case, the differential equation to obey and the conditions are given in eq. 24 and expressions 25 and 26. Before starting to derive the matrix equations for our element, notice that if equation 30 holds for our linear element, then

$$\frac{d\theta}{dx} = \frac{-1}{l}\theta_i + \frac{1}{l}\theta_j. \tag{34}$$

Because we assume θ within the element to be approached by a expression like eq. 30, we will always have an error respect what would be the exact solution. What we want is to minimize as far as possible the error when we compare the exact solution with the solution obtained via FEM. A nice method to get an approximate solution of the differential equation such that minimizes errors is known as the *Galerkin method* (and from there derives the name of the whole FEM method we are using). The Galerkin method applied to our case says that if our differential equation is eq. 24, our approximate solution θ will have the minimum possible error respect the exact solution if it obeys a set of equations:

$$\int_{rod} N_k \left(\frac{d^2\theta}{d\eta^2} - \mu^2\theta \right) d\eta = 0 \tag{35}$$

where the number of equations is equal to the number of shape functions N_k we have (in our case, two per each element we use). Notice the shape functions N_k act as weighting functions. Given N_k is valid only for the element at which the node k belongs (notice that intermediate nodes will for one element act as the i-node, and for the neighbor node they will act as a j-node), we can reduce the integration over the whole rod to the integration over the element associated to the shape function N_k we consider.

Let's define $\eta_e \equiv l/L$. Because we have previously defined $\eta \equiv x/L$ (where L is the length of the rod), the following equalities hold for a linear element:

$$\frac{d\theta}{d\eta} = L\,\frac{d\theta}{dx} = \frac{-1}{\eta_e}\theta_i + \frac{1}{\eta_e}\theta_j \tag{36}$$

$$N_i(\eta) = 1 - \eta/\eta_e \tag{37}$$

$$N_j(\eta) = \eta/\eta_e. \tag{38}$$

Then, let's compute eq. 35 for the first element using the weighting functions $N_i(\eta)$ and $N_j(\eta)$ which are associated to such element. After a little bit of algebra involving the integration by

parts, we get the following equations must be obeyed by the values of θ at nodes i and j of such element (i.e, θ_i and θ_j):

$$0 = \frac{1}{\eta_e}(\theta_i - \theta_j) + \frac{d\theta}{d\eta} + \frac{\mu^2\,\eta_e}{6}(2\theta_i + \theta_j) \quad (39)$$

$$0 = \frac{1}{\eta_e}(-\theta_i + \theta_j) - \frac{d\theta}{d\eta} + \frac{\mu^2\,\eta_e}{6}(\theta_i + 2\theta_j). \quad (40)$$

The first equation corresponds to use N_i as a weighting function and the second equation is obtained when N_j is used as a weighting function in eq 35 (applied to the first element). Previous equations can be rewritten together in a matrix-like notation as:

$$\frac{1}{\eta_e}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix} + \frac{\mu^2\,\eta_e}{6}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix} + \begin{Bmatrix} \frac{d\theta}{d\eta} \\ -\frac{d\theta}{d\eta} \end{Bmatrix} \quad (41)$$

which is the *element characteristics* of the first element. Thus for instance, when $\mu^2 = 3$ and $\eta_e = 0.2$ (i.e., the system is split in 5 elements) the element characteristics becomes:

$$\begin{pmatrix} 5.2 & -4.9 \\ -4.9 & 5.2 \end{pmatrix}\begin{Bmatrix} \theta_i \\ \theta_j \end{Bmatrix} + \begin{Bmatrix} \frac{d\theta}{d\eta} \\ -\frac{d\theta}{d\eta} \end{Bmatrix} \quad (42)$$

It is possible to show that for the other elements we can get a similar element characteristics.

Now in a fourth stage, we must ensemble all the equations for all the elements. This is very easily done as follows: if in a element characteristics matrix we have a row that refers to the same θ_x than a row in the characteristics matrix of another element, then we just add the two rows together. At the end we must get a matrix expression with as many rows as nodes we have. For instance, if we consider the first two elements (i.e. 3 nodes in total), the equations we get for the system will be:

$$\begin{pmatrix} 5.2 & -4.9 & 0 \\ -4.9 & 10.4 & -4.9 \\ 0 & -4.9 & 5.2 \end{pmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \frac{d\theta}{d\eta} \end{Bmatrix} \quad (43)$$

and if we consider the first three elements (4 nodes) the expression would be

$$\begin{pmatrix} 5.2 & -4.9 & 0 & 0 \\ -4.9 & 10.4 & -4.9 & 0 \\ 0 & -4.9 & 10.4 & -4.9 \\ 0 & 0 & -4.9 & 5.2 \end{pmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ \frac{d\theta}{d\eta} \end{Bmatrix} \quad (44)$$

and so on so forth. Notice that for the row that corresponds to θ_1 the $d\theta/d\eta$ is set to zero due to the boundary condition that we have for $\eta = 0$ (see eq.25).

Because the other boundary condition we have is that the last node should have $\theta = \theta_{tip}$, we can impose this directly in the system of equations to solve. Thus for instance, if we consider

the 5 elements (6 effective nodes):

$$\begin{pmatrix} 5.2 & -4.9 & 0 & 0 & 0 & 0 \\ -4.9 & 10.4 & -4.9 & 0 & 0 & 0 \\ 0 & -4.9 & 10.4 & -4.9 & 0 & 0 \\ 0 & 0 & -4.9 & 10.4 & -4.9 & 0 \\ 0 & 0 & 0 & -4.9 & 10.4 & -4.9 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \\ \theta_6 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \theta_{tip} \end{Bmatrix}. \tag{45}$$

This last expression is the system of equations we have to solve to get the solution via the Galerkin FEM. The result can be generalized to whatever number of elements we want to have (but mind to change η_e). The solution of the system of algebraic equations is the fifth and final stage of the FEM (of course, later, once we know the θ values at each node, we can compute secondary quantities like for instance heat fluxes, etc.). The resolution of the system of equations, as in the case of the FDM method is done via one of the several techniques available. Notice that the matrix is sparse, in fact, band diagonal. The larger is the number of nodes we use, the more sparser becomes the matrix.

Tasks

1. Write down for a rod with $\mu^2 = 3$ and $\eta_e = 0.2$ the element characteristics matrix for a FEM with $N = 5,\ 9$ elements with $\theta_{tip} = 1$.

2. Compute the FEM solution for $N = 5,\ 9,\ 19$ and 99 elements. Because the matrixes you will get for this 1D system are band diagonal matrixes, you can use for instance reuse the subroutines we were using in the file "example-finite-differences-2.c".

3. Check your numerical solution against the known exact solution:

$$\hat{\theta}(\eta) = \frac{e^{+\mu\eta} + e^{-\mu\eta}}{e^{\mu} + e^{-\mu}} \tag{46}$$

4. Solve the same problem using FDM. Use the approach you consider better for doing it. How many lattice points do you need in the FDM to have the same accuracy than using FEM?

5. VOLUNTARY TASK: Instead of using a 1D linear element, use a 1D quadratic element. Now, each element will have three nodes, one of them at the mid-point of the element. Redo the calculations in this section in order to get the shape functions, the element characteristics matrix, and subsequently then the system of algebraic equations to solve. Compare if given the same number of nodes in the system, the quadratic elements give a more accurate result than the linear elements.

4 To learn more

Suitable books to become more learned about FDM and FEM methods are:

- *Numerical Techniques in Electromagnetics*, Matthew N. O. Sadiku. CRC Press (2001). ISBN: 0-8493-1395-3.
- *Fundamentals of the Finite Element Method for Heat and Fluid Flow*, R.W. Lewis, P. Nithiarasu and K. N. Seetharamu. Wiley Ltd. (2004). ISBN: 0-470-84788-3.

There are also some nice web pages devoted to the finite element methods:

- The internet finite element resources page: *http://homepage.usask.ca /ĩjm451/finite/ fe_resources/ fe_resources.html*