Partial Derivatives in Physics

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Abstract

The usage of partial derivatives in physics is often not following the mathematical definition of partial derivatives. This is in a way sad but can sometimes shorten the notation of a mathematical idea. Also in statistical physics this (in a strict sense) improper usage of the partial derivative is widely distributed. Therefore it is important for the authors who use the partial derivative in a non-defined way to identify this case (at least in the text) in order to avoid confusion.

1 Partial derivative, mathematical definition

The partial derivative of a function is defined as[1]:

$$\frac{\partial f}{\partial x_i}(a_1,\dots,a_n)=\lim_{h\to 0}\frac{f(a_1,\dots,a_i+h,\dots,a_n)-f(a_1,\dots,a_i,\dots,a_n)}{h}. \quad (1)$$

This means that *absolutely all* variables that the function f has are fixed (with the exception of the variable x_i the partial derivative acts, clearly identified with $\frac{\partial}{\partial x_i}$.)

2 Non Standard Usage of Partial Derivatives in Physics

Let me begin this section with a common example: Let's consider the the internal energy in the canonical ensemble:

$$U(N,V,T)=\frac{\sum_E E\Omega(E)\exp(-\beta E)}{\sum_E \Omega(E)\exp(-\beta E)}, \quad (2)$$

where the standard abbreviation $\beta(T)=\frac{1}{k_B T}$ is introduced in order to write less symbols (which is the only reason for it). Now we can calculate the heat capacity at constant volume $C_V=\frac{\partial U(N,V,T)}{\partial T}$. At this point *it is superfluous* to note that the other variables are fixed since this is part of the definition of

[1] `http://mathworld.wolfram.com/PartialDerivative.html`

the partial derivative[2]. As an alternative one can also write $\frac{\partial U}{\partial T}|_{N,V}$ instead of $\frac{\partial U(N,V,T)}{\partial T}$. Denoting the function arguments is unimportant as soon as the function name is unique then also $\frac{\partial U}{\partial T}$ would be sufficient. However this is in physics typically not the case!

Now performing the partial derivative $C_V = \frac{\partial U(N,V,\beta)}{\partial T}$ in a strict mathematical sense would simply yield zero since obviously U is not depending on T, but only on β. However it is clear that β was only introduced for convenience and therefore the partial derivative with respect to T acts on β! Therefore the following holds:

$$C_V = \frac{\partial U(N,V,T)}{\partial T} \overset{\text{see footnote for physical notation}}{=} \color{red}{\frac{\partial U(N,V,\beta)}{\partial T}} = \frac{\partial U(N,V,\beta)}{\partial \beta}\frac{\partial \beta}{\partial T} \tag{4}$$

The red usage of the partial derivative is not the standard partial derivative from the definition above and the partial derivative sign ∂ has a (slightly) altered meaning (as given by the following equal sign). In such cases the authors should always state the way the (non standard) partial derivative is meant. This example was rather poor because it is common knowledge to a physics student. One has done this so many times before in the lecture and the corresponding excercises that the point seems trivial. This issue is always trivial if the reader knows that a variable is only introduced for convenience (in order to write less symbols). However trouble arises as soon as a variable was introduced by an other author and the author does not note this anywhere.

Let me give an exaggerated example for this: Let s assume author A defines a function f such that $f(x) = x^4$. Now he decides to save writing an exponent by introducing the shortcut $y = x^3$ and therefore he defines a new function $\tilde{f}(x,y) = xy$. As long as the author A only uses $y = x^3$ $f(x) = \tilde{f}(x,y)$ holds. Of course the derivative f' with respect to x has to yield: $f'(x) = 4x^3$ and the function $\tilde{f}$ should behave similarly: The physicist A writes:

$$\frac{\partial \tilde{f}(x,y)}{\partial x} = \frac{\partial \tilde{f}}{\partial x} \color{red}{+ \frac{\partial \tilde{f}}{\partial y}\frac{\partial y}{\partial x}} = y \color{red}{+ x \cdot 3x^2} \overset{y(x)=x^3}{=} 4x^3 \tag{5}$$

where he used the partial derivative in a non standard way (red) because he knew that he introduced y only for convenience, without any further meaning.

Now another author B comes who does not know about y being only defined for convenience. He only sees the definition $\tilde{f}(x,y) = xy$. He knows that the

[2] However *in statistical mechanics or thermodynamics* it is important to fully identify the internal energy U as a function of N,V,T since e.g. the function U(N,P,T) for the internal energy is not distinguished by the fucntion U(N,V,T) in the function name U, but only in the arguments. Mathematically it would be more stringent to use another function name, e.g. $\tilde{U}$ for the function which assigns the triple N,P,T a value:

$$\tilde{U} : N,P,T \mapsto \tilde{U}(N,P,T). \tag{3}$$

total differential of $\tilde{f}$ is given by:

$$d\tilde{f} = \frac{\partial \tilde{f}}{\partial x}|_y dx + \frac{\partial \tilde{f}}{\partial y}|_x dy. \tag{6}$$

Therefore B knows $\frac{\partial \tilde{f}}{\partial x}$ is given by $\frac{\partial \tilde{f}}{\partial x} = y$. He therefore comes to the conclusion $\frac{\partial \tilde{f}}{\partial x}(y(x)) = y \overset{y(x)=x^3}{=} x^3$ where he used the standard definition of the partial derivative. If author A had written that y is only introduced for convenience and the partial derivative with respect to x is meant to act on y as well, both authors would have gotten to the same result $4x^3$.

3 Conclusion

Using the partial derivative sign ∂ in a non standard way, one should always mention how this partial derivative is meant. Otherwise the reader will default to the standard definition of the partial derivative and will be missguided.