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Advanced Statistical Physics, SS 2017
Sheet 1

Problem 1:

Let $\mathbb{K} \subseteq \mathbb{R}^n$ be a convex set and $f\colon \mathbb{K} \to \mathbb{R}$ a convex function. Define

$$\mathbb{K}_f := \{(x, z) \in \mathbb{R}^{n+1} : x \in K,\ z \in \mathbb{R},\ z \geq f(x)\}. \tag{1}$$

a) Show that $\mathbb{K}_f$ is a convex subset of $\mathbb{R}^{n+1}$.

b) For a general function $f\colon \mathbb{K} \to \mathbb{R}$ show that f is convex on $\mathbb{K}$ if and only if $\mathbb{K}_f$ is a convex subset of $\mathbb{R}^{n+1}$.

Problem 2:

Let I be an open interval in $\mathbb{R}$ and $f\colon I \to \mathbb{R}$ a convex function. Prove the following statements:

a) For each $a, b \in I$, $a \leq b$ there exist $K_{a,b}, k_{a,b} \in \mathbb{R}$ such that $k_{a,b} \leq f(x) \leq K_{a,b}$ for all $a \leq x \leq b$.

b) For each $a, b \in I$, $a \leq b$ there exists $L_{a,b} \in \mathbb{R}$ such that $|f(x) - f(y)| \leq L_{a,b}|x - y|$ for all $a \leq x, y \leq b$.

c) The function f is continuous.

Hint: To prove statement b), first choose $c, d \in I$ such that $c + \epsilon \leq a$ and $b \leq d - \epsilon$ for some $\epsilon > 0$. Let $x, y \in [a, b]$ with $x < y$ and define $z = y + \epsilon$. Find λ such that $y = \lambda z + (1 - \lambda)x$. Use the inequality characterising the convexity of f and solve for $f(y) - f(x)$.

Remark: Similar statements hold for convex functions defined on convex subsets of $\mathbb{R}^n$.

Problem 3:

Show that each of the following functions is convex or concave and calculate their Legendre transforms.

a) The function $\sqrt{1 + x^2}$ defined on $\mathbb{R}$.

b) The function $\cos(x)$ defined on $(-\pi/2, \pi/2)$.

c) The function $|x|$ defined on $\mathbb{R}$.

Problem 4:

Consider the function $U = U(S, V) = S^{-2}\mathrm{e}^V$ for $V \in \mathbb{R}$ and $S > 0$.

a) Show that U is a convex function.

b) Define $T = \frac{\partial U}{\partial S}$. Calculate the partial Legendre transform $F(T, V) = U(S, V) - TS$ of U obtained by replacing S with T.

c) Define $P = -\frac{\partial U}{\partial V}$. Calculate the Legendre transform $G(T, P) = U(S, V) - TS - PV$ of U obtained by replacing S with T and V with P.